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CONGRUENCES FOR TRUNCATED HYPERGEOMETRIC SERIES ${}_2F_1$

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Abstract

Rodriguez-Villegas conjectured four supercongruences associated to certain elliptic curves, which were first confirmed by Mortenson by using the Gross–Koblitz formula. In this paper we prove four supercongruences between two truncated hypergeometric series ${}_2F_1$. The results generalise the four Rodriguez-Villegas supercongruences.

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1. Introduction

In 2003, Rodriguez-Villegas [13] studied hypergeometric families of Calabi–Yau manifolds. He observed numerically some remarkable supercongruences between the values of the truncated hypergeometric series and expressions derived from the number of F_p-points of the associated Calabi–Yau manifolds. A number of supercongruences for hypergeometric Calabi–Yau manifolds have been conjectured by Rodriguez-Villegas. For manifolds of dimension $d = 1$, he conjectured four supercongruences associated to certain elliptic curves. These four supercongruences were first confirmed by Mortenson [9, 10] by using the Gross–Koblitz formula.

To state these results, we first define the truncated hypergeometric series. For complex numbers a_i, b_j and z, with none of the b_j being negative integers or zero, the truncated hypergeometric series are given by

$$ {}_rF_s\begin{bmatrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{bmatrix}; z\Bigg]_n = \sum_{k=0}^{n-1} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{(b_1)_k (b_2)_k \cdots (b_s)_k} \cdot \frac{z^k}{k!}, $$

where $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geq 1$.

THEOREM 1.1 (Rodriguez-Villegas and Mortenson). *Let $p \geq 5$ be a prime. Then*

$$ {}_2F_1\begin{bmatrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{bmatrix}; 1\Bigg]_p \equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \quad {}_2F_1\begin{bmatrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{bmatrix}; 1\Bigg]_p \equiv \left(\frac{-3}{p}\right) \pmod{p^2}, $$

$$ {}_2F_1\begin{bmatrix} \frac{1}{4}, \frac{3}{4} \\ 1 \end{bmatrix}; 1\Bigg]_p \equiv \left(\frac{-2}{p}\right) \pmod{p^2}, \quad {}_2F_1\begin{bmatrix} \frac{1}{6}, \frac{5}{6} \\ 1 \end{bmatrix}; 1\Bigg]_p \equiv \left(\frac{-1}{p}\right) \pmod{p^2}, $$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

For more proofs of Theorem 1.1, see [3, 14, 18, 19]. Some extensions of the congruences in Theorem 1.1 to modulus p^3 were obtained in [15, 16]. For some interesting q-analogues of Theorem 1.1, see [6, 7]. By studying the generalised Legendre polynomials, Sun [14] extended Theorem 1.1 as follows.

THEOREM 1.2 (Sun). *Let $p \geq 5$ be a prime. For any p-adic integer x,*

$$ {}_2F_1\left[\begin{matrix} x, 1-x \\ 1 \end{matrix}; 1\right]_p \equiv (-1)^{\langle -x\rangle_p} \pmod{p^2}, \tag{1.1} $$

where $\langle a\rangle_p$ denotes the least non-negative integer r with $a \equiv r \pmod{p}$.

Observe that

$$ (-1)^{\langle -1/2\rangle_p} = \left(\frac{-1}{p}\right), \quad (-1)^{\langle -1/3\rangle_p} = \left(\frac{-3}{p}\right), \quad (-1)^{\langle -1/4\rangle_p} = \left(\frac{-2}{p}\right), \quad (-1)^{\langle -1/6\rangle_p} = \left(\frac{-1}{p}\right). $$

Thus Theorem 1.2 reduces to Theorem 1.1 when $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$.

Apéry introduced the numbers

$$ A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, $$

now known as Apéry numbers, in his ingenious proof [1] of the irrationality of $\zeta(3)$. Since the appearance of these numbers, some interesting arithmetic properties have gradually been discovered. For example, Gessel [5] proved that, for any prime $p \geq 5$,

$$ A_{np} \equiv A_n \pmod{p^3}, $$

which confirmed a conjecture by Chowla *et al.* [4].

We aim to prove similar supercongruences for the truncated hypergeometric series ${}_2F_1$, which generalise the four supercongruences in Theorem 1.1.

