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ON THE WEAKLY STRONGLY EXPOSED PROPERTY AND SOME SMOOTHNESS PROPERTIES OF ORLICZ SPACES

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The notion of a weakly strongly exposed Banach space is introduced and it is shown that this property is the dual property of very smoothness. Criteria for this property in Orlicz function spaces equipped with the Orlicz norm are presented. Criteria for strong smoothness and very smoothness of their subspaces of order continuous elements in the case of the Luxemburg norm are also given.

0. INTRODUCTION

Throughout this paper X denotes a Banach space, X^* denotes its dual space, $S(X)$ and $S(X^*)$ denote respectively the unit spheres of X and X^*. The triple (G, Σ, μ) stands for a nonatomic, complete and finite measure space and L^0 stands for the space of all (equivalence classes of) Σ-measurable real functions defined on G.

By Φ we denote an Orlicz function, that is, Φ is defined on the real line $\mathbb{R}$ with values in $\mathbb{R}_+ = [0, +\infty)$ and it vanishes only at zero, it is even, convex with $\Phi(u)/u \to 0$ as $u \to 0$ and $\Phi(u)/u \to +\infty$ as $u \to +\infty$.

For any Orlicz function Φ we define on L^0 the modular I_Φ by

$$I_\Phi(x) = \int_G \Phi(x(t))\, d\mu.$$

The Orlicz space L^Φ (or L_0^Φ) is the set

$$\{x \in L^0 : I_\Phi(\lambda x) < +\infty \quad \text{for some} \quad \lambda > 0\}$$

equipped with the Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \leqq 1\}$$

(or with the Orlicz norm)

$$\|x\|^0 = \sup\Big\{\int_G x(t)y(t)\, d\mu : y \in L^\Psi,\ I_\Psi(y) \leqq 1\Big\},$$

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where Ψ denotes the function complementary to Φ in the sense of Young, that is,

$$\Psi(u) = \sup\{uv - \Phi(v) : v \geqq 0\}$$

(see [**7, 8, 8, 13**]).

E^{Φ} (or E_0^{Φ}) denotes the subspace of L^{Φ} (or L_0^{Φ}) defined by

$$E^{\Phi} = \{x \in L^0 : I_{\Phi}(\lambda x) < +\infty \text{ for any } \lambda > 0\}$$

and equipped with the norm induced from L^{Φ} (or L_0^{Φ}).

It is more convenient to use the Amemiya formula for the Orlicz norm:

$$\|x\|^0 = \inf\Big\{\frac{1}{k}(1 + I_{\Phi}(kx)) : k > 0\Big\},$$

which does not use the complementary function of Φ (see [**7**] and [**13**]). The set of all $k > 0$ for which the infimum in the Amemiya formula for $x \in L^{\Phi} \setminus \{0\}$ is attained will be denoted by $K_{\Phi}(x)$. It is known that $K_{\Phi}(x) \neq \emptyset$ for any $x \in L^{\Phi} \setminus \{0\}$, since $\Phi(u)/u \to +\infty$ as $u \to +\infty$ (see [**6**]).

We say an Orlicz function Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$ for short) if there exist positive constants K and u_0 such that $\Phi(2u) \leqslant K\Phi(u)$ whenever $|u| \geqslant u_0$ (see [**7**] and [**13**]). It is obvious that the definition of the Δ_2-condition does not change if we replace the number 2 by any $\lambda > 1$.

