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DEGREES GIVING INDEPENDENT EDGES IN A HYPERGRAPH

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For r-partite and for r-uniform hypergraphs bounds are given for the minimum degree which ensures d independent edges.

1. Introduction and statement of results

(i) HYPERGRAPHS

Let c, r, s be positive integers with $2 \le r$ and let $S = \{1, 2, \ldots, s\}$. A set H of subsets of S is a hypergraph. The members of H are called edges. Two edges $\alpha, \beta \in H$ are independent if $\alpha \cap \beta = \emptyset$. The degree $\deg_H(x)$ of $x \in S$ in H is the number of members of H containing x. We write $\delta(H)$ for $\min\{\deg_H(x)\}$ over $x \in S$. Let B be the set of all $\alpha \subset S$ of cardinality $|\alpha| = r$. In this paper each $H \subset B$ so H is an r-graph or r-uniform hypergraph. We are concerned with the least number ω such that every H with $\omega < \delta(H)$ has more than d independent edges. Related problems are dealt with in the references.

(ii) r-PARTITE r-GRAPHS

Suppose S is a disjoint union $S = R_1 \cup \ldots \cup R_r$ with $|R_i| = c$ for $1 \le i \le r$ so $s = cr$. Let A be the set of all $\alpha \subset S$ such that $|\alpha \cap R_i| = 1$ for $1 \le i \le r$. In this case any $H \subset A \subset B$ is an r-partite hypergraph.

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THEOREM 1. *If* $0 \le d < c$ *and* H *is* r*-partite as above with*

$$\delta(H) > \{c^{r-1}-(c-d)^{r-1}\}(r-1)/r$$

then H *has more than* d *independent edges.*

To see how close this theorem gets to ω consider

EXAMPLE 1. Put $d = qr + p$ with $0 \le p < r$. For $1 \le i \le p$ select $q+1$ elements of R_i. For $p < i \le r$ select q elements of R_i. Let H consist of all $\alpha \in A$ which contain at least one of the d selected elements. Then $\delta(H)$ is approximately $c^{r-1} - (c-r^{-1}d)^{r-1}$ but H does not have $d+1$ independent edges.

(iii) GENERAL r-GRAPHS

EXAMPLE 2. Select d elements of S and let H consist of all $\alpha \in B$ which contain at least one of the selected elements. Then $\delta(H) = \binom{s-1}{r-1} - \binom{s-d-1}{r-1}$ but H does not have $d+1$ independent edges.

THEOREM 2 (Bollobás, Daykin and Erdös). *If* $0 \le d$ *and* $2r^3(d+2) < s$ *and*

$$\delta(H) > \binom{s-1}{r-1} - \binom{s-d-1}{r-1}$$

then H *has more than* d *independent edges.*

That this theorem has evaluated ω is shown by Example 2. It appears in [*1*] where it is in fact proved that all H with a fixed number of independent edges and high $\delta(H)$ are subhypergraphs of Example 2. In Theorem 2 it is required that s be large. Without this requirement we bound ω in

THEOREM 3. *If* r *divides* s *and*

$$\delta(H) > \left\{\binom{s-1}{r-1}-\binom{s-dr-1}{r-1}\right\}(r-1)/r$$

then H *has more than* d *independent edges.*

For Theorems 1 and 3 we prove slightly more than what is stated. Namely that if $C_1, \ldots, C_d$ is any maximum set of independent edges, and if E is any possible edge in $S \backslash \{C_1 \cup \ldots \cup C_d\}$ then E has low average degree. We believe the condition r divides s can be removed but were not able to do so.

2. Proof of Theorem 1

Part (i). Assume that $1 \le d < c$ and H has d independent edges $C_1, \ldots, C_d$ but not $d+1$. Choose arbitrarily members $C_{d+1}, \ldots, C_c$ of A so that S is the disjoint union $S = C_1 \cup \ldots \cup C_c$. We label the elements $x(i, j)$ of S so that

(1) $$C_j = \{x(1, j), \ldots, x(r, j)\} \text{ for } 1 \le j \le c ,$$

(2) $$R_i = \{x(i, 1), \ldots, x(i, c)\} \text{ for } 1 \le i \le r .$$

The reader will probably find it helpful to think of S as the elements of a matrix. Then c, C refer to columns and r, R to rows. We write D for the union of the d independent edges $D = C_1 \cup \ldots \cup C_d$ and E for C_c the end column in the matrix.