THEOREM 1.3. *Suppose $p \geq 5$ is a prime and n is a positive integer. For $x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$,*

$$ {}_2F_1\left[\begin{matrix} x, 1-x \\ 1 \end{matrix}; 1\right]_{np} \equiv (-1)^{\langle -x\rangle_p} \cdot {}_2F_1\left[\begin{matrix} x, 1-x \\ 1 \end{matrix}; 1\right]_n \pmod{p^2}. \tag{1.2} $$

Theorem 1.3 reduces to Theorem 1.1 when $n = 1$. Replacing n by p^{r-1} in (1.2) and then using induction, we immediately get the following result, which is a special case of [6, Theorem 1.1].

Corollary 1.4. *Suppose $p \ge 5$ is a prime and r is a positive integer. For $x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$,*

$$ {}_2F_1\left[\begin{matrix} x, 1-x \\ 1 \end{matrix}; 1\right]_{p^r} \equiv (-1)^{\langle -x\rangle_p \cdot r} \pmod{p^2}. $$

This paper is organised as follows. In the next section we first recall some properties of the Fermat quotients and some combinatorial identities involving harmonic numbers, and then prove two congruences. In Section 3 we give a new proof of Theorem 1.2 by using combinatorial identities. The proof of Theorem 1.3 is given in the Section 4. We make some concluding remarks in the final section.

2. Some lemmas

The Fermat quotient of an integer a with respect to an odd prime p is given by

$$ q_p(a) = \frac{a^{p-1} - 1}{p}. $$

The Fermat quotient plays an important role in the study of cyclotomic fields.

Lemma 2.1 (Eisenstein). *Suppose p is an odd prime and r is a positive integer. For nonzero p-adic integers a and b,*

$$ q_p(ab) \equiv q_p(a) + q_p(b) \pmod{p}, $$
$$ q_p(a^r) \equiv r q_p(a) \pmod{p}. $$

Lemma 2.2 (Lehmer [8]). *Let $H_n = \sum_{k=1}^{n} (1/k)$ be the nth harmonic number. For any prime $p \ge 5$,*

$$ H_{\lfloor p/2\rfloor} \equiv -2q_p(2) \pmod{p}, \quad H_{\lfloor p/3\rfloor} \equiv -\tfrac{3}{2}q_p(3) \pmod{p}, $$
$$ H_{\lfloor p/4\rfloor} \equiv -3q_p(2) \pmod{p}, \quad H_{\lfloor p/6\rfloor} \equiv -2q_p(2) - \tfrac{3}{2}q_p(3) \pmod{p}, $$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to a real number x.

Lemma 2.3. *If n is a positive integer, then*

$$ \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} = (-1)^n, \tag{2.1} $$

$$ \sum_{k=1}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} H_k = 2(-1)^n H_n, \tag{2.2} $$

$$ \sum_{k=1}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} \sum_{i=1}^{k} \frac{1}{n+i} = (-1)^n H_n. \tag{2.3} $$

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PROOF. Prodinger [12] has given a proof of (2.1)–(2.2) by partial fraction decomposition and creative telescoping (see also [11]). Using the same method, Prodinger [12] also obtained the identity

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}H_{n+k}=2(-1)^nH_n. \tag{2.4}$$

By (2.1) and (2.4),

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}\sum_{i=1}^{k}\frac{1}{n+i}=\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}(H_{n+k}-H_n)=(-1)^nH_n.$$

This proves (2.3). □

LEMMA 2.4. *Let $p\geq 5$ be a prime and let r and k be nonnegative integers with $0\leq k\leq p-1$. For $x\in\left\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}\right\}$,*

$$\begin{aligned}\frac{(x)_{k+rp}(1-x)_{k+rp}}{(1)^2_{k+rp}}&\equiv\frac{(x)_r(1-x)_r}{(1)^2_r}\cdot\frac{(x)_k(1-x)_k}{(1)^2_k}\\&\quad\times\left(1+2rpH_{\lfloor px\rfloor}-2rpH_k+rp\sum_{i=0}^{k-1}\left(\frac{1}{x+i}+\frac{1}{1-x+i}\right)\right)\pmod{p^2}.\end{aligned}\tag{2.5}$$

