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Fixed point theorems for nonexpansive mappings in a locally convex space

P. Srivastava and S.C. Srivastava

Several fixed point theorems for nonexpansive self mappings in metric spaces and in uniform spaces are known. In this context the concept of orbital diameters in a metric space was introduced by Belluce and Kirk. The concept of normal structure was utilized earlier by Brodskiĭ and Mil'man. In the present paper, both these concepts have been extended to obtain definitions of β-orbital diameter and β-normal structure in a uniform space having β as base for the uniformity. The closed symmetric neighbourhoods of zero in a locally convex space determine a base β of a compatible uniformity. For β-nonexpansive self mappings of a locally convex space, fixed point theorems have been obtained using the concepts of β-orbital diameter and β-normal structure. These theorems generalise certain theorems of Belluce and Kirk.

1. Introduction

While studying fixed point theorems for nonexpansive mappings of a metric space into itself Belluce and Kirk [*1*] introduced the concept of limiting orbital diameters and with its help and also using separately the concept of normal structure, obtained fixed point theorems for these mappings in the case of a Banach space. Uniform spaces form a natural extension of metric spaces and the concept of nonexpansiveness in this more

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general setting has been considered by several authors; namely, Brown and Comfort [3], Kammerer and Kasriel [4], and Reinermann [5]. On the other hand, locally convex topological vector spaces are extensions of normed linear spaces. Every normed space is a locally convex topological vector space, while the converse is not true. We further note that every topological vector space is completely regular and therefore it is unformisable. In case of a locally convex space, if we consider as local base at zero, the family β^* of all closed, convex and symmetric neighbourhoods of zero, this family gives rise to a family β of closed, symmetric members of a uniformity $\mathcal{U}$, where β is a base for $\mathcal{U}$. The uniform topology corresponding to $\mathcal{U}$ coincides with the given topology. Hence $\mathcal{U}$ is a natural uniformity for the given locally convex space.

In the present paper the concept of β-*orbital diameters* and of β-*normal structure* in a uniform space have been introduced. These concepts are analogous to those of orbital diameters [1] and of normal structure [2] in a metric space and a normed space.

Theorems 4.1 and 5.4 give fixed point theorems for β-nonexpansive mappings of a locally convex topological vector space into itself. These generalise Theorems 2 and 3 of Belluce and Kirk for a Banach space [1].

2. β-orbital diameter

DEFINITION 2.1. A map $f : X \to X$ is called β-nonexpansive if for any member V of β, $\big(f(x), f(y)\big) \in V$ whenever $(x, y) \in V$.

β-nonexpansive was termed as 'contraction with respect to β' by Brown and Comfort [3]. For a linear topological space Taylor [6] has used the term 'β-nonexpansive' in a different sense. Kammerer and Kasriel [4] have used the term 'weakly β-contractive'.

DEFINITION 2.2. Suppose β is a family of closed symmetric members of a uniformity. Let the map $\delta : \mathcal{D} \to \beta \cup \{\Delta\}$, $\mathcal{D} \subseteq P^*(X)$, where $P^*(X) = P(X) - \{\emptyset\}$ be defined by

(i) $\delta(A) = \Delta$ if and only if A is a singleton, and

(ii) $\delta(A)$ is the smallest member of β containing $A \times A$, if A is not a singleton.

The map exists if and only if for each A $(\in \mathcal{D})$ which is not a singleton,

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$\cap U \in \beta$, where the intersection is taken over all $U \in \beta$ such that $A \times A \subset U$.

The map δ is to be called the β-diametral map on $\mathcal{D}$. $\delta(A)$, for $A \in \mathcal{D}$, is to be called the β-diameter of A . It is clear that if $A \subset B$, then $\delta(A) \subset \delta(B)$. Thus δ is nonincreasing with respect to inclusion ordering.

EXAMPLE 2.3. In case of a metrisable uniformity $\mathcal{u}$ with metric d , $\beta = \{\{(x, y) : d(x, y) \leq r\} : r \in \mathbb{R}\}$ and $\mathcal{D} = P^*(X)$, the diametral map exists.

EXAMPLE 2.4. A discrete space which does not have a countable base provides an example of a non-metrisable space with a diametral map.