We will use the cyclic permutation σ on n distinct positive integers $w_1, \ldots, w_n$ defined by $\sigma w_n = w_1$ and $\sigma w_i = w_{i+1}$ otherwise. We proceed to partition A.

Part (ii). Given $\alpha = \{x(1, j_1), \ldots, x(r, j_r)\} \in A$ let $\{w_1, \ldots, w_n\} = \{j_1, \ldots, j_r\}$ with $1 \le w_1 < \ldots < w_n \le c$. Note that $n \le r$. Then put

(3) $$K(\alpha) = \left\{\left\{x\left(1, \sigma^e j_1\right), \ldots, x\left(r, \sigma^e j_r\right)\right\} : 1 \le e \le n\right\} .$$

We say that the members of $K(\alpha)$ are obtained by *rotating* α. The sets $K(\alpha)$ are the equivalence classes of our partition of A.

Part (iii). Let $X = \{\alpha : \alpha \in A, \alpha \cap D \ne \emptyset\}$. Then by definition of d we have $H \subset X$. Let $\mathcal{K}$ be the set of equivalence classes in the partition of A. If $K \in \mathcal{K}$ then either $K \subset X$ or $K \cap X = \emptyset$. For $L \subset A$ define

$$\Delta(L) = \sum (x \in E)\deg_L(x) .$$

Let $Y = \{\alpha : \alpha \in A, \alpha \cap E \ne \emptyset\}$. If $K \in \mathcal{K}$ then either $K \cap Y = \emptyset$ or

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$K \subset Y$ according as $0 = \Delta(K)$ or not. For all $L \subset A$ we have $\Delta(L) = \Delta(L \cap Y)$ and in particular $\Delta(H) = \Delta(H \cap X \cap Y)$.

Assume for the moment that

(4) $\quad r\Delta(H \cap K) \leq (r-1)\Delta(K)$ for all $K \in \mathcal{K}$ with $K \subset X \cap Y$.

Then we have

(5) $\quad r\Delta(H) = r \sum \Delta(H \cap K) \leq (r-1) \sum \Delta(K) = (r-1)\Delta(X \cap Y)$,

where summation is over $K \in \mathcal{K}$ with $K \subset X \cap Y$.

Part (iv). Clearly $\Delta(A) = rc^{r-1}$ and $\Delta(X \cap Y) = r\left(c^{r-1}-(c-d)^{r-1}\right)$. So the result follows by (5). It remains to prove (4).

Part (v). Suppose $K \in \mathcal{K}$ and $K \subset X \cap Y$. If $\alpha \in K$ then the other members of K are obtained by rotating α . Hence every $x \in E$ is in exactly one member of K and so $\Delta(K) = r$. If $k = |K|$ then K consists of k independent members of A . Again by the rotation $K \cap C_j \neq \emptyset$ for less than k of the j in $1 \leq j \leq d$. Therefore if $K \subset H$ we could remove these C_j from $C_1, \ldots, C_d$ and adjoin K to get more than d independent edges of H . Hence $K \not\subset H$ and so $\Delta(H \cap K) \leq r - 1$ and this proves (4).

3. Proof of Theorem 3

We use ideas from the last proof. In fact we have chosen our notation so that parts of the last proof carry over unchanged, provided A now means the set B of all $\alpha \subset S$ with $|\alpha| = r$. Do not be deceived. Although the writing is the same the meaning is different.

Part (i). As before. Note that before the R's were given but now they are defined by (2).