Let us recall now some geometric notions. A Banach space X is said to be *weakly locally uniformly rotund* ($WLUR$ for short) if for each $x \in S(X)$ and each sequence $\{x_n\}$ in $S(X)$ such that $\|x_n + x\| \to 2$, we have $x_n \to x$ weakly ($x_n \xrightarrow{w} x$ for short), see [**5**]. X is said to have property WM if for any $x \in S(X)$ and any sequence $\{x_n\}$ in $S(X)$ such that $\|x_n + x\| \to 2$ there exist a support functional f_x at x and a subsequence $\{x_{n_i}\}$ such that $f_x(x_{n_i}) \to 1$ (see [**3, 10, 11**]). Recall that $f_x \in X^*$ is called a support functional at $x \in X \setminus \{0\}$ if $\|f_x\| = 1$ and $f_x(x) = \|x\|$ (see [**12**]). The set of all support functionals at x is denoted by $\operatorname{Grad}(x)$. It is obvious that $WLUR$ Banach spaces have property WM. The question arises as to whether or not there is some property (denote it for the moment by A) such that WM and A are together equivalent to $WLUR$. The answer is affirmative. This leads us to the notion of *weakly strongly exposed* (WSE for short) Banach spaces. A Banach space X is said to be *weakly strongly exposed* if for any $x \in S(X)$ and any $\{x_n\}$ in $S(X)$, if $f_x(x_n) \to 1$ for some $f_x \in \operatorname{Grad}(x)$ then $x_n \xrightarrow{w} x$. As we shall see, this notion is dual to very smoothness.

A Banach space X is called *smooth* if for any $x \in S(X)$ there is exactly one element in $\operatorname{Grad}(x)$. X is said to be very smooth (strongly smooth) if it is smooth and for each $x \in S(X)$ and each $\{f_n\}$ in $S(X^*)$ such that $f_n(x) \to 1$, we have $f_n \xrightarrow{w} f_x(\|f_n - f_x\| \to 0)$, where $\{f_x\} = \text{Grad}(x)$. We denote these properties for short by VS and SS, respectively.

The rest of the paper will be divided into two parts. In the first part we shall give some general results on the properties WM, $WLUR$, WSE and VS. In the second one we shall present criteria for VS and SS of E^Φ and for the property WSE in L_0^Φ. Let us recall that criteria for VS and SS of L_0^Φ were given in [**2**]. To get these criteria, some duality arguments were used. Since E^Φ need not be a dual space, it was necessary to find a new method to get an analogue of the results from [**2**] for E^Φ in place of L_0^Φ.

1. Some general results

Let us start with the following theorem.

Theorem 1.1. *If X^* is VS, then X is WSE.*

Proof: Let $x \in S(X)$ and $\{x_n\}$ in $S(X)$ be such that $f_x(x_n) \to 1$ for some $f_x \in \text{Grad}(x)$. Since VS of X^* yields reflexivity of X there are a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $x' \in S(X)$ such that $x_{n_i} \xrightarrow{w} x'$. In particular, $f_x(x_{n_i}) \to f_x(x')$ and, by the uniqueness of the limit, we get $f_x(x') = 1$. Therefore, $f_x(x + x') = 2$ and consequently $\|x + x'\| = 2$. Since VS of X^* implies rotundity of X, we get $x' = x$. Therefore, $x_{n_i} \to x$ and by the double extract subsequence theorem, $x_n \to x$. This means that X is $WLUR$. ▯

Theorem 1.2. *If X^* is WSE, then X is VS.*

Proof: Let X^* be WSE, $x \in S(X)$, $\{f_n\}$ be in $S(X^*)$ and $f_n(x) \to 1$. By the Hahn-Banach theorem there is $f_x \in \text{Grad}(x)$, that is, $\|f_x\| = f_x(x) = \|x\| = 1$. Taking into account that X is canonically embedded into X^{**} we can write $x(f_n) \to 1 = x(f_x)$ so that $x \in \text{Grad}(f_x)$, again identifying x with its embedding. Now, by the assumption that X^* is WSE we get $f_x \xrightarrow{w} f_x$, which completes the proof. ▯

Theorem 1.3. *X is $WLUR$ if and only if X has property WM and X is WSE.*

Proof: The necessity is obvious. To prove the sufficiency, assume that X has both properties WM and WSE and that $x \in S(X)$, $\{x_n\}$ is a sequence in $S(X)$ and $\|x_n + x\| \to 2$. Since X has property WM there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $f_x \in \text{Grad}(x)$ such that $f_x(x_{n_i}) \to 1$. Since X is WSE, this implies that $x_{n_i} \xrightarrow{w} x$ and, by the double extract subsequence theorem, we get $x_n \xrightarrow{w} x$. ▯

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2. Very smoothness and strong smoothness of E^Φ and property WSE in L_0^Φ.