DEFINITION 2.5 [1]. For any map $f : X \to X$, the orbit $O(x)$ at $x \in X$ is defined by

$$O(x) = \{x, f(x), f^2(x), \ldots\} .$$

Suppose that the diametral map δ on $\mathcal{D}$ exists where

$$\mathcal{D} = \{O(f^n(x)) : n = 0, 1, 2, \ldots\} .$$

DEFINITION 2.6. Let f be a map on X to itself. f will be said to have β-diminishing orbital diameter at x if $\delta(O(f^n(x))) \neq \delta(O(x))$ for some n , whenever $\delta(O(x)) \neq \Delta$.

3. Lemmas

From now onwards, we assume that X is a locally convex topological vector space. The family β^* of closed, symmetric, and convex neighbourhoods of zero in X induces a base β for a uniformity $\mathcal{u}$ on X given by

$$\beta = \{U = \{(x, y) : x-y \in U^*\} : U^* \in \beta^*\} .$$

LEMMA 3.1. *Suppose* U *and* V *are members of* β . *Then* $U \circ V[z]$ *is convex, for every* $z \in X$.

Proof. Let $y_1, y_2 \in U \circ V[z]$, then $(z, y_1) \in U \circ V$ and $(z, y_2) \in U \circ V$. $(z, y_1) \in U \circ V$ implies that there is an $x_1 \in X$ such that $(z, x_1) \in U$ and $(x_1, y_1) \in V$. Similarly, there is an $x_2 \in X$

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such that $(z, x_2) \in U$ and $(x_2, y_2) \in V$. $(z, x_1) \in U$ implies that $z-x_1 \in U^*$ and $z-x_2 \in U^*$, where $U^* \in \beta^*$, as specified above. Similarly $x_1-y_1 \in V^*$ and $x_2-y_2 \in V^*$. Take $\lambda, \mu \geq 0$ such that $\lambda + \mu = 1$. Then $\lambda z-\lambda x_1 \in \lambda U^*$, $\mu z-\mu x_2 \in \mu U^*$ and so $(\lambda+\mu)z - (\lambda x_1+\mu x_2) \in (\lambda+\mu)U^* = U^*$. Thus $(z, \lambda x_1+\mu x_2) \in U$ and $(\lambda x_1+\mu x_2, \lambda y_1+\mu y_2) \in V$. Hence $(z, \lambda y_1+\mu y_2) \in U \circ V$ and $\lambda y_1 + \mu y_2 \in U \circ V[z]$.

LEMMA 3.2. *If* $U \in \beta$, *then* $U[z] = \bigcap_{V\in\beta} \overline{V \circ U[z]}$. *Consequently* $U[z]$ *is closed for every* $z \in X$.

Proof. It is clear that $U[z] \subset \bigcap \overline{V \circ U[z]}$. Conversely, let $x \in \bigcap_{V\in\beta} \overline{V \circ U[z]}$; that is to say that $x \in \overline{V \circ U[z]}$ for every $V \in \beta$. Hence, for every $V \in \beta$, there is a $y \in V \circ U[z]$ such that $y \in V[x]$. Thus $(z, y) \in V \circ U$ and $(x, y) \in V$, where V is symmetric. Therefore $(z, x) \in \bigcap_{V\in\beta} V \circ U \circ V = U$, because U is closed. This shows that $x \in U[z]$, and the lemma is proved.

4. A fixed point theorem

THEOREM 4.1. *Let* K *be a closed, convex subset of a locally convex space* X *and let* M *be a weakly compact subset of* X. *If* f *is a* β-*nonexpansive mapping of* K *into* K *such that*

(i) *for each* $x \in K$, $\operatorname{clco}(O(x)) \cap M \neq \emptyset$, *and*

(ii) f *has* β-*diminishing orbital diameter at each* $x \in K$,

then there is a point $x \in K$ *such that* $f(x) = x$.

Proof. If $\{K_\alpha\}$ is a descending chain of closed, convex (hence weakly closed) subsets of K, each of which intersects M, then weak-compactness of M implies that $\left(\bigcap K_\alpha\right) \cap M \neq \emptyset$. Thus we may use Zorn's Lemma to obtain a subset K_1 of K, which is minimal with respect to being closed, convex, invariant under f, and having points in common with M. Let $M_1 = K_1 \cap M$.

Let $x \in K_1$ and suppose $\delta(O(x)) \neq \Delta$. We shall show that this

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assumption leads to a contradiction.

