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Valuations for Matroid Polytope Subdivisions

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Abstract. We prove that the ranks of the subsets and the activities of the bases of a matroid define valuations for the subdivisions of a matroid polytope into smaller matroid polytopes.

1 Introduction

Aside from its wide applicability in many areas of mathematics, one of the pleasant features of matroid theory is the availability of a vast number of equivalent points of view. Among many others, one can think of a matroid as a notion of independence, a closure relation, or a lattice. One point of view has gained prominence due to its applications in algebraic geometry, combinatorial optimization, and Coxeter group theory: that of a matroid as a polytope. This paper is devoted to the study of functions of a matroid that are amenable to this point of view.

To each matroid M one can associate a (basis) *matroid polytope* $Q(M)$, which is the convex hull of the indicator vectors of the bases of M. One can recover M from $Q(M)$, and in certain instances $Q(M)$ is the fundamental object that one would like to work with. For instance, matroid polytopes play a crucial role in the matroid stratification of the Grassmannian [12]. They allow us to invoke the machinery of linear programming to study matroid optimization questions [22]. They are also the key to understanding that matroids are just the type A objects in the family of Coxeter matroids [5].

The subdivisions of a matroid polytope into smaller matroid polytopes have appeared prominently in different contexts: in compactifying the moduli space of hyperplane arrangements (Hacking, Keel, and Tevelev [13] and Kapranov [14]), in compactifying fine Schubert cells in the Grassmannian (Lafforgue [16, 17]), and in the study of tropical linear spaces (Speyer [23]).

Billera, Jia, and Reiner [3] and Speyer [23, 24] have shown that some important functions of a matroid, such as its quasisymmetric function and its Tutte polynomial, can be thought of as nice functions of their matroid polytopes. They act as valuations on the subdivisions of a matroid polytope into smaller matroid polytopes.

The purpose of this paper is to show that two much stronger functions are also valuations. Consider the matroid functions

$$f_1(M) = \sum_{A \subseteq [n]} (A, r_M(A)) \quad \text{and} \quad f_2(M) = \sum_{B \text{ basis of } M} (B, E(B), I(B)),$$

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regarded as formal sums. Here r_M denotes matroid rank, and $E(B)$ and $I(B)$ denote the sets of externally and internally active elements of B.

***Theorems* 5.1 *and* 5.4** *The functions f_1 and f_2 are valuations for matroid polytope subdivisions: for any subdivision of a matroid polytope $Q(M)$ into smaller matroid polytopes $Q(M_1), \dots, Q(M_m)$, these functions satisfy*

$$f(M) = \sum_i f(M_i) - \sum_{i<j} f(M_{ij}) + \sum_{i<j<k} f(M_{ijk}) - \cdots ,$$

where $M_{ab\dots c}$ is the matroid whose polytope is $Q(M_a) \cap Q(M_b) \cap \cdots \cap Q(M_c)$.

The paper is organized as follows. In Section 2, we present some background information on matroids and matroid polytope subdivisions. In Section 3, we define valuations under matroid subdivisions and prove an alternative characterization of them. In Section 4, we present a useful family of valuations, which we use to prove Theorems 5.1 and 5.4 in Section 5. Finally in Section 6, we discuss related work.

2 Preliminaries on Matroids and Matroid Subdivisions

A *matroid* is a combinatorial object that unifies several notions of independence. We start with some basic definitions; for more information on matroid theory, we refer the reader to [19].

There are many equivalent ways of defining a matroid. We will adopt the basis point of view, which is the most convenient for the study of matroid polytopes.

Definition 2.1 A *matroid M* is a pair $(\mathrm{E}, \mathcal{B})$ consisting of a finite set E and a collection of subsets $\mathcal{B}$ of E, called the *bases* of M, which satisfies the *basis exchange axiom:* if $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$, then there exists $b_2 \in B_2 - B_1$ such that $B_1 \setminus b_1 \cup b_2 \in \mathcal{B}$.

We will find it convenient to allow $(\mathrm{E}, \varnothing)$ to be a matroid; this is not customary.

A subset $A \subseteq \mathrm{E}$ is *independent* if it is a subset of a basis. All the maximal independent sets contained in a given set $A \subseteq \mathrm{E}$ have the same size, which is called the *rank* $r_M(A)$ of A. In particular, all the bases have the same size, which is called the rank $r(M)$ of M.

Example 2.2 If E is a finite set of vectors in a vector space, then the maximal linearly independent subsets of E are the bases of a matroid. The matroids arising in this way are called *representable* and motivate much of the theory of matroids.

Example 2.3 If $k \le n$ are positive integers, then the subsets of size k of $[n] = \{1, \dots, n\}$ are the bases of a matroid called the *uniform matroid $U_{k,n}$*.

Example 2.4 Given positive integers $1 \le s_1 < \cdots < s_r \le n$, the sets $\{a_1, \dots, a_r\}$ such that $a_1 \le s_1, \dots, a_r \le s_r$ are the bases of a matroid called the *Schubert matroid* $SM_n(s_1, \dots, s_r)$. These matroids were discovered by Crapo [7] and rediscovered in various contexts. They have been called shifted matroids [1, 15], PI-matroids [3], generalized Catalan matroids [4], and freedom matroids [8], among others. We prefer the name Schubert matroid, which highlights their relationship with the stratification of the Grassmannian into Schubert cells [2, Section 2.4].

The following geometric representation of a matroid is central to our study.