Theorem 2.1. *For the space E^Φ the following statements are equivalent:*

(1) *E^Φ is strongly smooth,*
(2) *E^Φ is very smooth,*
(3) *Φ is smooth and $\Psi \in \Delta_2$.*

Proof: The implication (1) $\Rightarrow$ (2) is obvious. We shall prove now that (2) $\Rightarrow$ (3). Since very smoothness implies smoothness, we conclude that Φ must be smooth on $\mathbb{R}$ (see [**1**, **6**, **14**]). So, we only need to prove that very smoothness of E^Φ implies that $\Psi \in \Delta_2$. Otherwise, we have $E_0^\Psi \subsetneqq L_0^\Psi$. By the Bishop-Phelps theorem there is $f \in S(L_0^\Psi) \setminus S(E_0^\Psi)$ and $x \in S(E^\Phi)$ such that $f(x) = \|f\|^0 = 1$, where $f(x)$ stands for $\langle x, f\rangle$. Define $f_n(t) = f(t)\chi_{G_n}(t)$, where $G_n = \{t \in G : |f(t)| \leqq n\}$. Then

$$f_n(x) = f(x) - \int_{G \setminus G_n} f(t)x(t)\, d\mu.$$

Since the norm in E^Φ is absolutely continuous, we have

$$\lim_{n\to\infty} \left| \int_{G\setminus G_n} f(t)x(t)\, d\mu \right| \leqq \lim_{n\to\infty} \|f\|^0 \left\| x\chi_{G\setminus G_n} \right\| = 0,$$

that is, $f_n(x) \to 1$. Since $f \in S(L_0^\Psi) \setminus S(E_0^\Psi)$, by the Hahn-Banach theorem there is a singular functional $s \neq 0$ such that $s(f) \neq 0$. Since $f_n \in E_0^\Psi$, we have $s(f_n) = 0$. Hence E^Φ is not very smooth, a contradicition which completes the proof.

(3) $\Rightarrow$ (1). For each $x \in S(E^\Phi)$ and $\{f_n\}$ in $S(E_0^\Psi)$ such that $f_n(x) \to 1 = f(x)$ we shall prove that $\|f_n - f\|_\Psi^0 \to 0$, where f is the unique support functional at x (by smoothness of E^Φ, which follows by smoothness of Φ, see [**6**]). Let $k_n \geqq 1$ $(n = 0, 1, 2, \ldots)$ be such that

$$\|f_n\|^0 = \frac{1}{k_n}(1 + I_\Psi(k_n f_n)), \quad \|f\|^0 = \frac{1}{k_0}(1 + I_\Psi(k_0 f)).$$

The existence of such constants k_n follows since $(\Phi(u)/u) \to +\infty$ as $u \to +\infty$ (see [**6**]). We shall present our proof in three steps.

I. We shall prove that $\{k_n\}_{n=0}^\infty$ is a bounded sequence.

(i) Since $(E^\Phi)^* = E_0^\Psi$, there is a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ and $f' \in E_0^\Psi$ such that $f_{n_i} \xrightarrow{E^\Phi} f'$. By $f_n(x) \to 1$ and $x \in E^\Phi$, we have $f_{n_i}(x) \to 1$, whence $f'(x) = f(x)$. Since E^Φ is smooth, we get $f = f'$ and so $f_n \xrightarrow{E^\Phi} f$.

(ii) We shall prove that $|f_n| \overset{E^\Phi}{\to} |f|$. From

$$1 = f(x) \leqslant |f|\,(|x|) \leqq \|f\|^0\,\|x\| = 1,$$

we get $|f|\,(|x|) = 1$. By the same argument as in (i), we obtain $|f_n| \overset{E^\Phi}{\to} |f|$.

