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A REMARK ON COMPLEX CONVEXITY

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ABSTRACT. We present a quasi-normed space which is locally H_∞-convex but is not locally PL-convex in the sense of Davis, Garling and Tomczak-Jaegermann.

In [2], Davis, Garling and Tomczak-Jaegermann define the notions of PL-convexity and H_∞-convexity, and ask whether a locally H_∞-convex space is necessarily locally PL-convex (problem 1, [2]). We show here that a quasi-normed 2-dimensional space described by Aleksandrov in [1] provides an example to answer this question negatively.

DEFINITION. A complex continuously quasi-normed space $(E, \|\cdot\|)$ is *locally PL-convex* (resp. *locally H_∞-convex*) if whenever x and y are in E, there exists $\delta = \delta(x, y) > 0$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} \|x + re^{i\theta}y\|d\theta \geqq \|x\|$$

(resp. $\sup\{\|x + re^{i\theta}y\|, 0 \leqq \theta \leqq 2\pi\} \geqq \|x\|$) for all $0 \leqq r \leqq \delta$. A complex continuously quasi-normed space $(E, \|\cdot\|)$ is *uniformly PL-convex* (resp. *uniformly H_∞-convex*) if

$$\inf\left\{\left(\frac{1}{2\pi}\int_0^{2\pi} \|x + e^{i\theta}y\|d\theta - 1\right):\|x\| = 1, \|y\| = \epsilon\right\} > 0$$

for all $\epsilon > 0$ (resp. if

$$\inf\left\{\left(\sup_{\theta\in[0,2\pi]} \|x + e^{i\theta}y\| - 1\right):\|x\| = 1, \|y\| = \epsilon\right\} > 0$$

for all $\epsilon > 0$).

Aleksandrov defines the following quasi-norm in $\mathbf{C}^2$

$$(1) \qquad \|(z_1, z_2)\|_0 = \max\left\{\frac{\max\{|z_1|, |z_2|\}}{B}, \frac{\min\{|z_1|, |z_2|\}}{A}\right\}$$

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where A and B are two fixed positive real numbers such that $A < B$ and $A + B \leqq 2^{1/p}A$ for some fixed $p < 1$.

PROPOSITION 1 (Aleksandrov). $(\mathbf{C}^2, \|\cdot\|_0)$ *is a non-locally PL-convex p-normed space.*

PROOF. We omit the easy proof that $\|\cdot\|_0$ is p-subadditive. Define

$$f(\epsilon) = \frac{1}{2\pi} \int_0^{2\pi} \|(1 + \epsilon e^{i\theta}, 1 - \epsilon e^{i\theta})\|_0 d\theta,$$

for every $\epsilon > 0$. For ϵ small enough

$$\|(1 + \epsilon e^{i\theta}, 1 - \epsilon e^{i\theta})\|_0 = \frac{1}{A} \min(|1 + \epsilon e^{i\theta}|, |1 - \epsilon e^{i\theta}|),$$

and therefore,

$$f(\epsilon) = \frac{2}{A\pi} \int_0^{\pi/2} \sqrt{1 + \epsilon^2 - 2|\epsilon| \cos \theta} d\theta.$$

There exists $\delta > 0$ such that f is continuous in $[0, \delta]$, differentiable in $(0, \delta)$ and $f'(\epsilon) < 0$ for every $\epsilon \in (0, \delta)$, therefore $f(\epsilon) < f(0)$ for every $\epsilon \in (0, \delta)$ i.e.,

$$\frac{1}{2\pi} \int_0^{2\pi} \|(1 + \epsilon e^{i\theta}, 1 - \epsilon e^{i\theta})\|_0 d\theta < \|(1, 1)\|_0$$

for every $\epsilon \in (0, \delta)$. Hence, $(\mathbf{C}^2, \|\cdot\|_0)$ is not locally PL-convex.

PROPOSITION 2. $(\mathbf{C}^2, \|\cdot\|_0)$ *is locally* H_∞*-convex.*

PROOF. Given $(z_1, z_2), (w_1, w_2) \in \mathbf{C}^2$, $\{z_k + e^{i\theta} w_k, \theta \in [0, 2\pi]\}$ is a circle of center z_k and radius $|w_k|$ $(k = 1, 2)$. It is clear that

$$|z_k + e^{i\theta} w_k| \geqq \sqrt{|z_k|^2 + |w_k|^2}$$

in a closed set of θ's of measure π, and $|z_k + e^{i\theta} w_k| \geqq |z_k|$ in an open set of θ's of measure greater than π $(k = 1, 2)$. Thus, there must exist $\theta_0 \in [0, 2\pi]$ such that $|z_1 + e^{i\theta_0} w_1| \geqq |z_1|$ and $|z_2 + e^{i\theta_0} w_2| \geqq |z_2|$. In particular,

$$\sup_{\theta \in [0,2\pi]} \|(z_1 + e^{i\theta} w_1, z_2 + e^{i\theta} w_2)\|_0 \geqq \|(z_1, z_2)\|_0.$$

$(\mathbf{C}^2, \|\cdot\|_0)$ is not uniformly H_∞-convex, but we can modify this example to show than even uniform H_∞-convexity does not imply local PL-convexity. We may define the following quasi-norm in $\mathbf{C}^2$

$$\|(z_1, z_2)\|_\delta = \|(z_1, z_2)\|_0 + \delta \sqrt{|z_1|^2 + |z_2|^2}$$

where $\|\cdot\|_0$ is the quasi-norm defined in (1) and $\delta > 0$. For δ small enough $(\mathbf{C}^2, \|\cdot\|_\delta)$ is not locally PL-convex, but it is uniformly H_∞-convex. We omit the details.

QUESTION. Can a locally H_∞-convex space be always renormed with an equivalent locally PL-convex quasi-norm?

REFERENCES

1. A. B. Aleksandrov, *Essays on non-locally convex Hardy classes*, Lecture Notes in Mathematics **864**, pp. 1-89 (Springer-Verlag, 1981).

2. W. J. Davis, D. J. H. Garling and N. Tomczak-Jaegermann, *The complex convexity of quasi-normed linear spaces*, J. Funct. Anal. **55** (1984), pp. 110-150.

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