Part (ii). Given a row vector $v = \left(v(1), \ldots, v(c)\right)$ of non-negative integers $v(j)$ let

$$W = \{w_1, \ldots, w_n\} = \{j : 1 \leq j \leq c \text{ and } 0 < v(j)\} ,$$

with $1 \leq w_1 < \ldots < w_n$. Note that $n \leq c$. Now define a permutation π of $\{1, \ldots, c\}$ by $\pi j = \sigma j$ if $j \in W$ but $\pi j = j$ otherwise. Finally

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put

$$V = V(v) = \{(v(\pi^e 1), \ldots, v(\pi^e c)) : 1 \le e \le n\} .$$

For example if $v = (1, 0, 2, 1, 0, 0, 2)$ then $n = 4$ and $W = \{1, 3, 4, 7\}$ and V is v and $(2, 0, 1, 2, 0, 0, 1)$.

Given $\alpha \in A$ put $v(j) = |\alpha \cap C_j|$ for $1 \le j \le c$. In this way α yields a row vector v . In turn v yields a set V of row vectors as above. We use $V = V(\alpha)$ to define $K \subset A$ by

$$K = K(\alpha) = \{\beta : \beta \in A, \text{ row vector of } \beta \in V(\alpha)\} .$$

Clearly the set $\mathcal{K}$ of all sets $K(\alpha)$ over $\alpha \in A$ are the equivalence classes of a partition of A .

Part (iii). As before.

Part (iv). Clearly $\Delta(A) = r\binom{s-1}{r-1}$ and $\Delta(X \cap Y) = r\left\{\binom{s-1}{r-1} - \binom{s-dr-1}{r-1}\right\}$. So the result follows by (5). It remains to prove (4).

Part (v). Choose any $K \in \mathcal{K}$ with $K \subset X \cap Y$ and fix it. An ordering of C_j is a bijection $\lambda_j : C_j \to \{1, 2, \ldots, r\}$ and the number of these is $r!$. For $1 \le j \le c$ let λ_j be an ordering of C_j . We say that $\alpha \in K$ is *good* in $\lambda = (\lambda_1, \ldots, \lambda_c)$ if

$$\bigcup_{1 \le j \le c} \left\{ \bigcup_{x \in \alpha \cap C_j} \lambda_j(x) \right\} = \{1, 2, \ldots, r\} .$$

If we think of λ as reordering the columns of S as a matrix then α is good in λ if it has exactly one element in each row of the reordered S .

If $\alpha, \beta \in K$ then the numbers $|\alpha \cap C_j|$ are the same as the numbers $|\beta \cap C_j|$ in some order. Hence α and β are good in the same number t of the λ . For each λ let $F(\lambda)$ and $G(\lambda)$ be the set of all α in K and $H \cap K$ respectively which are good in λ . Then

$$\Delta(H \cap K) = t \sum \Delta(G(\lambda)) \quad \text{and} \quad \Delta(K) = t \sum \Delta(F(\lambda)) , \tag{6}$$

where summation is over λ . Assume for the moment that

$$r\Delta(G(\lambda)) \le (r-1)\Delta(F(\lambda)) \quad \text{for all } \lambda . \tag{7}$$

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Then (4) follows immediately using (6).

Part (vi). Choose any λ and fix it. For simplicity write F, G instead of $F(\lambda)$, $G(\lambda)$. After S has been reordered by λ we renumber the elements $x(i, j)$ of S so that (1) and (2) again hold. Given any $\alpha \in F$ we define the set $K(\alpha)$ exactly as in (3). To avoid confusion let $K(\alpha)$ be called J. Because the members of J are obtained by rotating α they are all in K. Also by construction they are all good in λ. In fact the various J partition F. Exactly as in Part (v) of the proof of the last theorem we find that $\Delta(J) = r$ and $\Delta(H \cap J) \le r - 1$. Hence

$$r\Delta(G) = r \sum \Delta(H \cap J) \le r \sum (r-1) = (r-1) \sum r = (r-1) \sum \Delta(J) = (r-1)\Delta(F) ,$$

where summation is over the equivalence classes J which partition F, and this proves (7).

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