(iii) We shall show that there exist $a > 0$, $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that

$$\mu\{t \in G : |f_n(t)| \geqq a\} > \varepsilon \quad \text{whenever} \quad n \geqq n_0. \tag{1}$$

Define $\alpha = \int_G |f(t) =|\, d\mu$ and take $0 < \beta < \alpha/3\mu(G)$. By $I_\Psi(f_n) \leqq 1$ and the Vallée-Poussin theorem, the functions of the sequence $\{|f_n|\}$ have equi-absolutely continuous integrals (see [7] and [**13**]). So, for $\varepsilon_0 = \alpha/3$ there is a $\delta > 0$ such that

$$\int_F |f_n(t)|\, d\mu < \varepsilon_0 \quad \text{whenever} \quad \mu(F) < \delta.$$

If (1) does not hold, then for each $\beta > 0$ and $n \in \mathbb{N}$ there is $n_i \in \mathbb{N}$ such that $\mu\{t \in G : |f_{n_i}(t)| \geqq \beta\} < \delta$. Therefore,

$$\begin{aligned}\int_G \left|f_{n_i}(t)\right| d\mu = &\int_{G\setminus\{t\in G:\ |f_{n_i}(t)|\geqq\beta\}} \left|f_{n_i}(t)\right| d\mu + \int_{\{t\in G: |f_{n_i}(t)|\geqq\beta\}} \left|f_{n_i}(t)\right| d\mu \\ &< \beta\mu(G) + \varepsilon_0 < \frac{2}{3}\alpha,\end{aligned}$$

which contradicts (ii).

If the sequence $\{k_n\}$ is not bounded, there is a subsequence $\{k_{n_i}\}$ of $\{k_n\}$ such that $\lim_{i\to\infty} k_{n_i} = \infty$. Hence

$$\begin{aligned}1 = \left\|f_{n_i}\right\|^0 &= \frac{1}{k_{n_i}}\left(1 + I_\Psi\left(k_{n_i} f_{n_i}\right)\right) \\ &= \lim_{i\to\infty} \frac{1}{k_{n_i}}\left(1 + I_\Psi\left(k_{n_i} f_{n_i}\right)\right) \\ &= \lim_{i\to\infty} \frac{1}{k_{n_i}} \int_G \Psi\left(k_{n_i} f_{n_i}(t)\right) d\mu \\ &\geqq \lim_{i\to\infty} \frac{1}{k_{n_i}} \int_{\{t\in G: |f_{n_i}(t)|\geqq a\}} \Psi\left(k_{n_i} f_{n_i}(t)\right) d\mu \\ &\geqq \lim_{i\to\infty} \frac{1}{k_{n_i}} \Psi\left(k_{n_i} a\right) \mu\{t \in G : \left|f_{n_i}(t)\right| \geqq a\} \\ &\geqq \lim_{i\to\infty} \frac{\Psi\left(k_{n_i}\right)}{k_{n_i}} \varepsilon = \infty,\end{aligned}$$

which is a contradiction. So, the sequence $\{k_n\}$ is bounded.

II. We shall prove that the functions of the sequence $\{f_n\}$ have equi-absolutely continuous norms. By $\Psi \in \Delta_2$, we only need to prove that

$$\lim_{\mu(F)\to 0} \sup_n I_\Psi(f_n \chi_F) = 0.$$

Otherwise, we can assume without loss of generality that

$$I_\Psi(f_n\chi_{F_n}) \geqq \varepsilon_0 \quad \text{and} \quad \mu(F_n) < 2^{-n}\mu(G)$$

for a sequence $\{F_n\}$ of measurable subsets of G. Take a number $m \in \mathbb{N}$ such that

$$\left\|f\chi_{G\setminus F}\right\|^0 \geqq \|f\|^0 - \varepsilon_0/2 = 1 - \varepsilon_0/2$$