PROOF. Note that

$$\frac{(x)_n(1-x)_n}{(1)^2_n}=\frac{S_x(n)}{a_x^n},$$

where

$$S_x(n)=\begin{cases}\binom{2n}{n}^2 & \text{if } x=\frac{1}{2},\\ \binom{2n}{n}\binom{3n}{n} & \text{if } x=\frac{1}{3},\\ \binom{2n}{n}\binom{4n}{2n} & \text{if } x=\frac{1}{4},\\ \binom{3n}{n}\binom{6n}{3n} & \text{if } x=\frac{1}{6},\end{cases}$$

and $a_{1/2}=16, a_{1/3}=27, a_{1/4}=64$ and $a_{1/6}=432$. Thus (2.5) is equivalent to

$$\begin{aligned}S_x(k+rp)&\equiv a_x^{r(p-1)}S_x(r)S_x(k)\\&\quad\times\left(1+2rpH_{\lfloor px\rfloor}-2rpH_k+rp\sum_{i=0}^{k-1}\left(\frac{1}{x+i}+\frac{1}{1-x+i}\right)\right)\pmod{p^2}.\end{aligned}\tag{2.6}$$

By Lemmas 2.1 and 2.2,

$$a_x^{r(p-1)}\equiv 1+pq_p(a_x^r)\equiv 1+rpq_p(a_x)\equiv 1-2rpH_{\lfloor px\rfloor}\pmod{p^2}.$$

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So in order to prove (2.6), it suffices to show that

$$S_x(k+rp) \equiv S_x(r)S_x(k)\Big(1 - 2rpH_k + rp\sum_{i=0}^{k-1}\Big(\frac{1}{x+i} + \frac{1}{1-x+i}\Big)\Big) \pmod{p^2}. \quad (2.7)$$

Note that

$$\sum_{j=0}^{k-1}\Big(\frac{1}{j+1/2} + \frac{1}{j+1/2}\Big) = 4H_{2k} - 2H_k,$$

$$\sum_{j=0}^{k-1}\Big(\frac{1}{j+1/3} + \frac{1}{j+2/3}\Big) = 3H_{3k} - H_k,$$

$$\sum_{j=0}^{k-1}\Big(\frac{1}{j+1/4} + \frac{1}{j+3/4}\Big) = 4H_{4k} - 2H_{2k},$$

$$\sum_{j=0}^{k-1}\Big(\frac{1}{j+1/6} + \frac{1}{j+5/6}\Big) = 6H_{6k} - 3H_{3k} - 2H_{2k} + H_k.$$

Thus (2.7) becomes the following set of congruences modulus p^2:

$$\binom{2rp+2k}{rp+k}^2 \equiv \binom{2r}{r}^2\binom{2k}{k}^2(1 + rp(4H_{2k} - 4H_k)), \quad (2.8)$$

$$\binom{2rp+2k}{rp+k}\binom{3rp+3k}{rp+k} \equiv \binom{2r}{r}\binom{3r}{r}\binom{2k}{k}\binom{3k}{k}(1 + rp(3H_{3k} - 3H_k)), \quad (2.9)$$

$$\binom{2rp+2k}{rp+k}\binom{4rp+4k}{2rp+2k} \equiv \binom{2r}{r}\binom{4r}{2r}\binom{2k}{k}\binom{4k}{2k}(1 + rp(4H_{4k} - 2H_{2k} - 2H_k)), \quad (2.10)$$

$$\binom{3rp+3k}{rp+k}\binom{6rp+6k}{3rp+3k} \equiv \binom{3r}{r}\binom{6r}{3r}\binom{3k}{k}\binom{6k}{3k}(1 + rp(6H_{6k} - 3H_{3k} - 2H_{2k} - H_k)). \quad (2.11)$$

We give the proof of (2.8). The proofs of (2.9)–(2.11) run analogously.

Note that

$$\binom{2rp+2k}{rp+k} = \binom{2rp}{rp}\prod_{i=1}^{2k}(2rp+i) \Big/ \prod_{i=1}^{k}(rp+i)^2$$

$$\equiv \binom{2r}{r}\prod_{i=1}^{2k}(2rp+i) \Big/ \prod_{i=1}^{k}(rp+i)^2 \pmod{p^2}, \quad (2.12)$$

where we have utilised Babbage's theorem [2]

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^2}.$$

Now we consider the rational function

$$f(x) = \prod_{i=1}^{2k}(2rx+i) \Big/ \prod_{i=1}^{k}(rx+i)^2. \quad (2.13)$$

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Taking the logarithmic derivative on both sides of (2.13) gives

$$\frac{f'(x)}{f(x)} = \sum_{i=1}^{2k} \frac{2r}{2rx+i} - 2\sum_{i=1}^{k} \frac{r}{rx+i}. \tag{2.14}$$