Definition 2.5 Given a matroid $M = ([n], \mathcal{B})$, the (basis) *matroid polytope* $Q(M)$ of M is the convex hull of the indicator vectors of the bases of M:

$$Q(M) = \text{convex}\{e_B : B \in \mathcal{B}\}.$$

For any $B = \{b_1, \ldots, b_r\} \subseteq [n]$, by e_B we mean $e_{b_1} + \cdots + e_{b_r}$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

When we speak of a "matroid polytope", we will refer to the polytope of a specific matroid in its specific position in $\mathbb{R}^n$. The following elegant characterization is due to Gelfand, Goresky, MacPherson, and Serganova.

Theorem 2.6 ([12]) *Let $\mathcal{B}$ be a collection of subsets of $[n]$ and let $Q(\mathcal{B}) =$ convex$\{e_B : B \in \mathcal{B}\}$. The following are equivalent:*

(i) *$\mathcal{B}$ is the collection of bases of a matroid;*
(ii) *every edge of $Q(\mathcal{B})$ is a parallel translate of $e_i - e_j$ for some $i, j \in [n]$.*

When the statements of Theorem 2.6 are satisfied, the edges of $Q(\mathcal{B})$ correspond exactly to the pairs of different bases B, B' such that $B' = B \setminus i \cup j$ for some $i, j \in [n]$. Two such bases are called *adjacent bases*.

A *subdivision* of a polytope P is a set of polytopes $S = \{P_1, \ldots, P_m\}$ whose vertices are vertices of P, such that

- $P_1 \cup \cdots \cup P_m = P$, and
- for all $1 \leq i < j \leq m$, if the intersection $P_i \cap P_j$ is nonempty, then it is a proper face of both P_i and P_j.

The *faces* of the subdivision S are the faces of the P_i; it is easy to see that the interior faces of S (*i.e.,* faces not contained in the boundary of P) are obtainable as the nonempty intersections between some of the P_i.

Definition 2.7 A *matroid polytope subdivision* is a subdivision of a matroid polytope $Q = Q(M)$ into matroid polytopes $Q_1 = Q(M_1), \ldots, Q_m = Q(M_m)$. We will also refer to this as a *matroid subdivision* of the matroid M into $M_1, \ldots, M_m$.

The lower-dimensional faces of the subdivision, which are intersections of subcollections of the Q_i, are also of interest. Given a set of indices $A = \{a_1, \ldots, a_s\} \subseteq [m]$, we will write $Q_A = Q_{a_1 \cdots a_s} := \bigcap_{a \in A} Q_a$. By convention, $Q_\varnothing = Q$. Since any face of a matroid polytope is itself a matroid polytope, it follows that any nonempty Q_A is the matroid polytope of a matroid, which we denote M_A.

Because of the small number of matroid polytopes in low dimensions, there is a general lack of small examples of matroid subdivisions. In two dimensions, the only matroid polytopes are the equilateral triangle and the square, which have no nontrivial matroid subdivisions. In three dimensions, the only nontrivial example is the subdivision of a regular octahedron (with bases $\{12, 13, 14, 23, 24, 34\}$) into two square pyramids (with bases $\{12, 13, 14, 23, 24\}$ and $\{13, 14, 23, 24, 34\}$, respectively); this subdivision is shown in Figure 1.

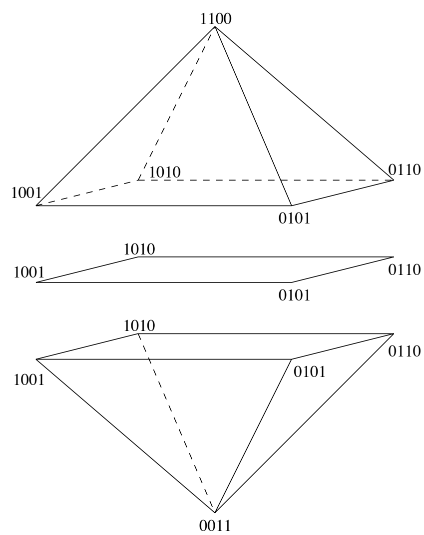

Figure 1: The matroid subdivision of a regular octahedron into two square pyramids.

Example 2.8 A more interesting example is the following subdivision [3, Example 7.13]: let $M_1 = SM_6(2,4,6)$ be the Schubert matroid whose bases are the sets $\{a,b,c\} \subseteq [6]$ such that $a \leq 2$, $b \leq 4$, and $c \leq 6$. The permutation $\sigma = 345612$ acts on the ground set $[6]$ of M_1, thus defining the matroids $M_2 = \sigma M_1$ and $M_3 = \sigma^2 M_1$. (Note that σ^3 is the identity.) Then, $\{M_1, M_2, M_3\}$ is a subdivision of $M = U_{3,6}$.[1]

3 Valuations under Matroid Subdivisions

We now turn to the study of matroid functions that are valuations under the subdivisions of a matroid polytope into smaller matroid polytopes. Throughout this section, $\mathrm{Mat} = \mathrm{Mat}_n$ will denote the set of matroids with ground set $[n]$, and G will denote an arbitrary abelian group. As before, given a subdivision $\{M_1, \ldots, M_m\}$ of a matroid M and a subset $A \subseteq [m]$, M_A is the matroid whose polytope is $\bigcap_{a \in A} Q(M_a)$.

[1]One can easily generalize this construction to obtain a subdivision of $U_{a,ab}$ into a isomorphic matroids.

Definition 3.1 A function $f\colon \mathrm{Mat} \to G$ is a *valuation under matroid subdivision*, or simply a *valuation*,[2] if for any subdivision $\{M_1, \dots, M_m\}$ of a matroid $M \in \mathrm{Mat}$, we have

$$\sum_{A \subseteq [m]} (-1)^{|A|} f(M_A) = 0,$$

or, equivalently,

$$f(M) = \sum_i f(M_i) - \sum_{i<j} f(M_{ij}) + \sum_{i<j<k} f(M_{ijk}) - \cdots . \tag{3.1}$$

Recall that, contrary to the usual convention, we have allowed $\varnothing = ([n], \varnothing)$ to be a matroid. We will also adopt the convention that $f(\varnothing) = 0$ for all the matroid functions considered in this paper.