whenever $\mu(F) < 2^{-m}\mu(G)$. We have for $n > m$,

$$\begin{aligned}
1 = \|f_n\|^0 &= \frac{1}{k_n}(1 + I_\Psi(k_n f_n)) \\
&= \frac{1}{k_n}\left(1 + I_\Psi\left(k_n f_n \chi_{G\setminus \bigcup_{i>m} F_i}\right) + I_\Psi\left(k_n f_n \chi_{\bigcup_{i>m} F_i}\right)\right) \\
&\geqq \left\|f_n\chi_{G\setminus \bigcup_{i>m} F_i}\right\|^0 + I_\Psi(f_n\chi_{F_n}) \\
&\geqq \left\|f_n\chi_{G\setminus \bigcup_{i>m} F_i}\right\|^0 + \varepsilon_0.
\end{aligned}$$

Since $f_n \xrightarrow{E^\Phi} f$ by (i), we have

$$\liminf_{n\to\infty}\left\|f_n\chi_{G\setminus \bigcup_{i>m} F_i}\right\|^0 \geqq \left\|f\chi_{G\setminus \bigcup_{i>m} F_i}\right\| > 1 - \varepsilon_0/2.$$

Hence, we get

$$1 \geqq 1 - \varepsilon_0/2 + \varepsilon_0 = 1 + \varepsilon_0/2,$$

which is a contradiction. So, f_n $(n = 1, 2, \dots)$ have equi-absolutely continuous norms.

III. We shall prove that $k_n f_n - k_0 f \xrightarrow{\mu} 0$. Otherwise, we can assume without loss of generality that there exist $\sigma_0 > 0$ and $\varepsilon_0 > 0$ such that

$$\mu\{t \in G : |k_n f_n(t) - k_0 f(t)| \geqq \sigma_0\} \geqq \varepsilon_0$$

for any $n \in \mathbb{N}$. By step I, we may assume that there is $d > 1$ such that $1 < k_n \leqq 1 + d$ for $n = 0, 1, 2\dots$. Define $k = \Psi^{-1}(3/\varepsilon_0)(1 + d)$ and

$$G_n = \{t \in G : |k_n f_n(t)| \leqq k,\ |k_0 f(t)| \leqq k \text{ and } |k_n f(t) - k_0 f(t)| \geqq \sigma_0\}.$$

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Since for any $u \in S(L_0^\Psi)$,

$$1 \geqq \int_G \Psi(u(t))\,d\mu \geqq \int_{\{t\in G:\,(1+d)|u(t)|>k\}} \Psi(u(t))\,d\mu$$
$$> \Psi\Big(\frac{k}{1+d}\Big)\mu(\{t\in G:\,(1+d)\,|u(t)|>k\}) = \frac{3}{\varepsilon_0}\mu(\{t\in G:\,(1+d)\,|u(t)|>k\}),$$

we get $\mu(\{t\in G:\,(1+d)\,|u(t)|>k\}) < \varepsilon_0/3$. Hence

$$\mu(G_n) \geqq \mu(\{t\in G:\,|k_nf_n(t)-k_0f(t)|\geqq\sigma_0\}) - \mu(\{t\in G:\,|k_0f(t)|>k\})$$
$$- \mu(\{t\in G:\,|k_nf_n(t)|>k\}) > \varepsilon_0-\varepsilon_0/3-\varepsilon_0/3>\varepsilon_0/3.$$

Since

$$0<\frac{1}{2+d}<\frac{k_n}{k_n+1+d}\leqq\frac{k_n}{k_n+k_0}<\frac{k_n}{k_n+1}\leqq\frac{1+d}{2+d}<1,$$
$$0<\frac{1}{2+d}<\frac{k_0}{k_0+1+d}\leqq\frac{k_0}{k_n+k_0}\leqq\frac{k_0}{k_0+1}\leqq\frac{1+d}{2+d}<1,$$

by strict convexity of Ψ (which is equivalent to smoothness of Φ), there is $\delta>0$ such that for every u,v with $|u|\leqslant k$, $|v|\leqslant k$, $|u-v|\geqq\sigma_0$ and for every $\alpha\in\big[1/(2+d),(1+d)/(2+d)\big]$, we have

$$\Psi(\alpha u+(1-\alpha)v)\leqslant(1-\delta)\{\alpha\Psi(u)+(1-\alpha)\Psi(v)\}.$$