From (2.13) and (2.14),

$$f(0) = \binom{2k}{k} \quad \text{and} \quad f'(0) = r\binom{2k}{k}(2H_{2k} - 2H_k),$$

which gives the first two terms of the Taylor expansion for $f(x)$:

$$f(x) = \binom{2k}{k} + rx\binom{2k}{k}(2H_{2k} - 2H_k) + O(x^2). \tag{2.15}$$

Combining (2.12) and (2.15),

$$\binom{2rp+2k}{rp+k} \equiv \binom{2r}{r}\binom{2k}{k}(1 + rp(2H_{2k} - 2H_k)) \pmod{p^2}.$$

It follows that

$$\binom{2rp+2k}{rp+k}^2 \equiv \binom{2r}{r}^2\binom{2k}{k}^2(1 + rp(4H_{2k} - 4H_k)) \pmod{p^2}.$$

This concludes the proof of (2.8). □

Lemma 2.5. *Suppose $p \geq 5$ is a prime and k is an integer with $0 \leq k \leq p-1$. For $x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right\}$,*

$$\frac{(x)_k(1-x)_k}{(1)_k^2} \equiv (-1)^k\binom{\lfloor px\rfloor}{k}\binom{\lfloor px\rfloor + k}{k} \pmod{p}. \tag{2.16}$$

Proof. It suffices to show that, for $0 \leq k \leq p-1$,

$$(\lfloor px\rfloor + 1)_k(-\lfloor px\rfloor)_k \equiv (x)_k(1-x)_k \pmod{p}. \tag{2.17}$$

For any prime $p \geq 5$, there exists $\varepsilon \in \{1, -1\}$ such that $p \equiv \varepsilon \pmod{2, 3, 4, 6}$. We give the proof of (2.17) for $x = \frac{1}{3}$. The proofs of the other three cases run similarly.

If $p \equiv 1 \pmod 3$, then $\lfloor p/3\rfloor = \frac{1}{3}(p-1)$ and hence

$$\left(\frac{p+2}{3}\right)_k\left(\frac{-p+1}{3}\right)_k \equiv \left(\frac{2}{3}\right)_k\left(\frac{1}{3}\right)_k \pmod{p}.$$

If $p \equiv -1 \pmod 3$, then $\lfloor p/3\rfloor = \frac{1}{3}(p-2)$ and so

$$\left(\frac{p+1}{3}\right)_k\left(\frac{-p+2}{3}\right)_k \equiv \left(\frac{2}{3}\right)_k\left(\frac{1}{3}\right)_k \pmod{p}.$$

This yields (2.17) for $x = \frac{1}{3}$. □

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3. A new proof of Theorem 1.2

Letting $x \to -x$ in Theorem 1.2, (1.1) is equivalent to

$$ {}_2F_1\left[\begin{matrix} -x, 1+x \\ 1 \end{matrix}; 1\right]_p \equiv (-1)^{\langle x\rangle_p} \pmod{p^2}. \tag{3.1}$$

It is easy to see that

$$\frac{(-x)_k(1+x)_k}{(1)_k^2} = (-1)^k\binom{x}{k}\binom{x+k}{k}.$$

Let δ denote the number $\delta = (x-\langle x\rangle_p)/p$. It is clear that δ is a p-adic integer and $x = \langle x\rangle_p + \delta p$. Note that

$$\begin{aligned}\binom{x}{k}\binom{x+k}{k} &= \binom{\langle x\rangle_p+\delta p}{k}\binom{\langle x\rangle_p+\delta p+k}{k}\\ &= \prod_{i=1}^{k}(\langle x\rangle_p+\delta p+1-i)\prod_{i=1}^{k}(\langle x\rangle_p+\delta p+i)\Big(\prod_{i=1}^{k} i^{-1}\Big)^2\\ &\equiv \binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p+k}{k}\Big(1+\delta p\Big(\sum_{i=1}^{k}\frac{1}{\langle x\rangle_p+i}+\sum_{i=1}^{k}\frac{1}{\langle x\rangle_p+1-i}\Big)\Big) \pmod{p^2}.\end{aligned}$$