Many important matroid functions are well-behaved under subdivision. Let us start with some easy examples.

Example 3.2 The function *Vol*, which assigns to each matroid $M \in \mathrm{Mat}$ the n-dimensional volume of its polytope $Q(M)$, is a valuation. This is clear since the lower-dimensional faces of a matroid subdivision have volume 0.

Example 3.3 The *Ehrhart polynomial* $E_P(x)$ of a lattice polytope P in $\mathbb{R}^d$ is the polynomial such that, for a positive integer n, $E_P(n) = |nP \cap \mathbb{Z}^d|$ is the number of lattice points contained in the n-th dilate nP of P [25, Section 4.6]. By the inclusion-exclusion formula, the function $E\colon \mathrm{Mat} \to \mathbb{R}[x]$ defined by $E(M) = E_{Q(M)}(x)$ is a valuation.

Example 3.4 The function $b(M) =$ (number of bases of M) is a valuation. This follows from the fact that the only lattice points in $Q(M)$ are its vertices, which are the indicator vectors of the bases of M, so $b(M)$ is the evaluation of $E(M)$ at $x = 1$.

Before encountering other important valuations, let us present an alternative way of characterizing them.

Theorem 3.5 *A function $f\colon \mathrm{Mat} \to G$ is a valuation if and only if, for any matroid subdivision S of $Q = Q(M)$,*

$$f(M) = \sum_{F \in \mathrm{int}(S)} (-1)^{\dim(Q) - \dim(F)} f(M(F)),$$

where the sum is over the interior faces of the subdivision S, and $M(F)$ denotes the matroid whose matroid polytope is F.

To prove Theorem 3.5, we first need to recall some facts from topological combinatorics. These can be found, for instance, in [25, Section 3.8].

[2] This use of the term *valuation* is standard in convex geometry [18]. It should not be confused with the unrelated notion of a matroid valuation found in the theory of valuated matroids [11].

Definition 3.6 A *regular cell complex* is a finite set $C = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ of pairwise disjoint and nonempty *cells* $\sigma_i \subseteq \mathbb{R}^d$ such that for any $i \in [s]$:

(i) $\overline{\sigma_i} \approx \mathbb{B}^{m_i}$ and $\overline{\sigma_i} \setminus \sigma_i \approx \mathbb{S}^{m_i-1}$ for some nonnegative integer m_i called the *dimension* of σ_i;

(ii) $\overline{\sigma_i} \setminus \sigma_i$ is the union of some other σ_js.

Here, $\overline{\sigma_i}$ denotes the topological closure of σ_i and $\approx$ denotes homeomorphism. Also, $\mathbb{B}^l$ and $\mathbb{S}^l$ are the l-dimensional closed unit ball and unit sphere, respectively. The *underlying space* $|C|$ of C is the topological space $\sigma_1 \cup \cdots \cup \sigma_s$.

Definition 3.7 Let C be a regular cell complex, and let c_i be the number of i-dimensional cells of C. The *Euler characteristic* of C is:

$$\chi(C) = \sum_{\sigma \in C} (-1)^{\dim(\sigma)} = \sum_{i \in \mathbb{N}} (-1)^i c_i = c_0 - c_1 + c_2 - c_3 + \cdots .$$

The *reduced Euler characteristic* of C is $\tilde{\chi}(C) = \chi(C) - 1$. A fundamental fact from algebraic topology is that the Euler characteristic of C depends solely on the homotopy type of the underlying space $|C|$.

Definition 3.8 For a regular cell complex C, let $P(C)$ be the poset of cells of C, ordered by $\sigma_i \leq \sigma_j$ if $\overline{\sigma_i} \subseteq \overline{\sigma_j}$. Let $\widehat{P}(C) = P(C) \cup \{\hat{0}, \hat{1}\}$ be obtained from $P(C)$ by adding a minimum and a maximum element.

Definition 3.9 The *Möbius function* $\mu\colon \operatorname{Int}(P) \to \mathbb{Z}$ of a poset P assigns an integer to each closed interval of P, defined recursively by

$$\mu_P(x, x) = 1, \qquad \sum_{x \leq a \leq y} \mu(x, a) = 0 \quad \text{for all } x < y.$$

It can equivalently be defined in the following dual way:

$$\mu_P(x, x) = 1, \qquad \sum_{x \leq a \leq y} \mu(a, y) = 0 \quad \text{for all } x < y.$$

The following special case of Rota's Crosscut Theorem is a powerful tool for computing the Möbius function of a lattice.

Theorem 3.10 ([20]) *Let L be any finite lattice. Then for all $x \in L$,*

$$\mu(\hat{0}, x) = \sum_B (-1)^{|B|},$$

where the sum is over all sets B of atoms of L such that $\bigvee B = x$.

Finally, we recall an important theorem that relates the topology and combinatorics of a regular cell complex.

Theorem 3.11 ([25, Proposition 3.8.9]) *Let C be a regular cell complex such that $|C|$ is a manifold, with or without boundary. Let $P = \widehat{P}(C)$. Then,*

$$\mu_P(x, y) = \begin{cases} \tilde{\chi}(|C|) & \text{if } x = \hat{0} \text{ and } y = \hat{1}, \\ 0 & \text{if } x \neq \hat{0},\, y = \hat{1}, \text{ and } x \text{ is on the boundary of } |C|, \\ (-1)^{l(x,y)} & \text{otherwise,} \end{cases}$$

where $l(x, y)$ is the number of elements in a maximal chain from x to y.

We are now in a position to prove Theorem 3.5.