Since $(f_n+f)(x)\to 2$, we have $\|f_n+f\|^0\to 2$. Noticing that $k_n/(k_n+k_0)+k_0/(k_n+k_0)=1$, we have

$$2\leftarrow\|f_n+f\|^0\leqslant\frac{k_n+k_0}{k_nk_0}\Big\{1+I_\Psi\Big(\frac{k_nk_0}{k_n+k_0}(f_n+f)\Big)\Big\}$$
$$\leqq\frac{k_n+k_0}{k_nk_0}\Big\{1+(1-\delta)\int_{G_n}\Big[\frac{k_0}{k_n+k_0}\Psi(k_nf_n(t))+\frac{k_n}{k_n+k_0}\Psi(k_0f(t))\Big]\,d\mu$$
$$+\int_{G\setminus G_n}\Big[\frac{k_0}{k_n+k_0}\Psi(k_nf_n(t))+\frac{k_n}{k_n+k_0}\Psi(k_0f(t))\Big]\,d\mu\Big\}$$
$$=\frac{1}{k_n}(1+I_\Psi(k_nf_n))+\frac{1}{k_0}(1+I_\Psi(k_0f))$$
$$-\delta\int_{G_n}\Big[\frac{1}{k_n}\Psi(k_nf_n(t))+\frac{1}{k_0}\Psi(k_0f(t))\Big]\,d\mu$$
$$\leqq 2-\frac{\delta}{1+d}\int_{G_n}[\Psi(k_nf_n(t))+\Psi(k_0f(t))]\,d\mu$$
$$\leqq 2-2\delta\frac{1}{1+d}\int_{G_n}\Psi\Big(\frac{1}{2}k_nf_n(t)-\frac{1}{2}k_0f(t)\Big)\,d\mu$$
$$\leqq 2-\frac{2\delta}{1+d}\Psi\Big(\frac{\sigma_0}{2}\Big)\mu(G_n)$$
$$\leqq 2-\frac{2\delta}{1+d}\Psi\Big(\frac{\sigma_0}{2}\Big)\frac{\varepsilon_0}{3},$$

a contradiction. So, we have proved that $k_n f - k_0 f \xrightarrow{\mu} 0$.

Thus, by equi-absolute continuity of the norms of all functions from the sequence $\{f_n\}$ and by $k_n f_n - k_0 f \xrightarrow{\mu} 0$, defining

$$F_n = \left\{ t \in G : \frac{6}{\varepsilon} |k_n f_n(t) - k_0 f(t)| \geqslant \Phi^{-1}\Big(\frac{1}{\mu(G)}\Big) \right\},$$

we have $\left\|(k_n f_n - k_0 f)\chi_{G\setminus F_n}\right\|^0 \leqq 2\left\|(k_n f_n - k_0 f)\chi_{G\setminus F_n}\right\| < 2\cdot\varepsilon/6 = \varepsilon/3$ for all $n \in \mathbb{N}$ and $\mu(F_n) \to 0$ as $n \to \infty$. Consequently, $\|k_n f_n \chi_{F_n}\|^0 < \varepsilon/3$ and $\|k_0 f \chi_{F_n}\|^0 < \varepsilon/3$ for all $n \in \mathbb{N}$ large enough, say for $n \geqq n_0$. Hence $\|k_n f_n - k_0 f\|^0 < \varepsilon$ for $n \geqq n_0$, that is, $\|k_n f_n - k_0 f\|^0 \to 0$. So, $\left\|(k_n/k_0) f_n - f\right\|^0 \to 0$ and consequently $\left\|(k_n/k_0) f_n\right\|^0 \to \|f\|^0$. Since $\|f_n\|^0 \to \|f\|^0$, this yields $k_n \to k_0$ and finally $\|f_n - f\|^0 \to 0$. So, the proof is complete. ∎