It follows that the left-hand side of (3.1) is congruent to

$$\sum_{k=0}^{p-1}(-1)^k\binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p+k}{k}\Big(1+\delta p\Big(\sum_{i=1}^{k}\frac{1}{\langle x\rangle_p+i}+\sum_{i=1}^{k}\frac{1}{\langle x\rangle_p+1-i}\Big)\Big) \pmod{p^2}. \tag{3.2}$$

Let $b = p-\langle x\rangle_p$. Clearly, $\langle x\rangle_p \equiv -b \pmod p$ and $0 \le b-1 \le p-1$. By (2.3),

$$\begin{aligned}\sum_{k=0}^{p-1}&(-1)^k\binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p+k}{k}\sum_{i=1}^{k}\frac{1}{\langle x\rangle_p+1-i}\\ &\equiv -\sum_{k=0}^{p-1}\binom{-b}{k}\binom{-b+k}{k}\sum_{i=1}^{k}\frac{1}{b-1+i} \pmod p\\ &= -\sum_{k=0}^{p-1}\binom{b-1}{k}\binom{b-1+k}{k}\sum_{i=1}^{k}\frac{1}{b-1+i}\\ &= (-1)^b H_{b-1}\\ &\equiv (-1)^{\langle x\rangle_p+1}H_{\langle x\rangle_p} \pmod p,\end{aligned} \tag{3.3}$$

where we have used the observations $\binom{-b}{k}\binom{-b+k}{k} = \binom{b-1}{k}\binom{b-1+k}{k}$ in the second step and $H_{p-k-1} \equiv H_k \pmod p$ for $0 \le k \le p-1$ in the last step. By (2.1) and (2.3),

$$\sum_{k=0}^{p-1}(-1)^k\binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p+k}{k} = (-1)^{\langle x\rangle_p} \tag{3.4}$$

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and

$$\sum_{k=0}^{p-1}(-1)^k\binom{\langle x\rangle_p}{k}\binom{\langle x\rangle_p+k}{k}\sum_{i=1}^{k}\frac{1}{\langle x\rangle_p+i}=(-1)^{\langle x\rangle_p}H_{\langle x\rangle_p}. \tag{3.5}$$

By substituting (3.3)–(3.5) into (3.2), we complete the proof of (3.1).

4. Proof of Theorem 1.3

Assume $x\in\left\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}\right\}$. We first prove that for any nonnegative integer r,

$$\sum_{k=rp}^{(r+1)p-1}\frac{(x)_k(1-x)_k}{(1)_k^2}\equiv\frac{(x)_r(1-x)_r}{(1)_r^2}\cdot{}_2F_1\left[\begin{matrix}x,1-x\\1\end{matrix};1\right]_p \pmod{p^2}. \tag{4.1}$$

Letting $k\to k+rp$ on the left-hand side of (4.1) and then applying Lemma 2.4,

$$\sum_{k=rp}^{(r+1)p-1}\frac{(x)_k(1-x)_k}{(1)_k^2}\equiv\frac{(x)_r(1-x)_r}{(1)_r^2}\sum_{k=0}^{p-1}\frac{(x)_k(1-x)_k}{(1)_k^2}$$
$$\times\left(1+2rpH_{\lfloor px\rfloor}-2rpH_k+rp\sum_{i=0}^{k-1}\left(\frac{1}{x+i}+\frac{1}{1-x+i}\right)\right) \pmod{p^2}. \tag{4.2}$$

Here, we apply the following identity [19, Theorem 1]:

$$\frac{(x)_k(1-x)_k}{(1)_k^2}\sum_{i=0}^{k-1}\left(\frac{1}{x+i}+\frac{1}{1-x+i}\right)=\sum_{i=0}^{k-1}\frac{(x)_i(1-x)_i}{(1)_i^2}\cdot\frac{1}{k-i},$$

which yields

$$\sum_{k=0}^{p-1}\frac{(x)_k(1-x)_k}{(1)_k^2}\sum_{i=0}^{k-1}\left(\frac{1}{x+i}+\frac{1}{1-x+i}\right)=\sum_{k=0}^{p-1}\sum_{i=0}^{k-1}\frac{(x)_i(1-x)_i}{(1)_i^2}\cdot\frac{1}{k-i}$$
$$=\sum_{i=0}^{p-2}\frac{(x)_i(1-x)_i}{(1)_i^2}\sum_{k=i+1}^{p-1}\frac{1}{k-i}\equiv\sum_{i=0}^{p-1}\frac{(x)_i(1-x)_i}{(1)_i^2}H_i \pmod{p}, \tag{4.3}$$

since $H_{p-1-i}\equiv H_i \pmod{p}$ and $(x)_i(1-x)_i\equiv 0 \pmod{p}$ for $i=p-1$.