Proof of Theorem 3.5 Let $S = \{M_1, \dots, M_m\}$ be a matroid subdivision of M. Let $\{Q_1, \dots, Q_m\}$ and Q be the corresponding polytopes. Notice that the (relative interiors of the) faces of the subdivision S form a regular cell complex whose underlying space is Q. Additionally, the poset $\widehat{P}(S)$ is a lattice, since it has a meet operation, $\sigma_i \wedge \sigma_j = \operatorname{int}(\overline{\sigma}_i \cap \overline{\sigma}_j)$, and a maximum element.

We will show that

$$\sum_{F \in \operatorname{int}(S)} (-1)^{\dim(Q) - \dim(F)} f(M(F)) = \sum_i f(M_i) - \sum_{i<j} f(M_{ij}) + \sum_{i<j<k} f(M_{ijk}) - \cdots ,$$

which will establish the desired result in view of (3.1). On the right hand side, each term is of the form $f(M(F))$ for a face F of the subdivision S and moreover, all interior faces F appear. The term $f(M(F))$ appears with coefficient

$$\sum_{A \subseteq [m] \,:\, M_A = M(F)} (-1)^{|A|+1}.$$

This is equivalent to summing over the sets of coatoms of the lattice $\widehat{P}(S)$ whose meet is F. By Rota's Crosscut Theorem 3.10, when applied to the poset $\widehat{P}(S)$ turned upside down, this sum equals $-\mu_{\widehat{P}(S)}(F, \hat{1})$. Theorem 3.11 tells us that this is equal to 0 if F is in the boundary of Q, and $(-1)^{l(F,\hat{1})-1} = (-1)^{\dim(Q)-\dim(F)}$ if F is an interior face, as desired. ∎

4 A Powerful Family of Valuations

Definition 4.1 Given $X \subseteq \mathbb{R}^n$, let $i_X \colon \mathrm{Mat} \to \mathbb{Z}$ be defined by

$$i_X(M) = \begin{cases} 1 & \text{if } Q(M) \cap X \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Our interest in these functions is that, under certain hypotheses, they are valuations under matroid subdivisions. They are a powerful family for our purposes because many valuations of interest, in particular those in Section 5, can be obtained as linear combinations of evaluations of these valuations.

Theorem 4.2 *If $X \subseteq \mathbb{R}^n$ is convex and open, then i_X is a valuation.*

Proof Let $M \in \mathrm{Mat}$ be a matroid and S be a subdivision of $Q = Q(M)$. We can assume that $Q \cap X \neq \varnothing$, or else the result is trivial. One can easily reduce to the case when X is bounded, since matroid polytopes are contained in the unit cube.

We will first reduce the proof to the case when X is an open polytope in $\mathbb{R}^n$. By the Hahn–Banach separation theorem [21, Theorem 3.4], for each face F of S such that $F \cap X = \varnothing$, there exists an open halfspace H_F containing X and disjoint from F. Let

$$X' = \bigcap_{F \cap X = \varnothing} H_F$$

be the intersection of these halfspaces. Then $X' \supseteq X$ and $X' \cap F = \varnothing$ for each face F not intersecting X, so $i_{X'}$ and i_X agree on all the matroids of this subdivision. If we define X'' as the intersection of X' with some open cube containing Q, $i_{X''}$ and i_X agree on this subdivision and X'' is an open polytope.

We can therefore assume that X is an open polytope in $\mathbb{R}^n$; in particular, it is full-dimensional. Note that $X \cap \mathrm{int}(Q)$ is the interior $\mathrm{int}(R)$ of some polytope $R \subseteq Q$. Since R and Q have the same dimension, $R \approx \mathbb{B}^{\dim(Q)}$ and $\partial R \approx \mathbb{S}^{\dim(Q)-1}$. If F is a face of the subdivision S and σ is a face of the polytope R, let $c_{F,\sigma} = \mathrm{int}(F) \cap \mathrm{int}(\sigma)$. Since $c_{F,\sigma}$ is the interior of a polytope, it is homeomorphic to a closed ball and its boundary to the corresponding sphere. Define

$$C = \{c_{F,\sigma} : c_{F,\sigma} \neq \varnothing\}$$
$$\partial C = \{c_{F,\sigma} : c_{F,\sigma} \neq \varnothing \text{ and } \sigma \neq R\}.$$

The elements of C form a partition of R, and in this way C is a regular cell complex whose underlying space is R. Similarly, ∂C is a regular subcomplex whose underlying space is ∂R. Note that if F is an interior face of S, $c_{F,R} = \mathrm{int}(F) \cap \mathrm{int}(R) \neq \varnothing$ if and only if $F \cap X \neq \varnothing$, and in this case $\dim(c_{F,R}) = \dim(F)$.

We then have

$$\begin{aligned}
\sum_{F \in \mathrm{int}(S)} (-1)^{\dim(F)} i_X(M(F)) &= \sum_{\substack{F \in \mathrm{int}(S) \\ F \cap X \neq \varnothing}} (-1)^{\dim(F)} \\
&= \sum_{c_{F,R} \neq \varnothing} (-1)^{\dim(c_{F,R})} \\
&= \sum_{c \in C} (-1)^{\dim(c)} - \sum_{c \in \partial C} (-1)^{\dim(c)} \\
&= \chi(R) - \chi(\partial R) \\
&= 1 - \left(1 + (-1)^{\dim(Q)-1}\right) \\
&= (-1)^{\dim(Q)} i_X(M),
\end{aligned}$$

which finishes the proof in view of Theorem 3.5. ■

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Corollary 4.3 *If $X \subseteq \mathbb{R}^n$ is convex and closed, then i_X is a valuation.*