Theorem 2.2. *An Orlicz space L_0^Φ is weakly strongly exposed if and only if $\Phi \in \Delta_2$, $\Psi \in \Delta_2$, and Φ is strictly convex.*

Proof: *Sufficiency.* Suppose $\{x_n\}_{n=0}^\infty$ is a sequence in $S(L_0^\Phi)$ such that $f(x_n) \to f(x_0) = 1$ for some $f \in \operatorname{Grad}(x)$. Since, by $\Phi \in \Delta_2$ and $\Psi \in \Delta_2$, L_0^Φ is reflexive, and we conclude that there are a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $x \in L_0^\Phi$ such that $x_{n_i} \xrightarrow{w} x$. In particular, $f(x_{n_i}) \to f(x)$. By $f(x_{n_i}) \to 1$, we have $f(x) = 1$. Thus $f(x_0 + x) = 2$, whence $\|x_0 + x\| = 2$. Since L_0^Φ is strictly convex (see [1] and [**13**]), this yields $x = x_0$. So, $x_{n_i} \xrightarrow{w} x_0$ and by the double extract subsequence theorem, we get $x_n \xrightarrow{w} x_0$.

Necessity. By Theorems 1.1 and 2.1, we only need to prove that $\Psi \in \Delta_2$. Assume, for the contrary that $\Psi \notin \Delta_2$. Then there is a sequence $\{u_n\}_{n=1}^\infty$ of positive numbers such that $u_n \to +\infty$ as $n \to \infty$ and

$$\Psi\left(\Big(1 + \frac{1}{n}\Big)u_n\right) \geqslant 2^n \Psi(u_n) \quad (n = 1, 2, \dots).$$

Since $u_n \to +\infty$ as $n \to \infty$, we can find (passing to a subsequence if necessary) a sequence $\{G_n\}_{n=1}^\infty$ of measurable pairwise disjoint sets in G such that $\mu(G_n) = 1/(2^n \Psi(u_n))$. Define

$$y = \sum_{n=1}^\infty u_n \chi_{G_n}.$$

Then $I_\Psi(y) = 1$ and for any $\tau > 0$, $I_\Psi((1+\tau)y) = +\infty$. Let $\{v_n\}_{n=1}^\infty$ be a sequence of positive numbers such that $u_n \in \partial\Phi(v_n) = [\Phi'_-(v_n), \Phi'_+(v_n)]$ and define

$$x = \sum_{n=1}^\infty v_n \chi_{G_n}.$$

Then we have (see [4])

$$\|x\|^0 = \int_G x(t)y(t)\,d\mu.$$

Define $x_0 = x/\|x\|^0$. Then $x_0 \in S(L_0^\Phi)$ and $y \in \operatorname{Grad}(x_0)$. Assume $A_n = \{t \in G : |y(t)| \leqslant n\}$. Then $\lim_{n\to\infty} \|y\chi_{G\setminus A_n}\|^0 = 1$. By

$$\|y\chi_{G\setminus A_n}\|^0 = \sup\Big\{\int_{G\setminus A_n} z(t)y(t)\,d\mu : \|z\| = 1\Big\},$$

there exists a sequence $\{z_n\}$ in $S(L^\Phi)$ such that

$$z_n = z_n\chi_{G\setminus A_n} \quad \text{and} \quad \int_{G\setminus A_n} z_n(t)y(t)\,d\mu \to 1 = \langle x_0, y\rangle.$$

Taking into account that, in view of the Vallée-Poussin theorem, all the functions from $\{z_n\}_{n=1}^\infty$ have equi-absolutely continuous integrals, we get by $\mu(G \setminus A_n) \to 0$ that

$$\lim_{n\to\infty} \int_{G\setminus A_n} z_n(t)\,d\mu = 0.$$

Hence

$$\lim_{n\to\infty} \int_G (x(t) - z_n(t))\,d\mu = \int_G x(t)\,d\mu > 0,$$

that is, $z_n \overset{w}{\nrightarrow} x$, which means that L_0^Φ is not WSE, and the proof is complete. □

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