It follows from (4.2) and (4.3) that

$$\sum_{k=rp}^{(r+1)p-1}\frac{(x)_k(1-x)_k}{(1)_k^2}$$
$$\equiv\frac{(x)_r(1-x)_r}{(1)_r^2}\left(\sum_{k=0}^{p-1}\frac{(x)_k(1-x)_k}{(1)_k^2}+rp\sum_{k=0}^{p-1}\frac{(x)_k(1-x)_k}{(1)_k^2}(2H_{\lfloor px\rfloor}-H_k)\right) \pmod{p^2}. \tag{4.4}$$

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By (2.16), (2.1) and (2.2),

$$\sum_{k=0}^{p-1}\frac{(x)_k(1-x)_k}{(1)_k^2}(2H_{\lfloor px\rfloor}-H_k)\equiv\sum_{k=0}^{p-1}(-1)^k\binom{\lfloor px\rfloor}{k}\binom{\lfloor px\rfloor+k}{k}(2H_{\lfloor px\rfloor}-H_k)\pmod p$$
$$=0. \tag{4.5}$$

Substituting (4.5) into (4.4) completes the proof of (4.1).

Taking the sum over r from 0 to $n-1$ on both sides of (4.1) gives

$${}_2F_1\left[\begin{matrix}x,1-x\\1\end{matrix};1\right]_{np}\equiv{}_2F_1\left[\begin{matrix}x,1-x\\1\end{matrix};1\right]_{p}\cdot{}_2F_1\left[\begin{matrix}x,1-x\\1\end{matrix};1\right]_{n}\pmod{p^2}. \tag{4.6}$$

Theorem 1.3 follows from (1.1) and (4.6).

5. Concluding remarks

Numerical calculation suggests that the supercongruence (1.2) cannot be extended to any p-adic integer x in the direction of Theorem 1.2.

Recently, Sun [17, Conjecture 5.4] made four challenging conjectures which extend Theorem 1.3 and some results proved by Zhi-Hong Sun [15] and Zhi-Wei Sun [16].

CONJECTURE 5.1 (Sun). Let $p\geq 5$ be a prime and n be a positive integer. Then

$$\frac{16^n}{n^2\binom{2n}{n}^2}\left(\sum_{k=0}^{np-1}\frac{\binom{2k}{k}^2}{16^k}-\left(\frac{-1}{p}\right)\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{16^k}\right)\equiv -4p^2E_{p-3}\pmod{p^3},$$

$$\frac{27^n}{n^2\binom{2n}{n}\binom{3n}{n}}\left(\sum_{k=0}^{np-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}-\left(\frac{-3}{p}\right)\sum_{k=0}^{n-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\right)\equiv -\frac{3}{2}p^2B_{p-2}\left(\frac13\right)\pmod{p^3},$$

$$\frac{64^n}{n^2\binom{2n}{n}\binom{4n}{2n}}\left(\sum_{k=0}^{np-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}-\left(\frac{-2}{p}\right)\sum_{k=0}^{n-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\right)\equiv -p^2E_{p-3}\left(\frac14\right)\pmod{p^3},$$

$$\frac{432^n}{n^2\binom{3n}{n}\binom{6n}{3n}}\left(\sum_{k=0}^{np-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}-\left(\frac{-1}{p}\right)\sum_{k=0}^{n-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}\right)\equiv -20p^2E_{p-3}\pmod{p^3},$$

where E_m is the mth Euler number and $E_m(x)$ and $B_m(x)$ denote the Euler polynomial and the Bernoulli polynomial of degree m, respectively.

Noting that

$$\frac{\binom{2k}{k}^2}{16^k}=\frac{(\frac12)_k^2}{(1)_k^2},\quad \frac{\binom{2k}{k}\binom{3k}{k}}{27^k}=\frac{(\frac13)_k(\frac23)_k}{(1)_k^2},$$
$$\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}=\frac{(\frac14)_k(\frac34)_k}{(1)_k^2},\quad \frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}=\frac{(\frac16)_k(\frac56)_k}{(1)_k^2},$$

we can directly deduce Theorem 1.3 from Conjecture 5.1. Unfortunately, the method in this paper is not applicable for proving Conjecture 5.1.

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