By *(ii)*, there is an integer N such that

$$(*) \qquad \delta\big(O\big(f^N(x)\big)\big) = U_N \neq \delta\big(O(x)\big) .$$

Let $L = \Big\{z \in K_1 : \big(z, f^n(x)\big) \in U_N \text{ for almost all } n\Big\}$. $\delta\big(O\big(f^N(x)\big)\big) = U_N$ implies that $O\big(f^N(x)\big) \subset L$, and thus $L \neq \emptyset$. If $y \in L$, then for some N_1, $\big(y, f^n(x)\big) \in U_N$ whenever $n \geq N_1$. By β-nonexpansiveness of f, $\big(f(y), f^{n+1}(x)\big) \in U_N$ for $n \geq N_1$. Thus f maps L into itself. L is also convex. For suppose $z_1, z_2 \in L$ and take $\lambda, \mu \geq 0$ such that $\lambda + \mu = 1$. We obtain an integer N_2 such that $\big(z_1, f^n(x)\big) \in U_N$ and $\big(z_2, f^n(x)\big) \in U_N$ for $n \geq N_2$. We have $\big(z_1-f^n(x)\big) \in U_N^*$ and $\big(z_2-f^n(x)\big) \in U_N^*$; therefore $\lambda z_1 - \lambda f^n(x) \in \lambda U_N^*$ and $\mu z_1 - \mu f^n(x) \in \mu U_N^*$. Adding, we get $\lambda z_1 + \mu z_2 - f^n(x) \in (\lambda+\mu)U_N^* = U_N^*$, on account of convexity of U_N^*, for $n \geq N_2$; that is, for almost all n. Hence $\big(\lambda z_1+\mu z_2, f^n(x)\big) \in U_N$.

The closure $\bar{L}$ of L is also convex, as L is convex. Moreover, L is invariant under f. Since f, being β-nonexpansive, is continuous, $\bar{L}$ is also invariant under f. By *(i)*, $\bar{L} \cap M \neq \emptyset$. By minimality of K_1, $\bar{L} = K_1$.

Let $p \in K_1$. Since $p \in \bar{L}$, for $U \in \beta$, there is a $p' \in L$, such that $(p, p') \in U$. Moreover there exists N_3 such that $\big(p', f^n(x)\big) \in U_N$ for $n \geq N_3$. Thus, for $n \geq N_3$, $\big(p, f^n(x)\big) \in U \circ U_N$ and therefore $f^n(x) \in U \circ U_N[p]$. By Lemma 3.1, $U \circ U_N[p]$ is convex and consequently closure $\overline{U \circ U_N[p]}$ is also convex. Hence for $n \geq N_3$, $\operatorname{clco}\big(O\big(f^n(x)\big)\big) \subset \overline{U \circ U_N[p]}$. By *(i)*, for all n,

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$\operatorname{clco}(O(f^n(x))) \cap M_1 \neq \emptyset$. Since M_1 is weakly compact, there is a point $t \in \bigcap_{n=0}^{\infty} \operatorname{clco}(O(f^n(x))) \cap M_1$. For every $U \in \beta$, $t \in U \circ U_N[p]$. By Lemma 3.2, $U_N[p] = \bigcap_{U\in\beta} \overline{U \circ U_N[p]}$ and $t \in U_N[p]$. Since this is true for $p \in K_1$, it follows that $t \in \bigcap_{p\in K_1} U_N[p]$. Therefore the set

$$S = \{z \in K_1 : K_1 \subset U_N[z]\}$$

is nonempty $(t \in S)$. We first show that S is closed. Let $p \in \overline{S}$. Then for each $U \in \beta$, there is a $p' \in S$, such that $(p, p') \in U$. Also for every $y \in K_1$, $(p', y) \in U_N$. Then $(p, y) \in \bigcap_{U\in\beta} U \circ U_N$. By Lemma 3.2, we obtain that $y \in U_N[p]$. Thus S is closed. S is clearly convex. Next suppose for some $z \in S$, $f(z) \notin S$. Define $H = U_N[f(z)] \cap K_1$. By definition, H is a proper subset of K_1 . Then $(z, x) \in U_N$ and by β-nonexpansiveness of f , $(f(z), f(x)) \in U_N$. Because K_1 is invariant under f , we have $f(x) \in H$; that is to say that H is invariant under f . By hypothesis *(i)*, $H \cap M$ is nonempty. Since H is a proper subset of K_1 , this contradicts the minimality of K_1 . Therefore $f(S) \subset S$. Since $z_1 \in U_N[z_2]$ for $z_1, z_2 \in S$, $S \times S \subset U_N$, we obtain, by (*),

$$\delta(S) \subset U_N = \delta(O(f^N(x))) \subsetneq \delta(O(x)) \subset \delta(K_1) .$$

Thus S is a proper subset of K_1 . Again the minimality of K_1 is contradicted. Therefore our assumption that $\delta(O(x)) \neq \Delta$ is not true. Hence $\delta(O(x)) = \Delta$ and $f(x) = x$.

