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Klingen Eisenstein congruences and modularity

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Abstract. We construct a mod ℓ congruence between a Klingen Eisenstein series (associated with a classical newform ϕ of weight k) and a Siegel cusp form f with irreducible Galois representation. We use this congruence to show non-vanishing of the Bloch–Kato Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho_\phi(2-k)\otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ under certain assumptions and provide an example. We then prove an $R = dvr$ theorem for the Fontaine–Laffaille universal deformation ring of $\overline{\rho}_f$ under some assumptions, in particular, that the residual Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0 \overline{\rho}_\phi(k-2))$ is cyclic. For this, we prove a result about extensions of Fontaine–Laffaille modules. We end by formulating conditions for when $H^1_f(\mathbf{Q}, \mathrm{ad}^0 \overline{\rho}_\phi(k-2))$ is non-cyclic and the Eisenstein ideal is non-principal.

1 Introduction

The construction of Eisenstein congruences has a long and consequential history. Interesting in their own right, their significance is amplified by the existence of Galois representations attached to the congruent forms, as the ones attached to Eisenstein series are always reducible, while the ones attached to cusp forms are often irreducible. Using various generalizations of the result known as Ribet's Lemma, they lead to the construction of non-zero elements in Selmer groups. This direction was first explored by Ribet himself in the context of the group GL_2 in [45] and later used by many other authors in a variety of different settings (e.g., [16, 49, 60]).

In a different direction, such congruences can play a crucial role in proving modularity of deformations of reducible residual Galois representations $\overline{\rho}$ (see, e.g., [6, 9, 10, 17, 50, 54, 56]). In [17] Calegari introduced a method of proving modularity assuming $\overline{\rho}$ is unique up to isomorphism, which relies on proving the principality of the ideal of reducibility of the universal deformation ring R of $\overline{\rho}$. This method was developed further by Berger and Klosin [5, 6, 9] and Wake and Wang-Erickson [56] and successfully applied in many contexts (see also [1, 29]). It relies heavily on the ideas of Bellaiche and Chenevier [4] and their study of generalized matrix algebras (GMAs).

In this article, we pursue both of these directions in the case of Klingen Eisenstein series of level one on the group Sp_4. More precisely, let $k \geq 12$ be an even integer and ϕ be a classical weight k Hecke eigenform of level 1 (i.e., on the group $\mathrm{GL}_{2/\mathbf{Q}}$). Write $E^{2,1}_\phi$ for the (appropriately normalized) Klingen Eisenstein series on Sp_4 induced

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from ϕ. It is a Siegel modular form of weight k and full level. Congruences between Klingen Eisenstein series and cusp forms have been studied previously by Kurokawa [35, 36], Katsurada and Mizumoto [32, 39], Takeda [52], and Urban (unpublished). Katsurada and Mizumoto obtain congruences as an application of the doubling method. In this article, we produce congruences via a much shorter argument using results of Yamauchi [61]. The trade-off is that while our proof is much shorter, we obtain congruences only modulo a prime ℓ, whereas Katsurada and Mizumoto obtain congruences modulo powers of ℓ. However, the hypotheses required for our result are different and less restrictive than those needed in [32]. We show that under certain conditions $E^{2,1}_{\phi}$ is congruent to some cusp form f of the same weight and level with irreducible Galois representation (Theorem 3.5). This is the first main result of the article. These congruences are governed by the numerator of the (algebraic part) of the symmetric square L-function $L_{\mathrm{alg}}(2k-2, \mathrm{Sym}^2\phi)$ of ϕ. We also exhibit a concrete example when the assumptions of Theorem 3.5 are satisfied (see Example 3.6).