Proof As before, we can assume that X is bounded since $i_X = i_{X\cap[0,1]^n}$. Now, let S be a subdivision of $Q = Q(M)$ into m parts. For all $A \subseteq [m]$ such that $X \cap Q_A = \varnothing$, the distance $d(X, Q_A)$ is positive since X is compact and Q_A is closed. Let $\epsilon > 0$ be smaller than all those distances, and define the convex open set

$$U = \{x \in \mathbb{R}^n : d(x, X) < \epsilon\}.$$

For all $A \subseteq [m]$ we have that $X \cap Q_A \neq \varnothing$ if and only if $U \cap Q_A \neq \varnothing$. By Theorem 4.2,

$$\sum_{A\subseteq[m]} (-1)^{|A|} i_X(M_A) = \sum_{A\subseteq[m]} (-1)^{|A|} i_U(M_A) = 0$$

as desired. ■

In particular, i_P is a valuation for any polytope $P \subseteq \mathbb{R}^n$.

Proposition 4.4 *The constant function $c(M) = 1$ for $M \in \mathsf{Mat}$ is a valuation.*

Proof This follows from $c(M) = i_{[0,1]^n}$. ■

Proposition 4.5 *If $X \subseteq \mathbb{R}^n$ is convex and is either open or closed, then the function $\overline{i_X}\colon \mathsf{Mat} \to \mathbb{Z}$ defined by*

$$\overline{i_X}(M) = \begin{cases} 0 & \text{if } Q(M) \cap X \neq \varnothing, \\ 1 & \text{otherwise,} \end{cases}$$

is a valuation.

Proof Notice that $\overline{i_X} = 1 - i_X$, which is the sum of two valuations. ■

5 Subset Ranks and Basis Activities are Valuations

We now show that there are two surprisingly fine valuations of a matroid: the ranks of the subsets and the activities of the bases.

5.1 Rank Functions

Theorem 5.1 *Let G be the free abelian group on symbols of the form (A, s), $A \subseteq [n]$, $s \in \mathbb{Z}_{\geq 0}$. The function $F\colon \mathsf{Mat} \to G$ defined by*

$$F(M) = \sum_{A\subseteq[n]} (A, r_M(A))$$

is a valuation.

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Proof It is equivalent to show that the function $f_{A,s}\colon \mathrm{Mat} \to \mathbb{Z}$ defined by

$$f_{A,s}(M) = \begin{cases} 1 & \text{if } r_M(A) = s, \\ 0 & \text{otherwise,} \end{cases}$$

is a valuation. Define the polytope

$$P_{A,s} = \Big\{ x \in [0,1]^n : \sum_{i \in A} x_i \geq s \Big\}.$$

A matroid M satisfies that $r_M(A) = s$ if and only if it has a basis B with $|A \cap B| \geq s$, and it has no basis B such that $|A \cap B| \geq s+1$. This is equivalent to $Q(M) \cap P_{A,s} \neq \varnothing$ and $Q(M) \cap P_{A,s+1} = \varnothing$. It follows that $f_{A,s} = i_{P_{A,s}} - i_{P_{A,s+1}}$, which is the sum of two valuations. ∎

5.2 Basis Activities

One of the most powerful invariants of a matroid is its *Tutte polynomial*:

$$T_M(x,y) = \sum_{A \subseteq [n]} (x-1)^{r(M)-r(A)} (y-1)^{|A|-r(A)}.$$

Its importance stems from the fact that many interesting invariants of a matroid satisfy the *deletion-contraction recursion*, and every such invariant is an evaluation of the Tutte polynomial [6].

Definition 5.2 Let B be a basis of the matroid $M = ([n], \mathcal{B})$. An element $i \in B$ is said to be *internally active* with respect to B if $i < j$ for all $j \notin B$ such that $B \setminus i \cup j \in \mathcal{B}$. Similarly, an element $i \notin B$ is said to be *externally active* with respect to B if $i < j$ for all $j \in B$ such that $B \setminus j \cup i \in \mathcal{B}$. Let $I(B)$ and $E(B)$ be the sets of internally and externally active elements with respect to B.

Theorem 5.3 (Tutte, Crapo [6]) *The Tutte polynomial of a matroid is*

$$T_M(x,y) = \sum_{B \text{ basis of } M} x^{|I(B)|} y^{|E(B)|}.$$

Theorem 5.4 *Let G be the free abelian group generated by the triples (B, E, I), where $B \subseteq [n]$, $E \subseteq [n] \setminus B$ and $I \subseteq B$. The function $F\colon \mathrm{Mat} \to G$ defined by*

$$F(M) = \sum_{B \text{ basis of } M} (B, E(B), I(B)) \tag{5.1}$$

is a valuation.

Before proving this result, let us illustrate its strength with an example. Consider the subdivision of $M = U_{3,6}$ into three matroids M_1, M_2, and M_3 described in Example 2.8. Table 1 shows the external and internal activity with respect to each basis in each one of the eight matroids M_A arising in the subdivision. The combinatorics prescribed by Theorem 5.4 are extremely restrictive: in any row, any choice of (E, I) must appear the same number of times in the M_As with $|A|$ even and in the M_As with $|A|$ odd.