COROLLARY 4.2. *If K is a closed, convex, weakly compact subset of X and if f is a β-nonexpansive mapping of K into itself, which has β-diminishing orbital diameters, then f has a fixed point.*

This corollary is obtained by putting $M = K$ in the above theorem.

5. β-normal structure

We assume that the β-diametral map on $P^*(X)$ exists.

DEFINITION 5.1. Let A be a nonempty subset of X having at least two elements. A point $a \in A$ is to be called non β-diametral if
$$\bigcup_{x \in A} \delta\{x, a\} \neq \delta(A) .$$

DEFINITION 5.2. A subset A of X having at least two elements will be said to have β-normal structure, if for each subset H of A which contains more than one point, there is a point $x \in H$, which is a β-non diametral point of H.

EXAMPLE 5.3. A metrisable space X which has normal structure, possesses β-normal structure.

THEOREM 5.4. *Let K be a closed, convex subset of a locally convex topological vector space X, and let M be a weakly compact subset of K. If $f : K \to K$ is β-nonexpansive such that for each $x \in K$,*

(i) $\operatorname{clco}(O(x)) \cap M \neq \emptyset$, *and*

(ii) $\operatorname{clco}(O(x))$ *has β-normal structure,*

then there is a point $x \in M$ such that $f(x) = x$.

Proof. Let us define K_1 as in the proof of Theorem 4.1 and obtain the set L as follows. Suppose $\delta(K_1) \neq \Delta$. Let $x \in K_1$. By *(ii)* there is a point $y \in \operatorname{clco}(O(x))$ such that
$$\bigcup \delta\{y, \omega\} \neq \delta\big(\operatorname{clco}(O(x))\big) = U_x ,$$
say, where ω runs through $\operatorname{clco}(O(x))$.

Let $L = \left\{z \in K_1 : O\big(f^n(x)\big) \subset U_x[z] \text{ for some } n\right\}$. Then $y \in L$ and L is nonempty. L is convex, invariant under f, and $\bar{L} \cap M \neq \emptyset$ as in the proof of Theorem 4.1. Accordingly, $\bar{L} = K_1$.

Following the argument of Theorem 4.1, one can see that
$$S = \{z \in K_1 : K_1 \subset U_x[z]\}$$
is nonempty, closed, convex, and invariant under f. But

$$\delta(S) \subset U_x \subsetneqq \delta(O(x)) \subset \delta(K_1) \ .$$

Thus S is a proper subset of K_1, contradicting the minimality of K_1. Hence $\delta(K_1) = \Delta$ and K_1 consists of a single point which is fixed under f.

References

[1] L.P. Belluce and W.A. Kirk, "Fixed-point theorems for certain classes of nonexpansive mappings", *Proc. Amer. Math. Soc.* 20 (1969), 141-146.

[2] М.С. Бродский и Д.П. Мильман [M.S. Brodskii and D.P. Mil'man], "О центре выпуклого множества" [On the center of a convex set], *Dokl. Akad. Nauk SSSR (N.S.)* 59 (1948), 837-840.

[3] Thomas A. Brown and W.W. Comfort, "New method for expansion and contraction maps in uniform spaces", *Proc. Amer. Math. Soc.* 11 (1960), 483-486.

[4] W.H. Kammerer and R.H. Kasriel, "On contractive mappings in uniform spaces", *Proc. Amer. Math. Soc.* 15 (1964), 288-290.

[5] J. Reinermann, "On a fixed-point theorem of Banach-type for uniform spaces", *Mat. Vesnik* 6 (21), (1969), 211-213. Quoted from MR42#3776.

[6] W.W. Taylor, "Fixed-point theorems for nonexpansive mappings in linear topological spaces", *J. Math. Anal. Appl.* 40 (1972), 164-173.

Department of Mathematics,
University of Allahabad,
Allahabad, Uttar Pradesh,
India;

Department of Mathematics,
Kulbhaskar Ashram Degree College,
University of Kanpur,
Allahabad, Uttar Pradesh,
India.

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