	M		M_1		M_2		$M_{1,2}$		M_3		$M_{1,3}$		$M_{2,3}$		$M_{1,2,3}$	
B	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$
123	∅	123	∅	123												
124	∅	12	∅	12												
125	∅	12	∅	12					∅	125	∅	125				
126	∅	12	5	12					∅	12	5	12				
134	∅	1	∅	1	∅	134	∅	134								
135	∅	1	∅	1	∅	13	∅	13	∅	15	∅	15	∅	135	∅	135
136	∅	1	5	1	∅	13	5	13	∅	1	5	1	∅	13	5	13
145	∅	1	3	1	∅	1	3	1	3	15	3	15	3	15	3	15
146	∅	1	35	1	∅	1	35	1	3	1	35	1	3	1	35	1
156	∅	1							∅	1						
234	1	∅	1	∅	1	34	1	34								
235	1	∅	1	∅	1	3	1	3	1	5	1	5	1	35	1	35
236	1	∅	15	∅	1	3	15	3	1	∅	15	∅	1	3	15	3
245	1	∅	13	∅	1	∅	13	∅	13	5	13	5	13	5	13	5
246	1	∅	135	∅	1	∅	135	∅	13	∅	135	∅	13	∅	135	∅
256	1	∅							1	∅						
345	12	∅			12	∅										
346	12	∅			12	∅										
356	12	∅			12	3			12	∅			12	3		
456	123	∅			123	∅			123	∅			123	∅		

Table 1: External and internal activities for a subdivision of $U_{3,6}$

We will divide the proof of Theorem 5.4 into two lemmas.

Lemma 5.5 *Let $B \subseteq [n]$, $E \subseteq [n] \setminus B$ and $I \subseteq B$. Let*

$$V(B,E,I) = \\ \{A \subseteq [n] : e_A - e_B = e_a - e_b \text{ with } a \in E \text{ and } a > b, \text{ or with } b \in I \text{ and } a < b\}$$

and

$$P(B,E,I) = \operatorname{convex}\Big\{ \frac{e_A + e_B}{2} : A \in V(B,E,I) \Big\}.$$

Then, for any matroid $M \in \mathrm{Mat}$, we have that $Q(M) \cap P(B,E,I) = \varnothing$ if and only if

- *B is not a basis of M or*
- *B is a basis of M with $E \subseteq E(B)$ and $I \subseteq I(B)$.*

To illustrate this lemma with an example, consider the case $n = 4$, $B = \{1,3\}$, $E = \{2\}$ and $I = \{3\}$. Then $V(B,E,I) = \{\{1,2\},\{2,3\}\}$. Figure 2 shows the polytope $P = P(B,E,I)$ inside the hypersimplex whose vertices are the characteristic vectors of the 2-subsets of $[4]$. The polytope of the matroid M_1 with bases $\mathcal{B}_1 = \{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$ does not intersect P because B is not a basis of M_1. The polytope of the matroid M_2 with bases $\mathcal{B}_2 = \{\{1,3\},\{1,4\},\{3,4\}\}$ does not intersect P either, because B is a basis of M_2, but 2 is externally active with respect to B and 3 is internally active with respect to B. Finally, the polytope of the matroid M_3 with bases $\mathcal{B}_3 = \{\{1,3\},\{2,3\},\{3,4\}\}$ does intersect P, since B is a basis of M_3 and 2 is not externally active with respect to B; the intersection point $\frac{1}{2}(0110 + 1010)$ "certifies" this.

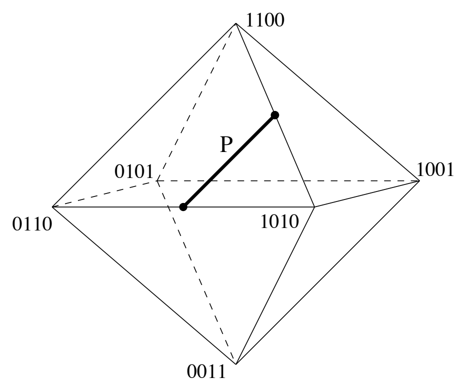

Figure 2: The polytope $P = P(B, E, I)$ inside $Q(U_{2,4})$

Proof Assume B is a basis of M. For $a \notin B$, a is externally active with respect to B if and only if there are no edges in $Q(M)$ parallel to $e_a - e_b$ with $a > b$ which are incident to e_B. In the same way, for $b \in B$, b is internally active with respect to B if and only if there are no edges in $Q(M)$ parallel to $e_a - e_b$ with $a < b$ which are incident to e_B. Since the vertices of $P(B, E, I)$ are precisely the midpoints of these edges when $a \in E$ and $b \in I$, if $Q(M) \cap P(B, E, I) = \varnothing$ then $E \subseteq E(B)$ and $I \subseteq I(B)$.

To prove the other direction, suppose that $Q(M) \cap P(B, E, I) \neq \varnothing$. First notice that, since $P(B, E, I)$ is on the hyperplane $x_1 + x_2 + \cdots + x_n = |B|$ and $Q(M)$ is on the hyperplane $x_1 + x_2 + \cdots + x_n = r(M)$, we must have $|B| = r(M)$. Moreover, since the vertices v of $P(B, E, I)$ satisfy $e_B \cdot v = r(M) - 1/2$, we know that B must be a basis of M, or else the vertices w of $Q(M)$ would all satisfy $e_B \cdot w \leq r(M) - 1$.

Now, let $q \in Q(M) \cap P(B, E, I)$. Since $q \in Q(M)$, we know that q is in the cone with vertex e_B generated by the edges of $Q(M)$ incident to e_B. In other words, if $A_1, A_2, \ldots, A_m$ are the bases adjacent to B,

$$q = e_B + \sum_{i=1}^{m} \lambda_i (e_{A_i} - e_B),$$

where the λ_i are all nonnegative. If we let $e_{c_i} - e_{d_i} = e_{A_i} - e_B$, where $c_i, d_i \in [n]$, then

$$q = e_B + \sum_{i=1}^{m} \lambda_i (e_{c_i} - e_{d_i}).$$

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On the other hand, since $q \in P(B, E, I)$,

$$q = \sum_{A \in V(B,E,I)} \gamma_A \frac{e_A + e_B}{2},$$

where the γ_A are nonnegative and add up to 1. Setting these two expressions equal to each other, we obtain

$$q = e_B + \sum_{i=1}^{m} \lambda_i (e_{c_i} - e_{d_i}) = \sum_{A \in V(B,E,I)} \gamma_A \frac{e_A + e_B}{2},$$

and therefore

$$r = q - e_B = \sum_{i=1}^{m} \lambda_i (e_{c_i} - e_{d_i}) = \sum_{A \in V(B,E,I)} \gamma_A \frac{e_A - e_B}{2}.$$

For $A \in V(B, E, I)$, we will let $e_{a_A} - e_{b_A} = e_A - e_B$. We have

$$r = \sum_{i=1}^{m} \lambda_i (e_{c_i} - e_{d_i}) = \sum_{A \in V(B,E,I)} \gamma_A \frac{e_{a_A} - e_{b_A}}{2}. \tag{5.2}$$

Notice that there is no cancellation of terms on either side of (5.2), since the d_is and the b_As are elements of B, while the c_is and the a_As are not. Let $r = (r_1, r_2, \ldots, r_n)$ and let k be the largest integer for which r_k is nonzero.

Assume that $k \notin B$. From the right hand side of (5.2) and taking into account the definition of $V(B, E, I)$, we have that $k \in E$. From the left hand side, we know there is an i such that $c_i = k$. But then $e_{c_i} - e_{d_i}$ is an edge of $Q(M)$ incident to e_B, and $d_i < k = c_i$ by our choice of k. It follows that k is not externally active with respect to B. In the case that $k \in B$, we obtain similarly that $k \in I$ and that $d_j = k$ for some j. Thus, $e_{c_j} - e_{d_j}$ is an edge of $Q(M)$ incident to e_B and $c_j < k = d_j$, so k is not internally active with respect to B. In either case, we conclude that $E \nsubseteq E(B)$ or $I \nsubseteq I(B)$, which finishes the proof. ∎

Lemma 5.6 *Let B be a subset of $[n]$, and let $E \subseteq [n] \setminus B$ and $I \subseteq B$. The function $G_{B,E,I} \colon \mathrm{Mat} \to \mathbb{Z}$ defined by*

$$G_{B,E,I}(M) = \begin{cases} 1 & \text{if } B \text{ is a basis of } M, E = E(B) \text{ and } I = I(B), \\ 0 & \text{otherwise}, \end{cases}$$

is a valuation.

Proof To simplify the notation, we will write $\overline{i_B}$ instead of $\overline{i_{\{e_B\}}}$. We will prove that $G(B, E, I) = G'(B, E, I)$, where

$$G'_{B,E,I}(M) = (-1)^{|E|+|I|} \cdot \sum_{\substack{E \subseteq X \subseteq [n] \\ I \subseteq Y \subseteq [n]}} (-1)^{|X|+|Y|} \left(\overline{i_{P(B,X,Y)}}(M) - \overline{i_B}(M) \right), \tag{5.3}$$

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which is a sum of valuations.

Let $M \in \text{Mat}$. If B is not a basis of M, then $\overline{i_B}(M) = 1$, and by Lemma 5.5, we have $\overline{i_{P(B,X,Y)}}(M) = 1$ for all X and Y. Therefore $G'_{B,E,I}(M) = 0 = G_{B,E,I}(M)$ as desired. If B is a basis of M, then $\overline{i_B}(M) = 0$; and we use Lemma 5.5 to rewrite (5.3) as

$$\begin{aligned} G'_{B,E,I}(M) &= (-1)^{|E|+|I|} \sum_{\substack{E \subseteq X \subseteq E(B) \\ I \subseteq Y \subseteq I(B)}} (-1)^{|X|+|Y|} \\ &= (-1)^{|E|+|I|} \sum_{E \subseteq X \subseteq E(B)} (-1)^{|X|} \sum_{I \subseteq Y \subseteq I(B)} (-1)^{|Y|} \\ &= \begin{cases} 1 & \text{if } E = E(B) \text{ and } I = I(B), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

as desired. ■

Proof of Theorem 5.4 The coefficient of (B, E, I) in the definition of (5.1) is $G_{B,E,I}(M)$, so the result follows from Lemma 5.6. ■

Theorem 5.4 is significantly stronger than the following result of Speyer that motivated it.

Corollary 5.7 (Speyer, [23]) *The Tutte polynomial (and therefore any of its evaluations) is a valuation under matroid subdivisions.*

Proof By Theorem 5.3, $T_M(x, y)$ is the composition of the function $h \colon G \to \mathbb{Z}[x, y]$ defined by $h(B, E, I) = x^{|I|} y^{|E|}$ with the function F of Theorem 5.4. ■

6 Related Work

Previous to our work, Billera, Jia, and Reiner [3] and Speyer [23,24] had studied various valuations of matroid polytopes. A few months after our paper was submitted, we learned about Derksen's results on this topic [9], which were obtained independently and roughly simultaneously. Their approaches differ from ours in the basic fact that they are concerned with matroid invariants which are valuations, whereas our matroid functions are not necessarily constant under matroid isomorphism; however, there are similarities. We outline their main invariants here.

In his work on tropical linear spaces [23], Speyer shows that the Tutte polynomial is a valuative invariant. He also defines in [24] a polynomial invariant $g_M(t)$ of a matroid M that arises in the K-theory of the Grassmannian. It is not known how to describe $g_M(t)$ combinatorially in terms of M.

Given a matroid $M = (\mathsf{E}, \mathcal{B})$, a function $f \colon \mathsf{E} \to \mathbb{Z}_{>0}$ is said to be *M-generic* if the minimum value of $\sum_{b \in B} f(b)$ over all bases $B \in \mathcal{B}$ is attained just once. Billera, Jia, and Reiner study the valuation

$$QS(M) = \sum_{f\ M\text{-generic}} \prod_{b \in \mathsf{E}} x_{f(b)},$$

which takes values in the ring of *quasi-symmetric functions* in the variables $x_1, x_2, \dots$, *i.e.*, the ring generated by

$$\sum_{i_1<\cdots<i_r} x_{i_1}^{\alpha_1} \cdots x_{i_r}^{\alpha_r}$$

for all tuples $(\alpha_1, \dots, \alpha_r)$ of positive integers.

Derksen's invariant is given by

$$G(M) := \sum_{\mathbf{A}} U(r_M(A_1) - r_M(A_0), \dots, r_M(A_n) - r_M(A_{n-1}))$$

where $\mathbf{A} = (A_0, \dots, A_n)$ ranges over all maximal flags of M, and

$$\{U(\mathbf{r}) : \mathbf{r} \text{ a finite sequence of nonnegative integers}\}$$

is a particular basis for the ring of quasi-symmetric functions. Derksen's invariant can be defined more generally on polymatroids. He shows that the Tutte polynomial and the quasisymmetric function of Billera, Jia, and Reiner are specialisations of $G(M)$, and asks whether $G(M)$ is universal for valuative invariants in this setting. He and the second author answer this question affirmatively in [10].

For the remainder of this section, $F(M)$ will denote the function of our Theorem 5.1. Since $F(M)$ is not a matroid invariant, it cannot be a specialisation of $g_M(t)$, $QS(M)$, or $G(M)$. In the other direction, we do not know whether, like the Tutte polynomial, Speyer's polynomial $g_M(t)$ is a specialisation of $F(M)$. As one would expect, $G(M)$ and $QS(M)$ are not specialisations of $F(M)$. One linear combination that certifies this is set out in Table 2, in which, to facilitate carrying out the relevant checks for $F(M)$, the relevant matroids are specified via their rank functions.

However, one can give a valuation that is similar in spirit to our $F(M)$ and specialises to Derksen's $G(M)$.

Proposition 6.1 *The function $H\colon \mathsf{Mat} \to G^n$ defined by*

$$H(M) = \sum_{\mathbf{A}} \big((A_1, r(A_1)), \dots, (A_n, r(A_n)) \big),$$

where $\mathbf{A} = (A_1, \dots, A_n)$ ranges over all maximal flags of M, is a valuation.

Proof The proof is a straightforward extension of our argument for Theorem 5.1. With the notation of that proof, checking whether a matroid M satisfies $r_M(A_i) = r_i$ for some fixed vector $\mathbf{r} = (r_i)$, *i.e.*, whether the term $((A_1, r_1), \dots, (A_n, r_n))$ is present in $H(M)$, is equivalent to checking that $Q(M)$ intersects P_{A_i, r_i} and not P_{A_i, r_i+1} for each i.

Observe that if $Q(M)$ intersects P_{A_i, s_i} for all i, then $r(A_i) \geq s_i$ and, since $\mathbf{A}$ is a flag, we can choose a single basis of M whose intersection with A_i has at least s_i elements for each i. Therefore, $Q(M)$ intersects $P_{A_1, s_1} \cap \cdots \cap P_{A_n, s_n}$.

Consider the sum

$$\sum (-1)^{e_1+\cdots+e_n} i_{P_{\mathbf{A}, \mathbf{r}+\mathbf{e}}}(M), \tag{6.1}$$

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S	$\varnothing$	1	2	12	3	13	23	123	4	14	24	124	34	134	234	1234
$r_{M_1}(S)$	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1
$r_{M_2}(S)$	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
$r_{M_3}(S)$	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$r_{M_4}(S)$	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$r_{M_5}(S)$	0	1	1	2	0	1	1	2	0	1	1	2	0	1	1	2
$r_{M_6}(S)$	0	1	0	1	1	2	1	2	1	2	1	2	1	2	1	2
$r_{M_7}(S)$	0	0	1	1	1	1	2	2	1	1	2	2	1	1	2	2
$r_{M_8}(S)$	0	1	1	2	1	2	2	2	1	2	2	2	1	2	2	2
$r_{M_9}(S)$	0	0	1	1	1	1	2	2	1	1	2	2	2	2	3	3
$r_{M_{10}}(S)$	0	1	1	2	1	2	2	3	1	1	2	2	2	2	3	3
$r_{M_{11}}(S)$	0	1	1	2	1	1	2	2	1	2	2	3	2	2	3	3
$r_{M_{12}}(S)$	0	1	1	2	1	2	2	3	1	2	2	3	2	2	3	3

i	1	2	3	4	5	6	7	8	9	10	11	12
c_i	-1	1	-1	1	1	-1	-1	1	2	-2	-2	2

Table 2: The top table contains the rank functions of twelve matroids M_i on $[4]$, $i = 1, \dots, 12$. The bottom table shows coefficients c_i such that $\sum c_i F(M_i) = 0$ but $\sum c_i G(M_i) \neq 0$ and $\sum c_i QS(M_i) \neq 0$.

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where the sum is over all $\mathbf{e} = (e_1, \dots, e_n) \in \{0,1\}^n$, and where $P_{\mathbf{A},\mathbf{r}+\mathbf{e}}$ is the intersection $P_{A_1,r_1+e_1} \cap \cdots \cap P_{A_n,r_n+e_n}$. By our previous observation, this sum equals

$$\Big(\sum_{e_1}(-1)^{e_1} i_{P_{A_1,r_1+e_1}}(M)\Big) \cdots \Big(\sum_{e_n}(-1)^{e_n} i_{P_{A_n,r_n+e_n}}(M)\Big),$$

which is 1 if the term $((A_1, r_1), \dots, (A_n, r_n))$ is present in $H(M)$, and is 0 otherwise. All the terms in (6.1) are valuations, hence H is a valuation. ■

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