We then proceed to show that these congruences give rise (under some assumptions) to non-trivial elements in the Selmer group $H_{2-k} := H^1_f(\mathbf{Q}, \mathrm{ad}\,\rho_\phi(2-k) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$. Here, ρ_ϕ is the Galois representation attached to ϕ by Deligne and we use the Fontaine–Laffaille condition at ℓ. Assuming the Vandiver Conjecture for ℓ we also deduce the non-triviality of the Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho_\phi(2-k) \otimes \mathbf{Q}_\ell/\mathbf{Z}_\ell)$ (Corollary 5.7 and Remark 5.8). This is our second main result and gives evidence for new cases of the Bloch–Kato conjecture. This conjecture was studied for other twists of $\mathrm{ad}\,\rho_\phi$ by [20, 34]. In [53] Urban assumed the existence of Klingen Eisenstein congruences to prove a result toward the main conjecture of Iwasawa theory for the adjoint L-function.

To properly analyze these Selmer groups, we require some results on extensions of Fontaine–Laffaille modules whose proofs appear to be absent in the literature. In Section 4, we carefully study certain aspects of Fontaine–Laffaille theory, in particular, prove the Hom-tensor adjunction formula and give a precise definition of Selmer groups with coefficient rings of finite length.

Given the eigenvalue congruence $E^{2,1}_{\phi} \equiv f \pmod{\ell}$, we also study deformations of a non-semi-simple Galois representation $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_4(\overline{\mathbf{F}}_\ell)$ whose semi-simplification arises from the Klingen Eisenstein series. Such a representation is reducible with two two-dimensional Jordan–Holder blocks and more precisely, one has

$$\overline{\rho} = \begin{bmatrix} \overline{\rho}_\phi & * \\ & \overline{\rho}_\phi(k-2) \end{bmatrix}.$$

Conjecturally such representations should arise as mod ℓ reductions of Galois representations attached to Siegel cusp forms which are congruent to $E^{2,1}_{\phi}$ mod ℓ. We assume that $\dim H_{2-k}[\ell] = 1$, where $[\ell]$ indicates ℓ-torsion. This can be seen as a refinement of the uniqueness assumption of [50] similar to the one in [6] and as in [6, 17] we prove the principality of the reducibility ideal of the universal deformation. However, this principality cannot be achieved through the method of [6] because the representation in question fails to satisfy the strong self-duality property required for the method of [loc.cit.]. Instead we improve on a recent result of Akers [1] which replaces the self-duality condition with a one-dimensionality assumption on the Selmer group $H_{k-2} :=$

$H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2))$ of the "opposite" Tate twist of $\mathrm{ad}\,\rho_\phi$. With these assumptions in place, we are able to show that the universal deformation ring R is a discrete valuation ring and prove a modularity result guaranteeing that the unique deformation of $\overline{\rho}$ indeed arises from a Siegel cusp form congruent to $E^{2,1}_\phi$ (Theorem 6.20). This is the third main result of the article.

We then proceed to formulate conditions for non-cyclicity of the Selmer group H_{k-2}. While many results in the literature give bounds on the orders of Selmer groups (in particular, Corollary 5.7 gives such a lower bound on H_{2-k}), the structure of these groups is notoriously mysterious. In this article, we prove that if the (local) Klingen Eisenstein ideal $J_\mathfrak{m}$ is not principal then H_{k-2} is not cyclic (Corollary 7.3). We further refine this result by providing a criterion for non-principality in terms of the depth of congruences between cusp forms and $E^{2,1}_\phi$ (Corollary 7.5). An intriguing feature of these results is that H_{k-2} is non-critical, i.e., this Selmer group is not controlled by a critical L-value in the sense of Deligne.

2 Background and notation

Given a field F, we denote by G_F its absolute Galois group. Fix a rational prime $\ell > 2$. If M is a topological $\mathbf{Z}_\ell[G_F]$-module, we will write $M(n) = M \otimes \varepsilon^n$ for the n-th Tate twist where ε denotes the ℓ-adic cyclotomic character.

For each prime p, we fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. This is equivalent to choosing a prime $\overline{p}$ of $\overline{\mathbf{Q}}$ lying over p and fixes an isomorphism $D_p \cong G_{\mathbf{Q}_p}$, where D_p is the decomposition group of $\overline{p}$. We will denote by $I_p \subset D_p$ the corresponding inertia group. We also fix an isomorphism $\overline{\mathbf{Q}}_\ell \cong \mathbf{C}$.

Let E denote a finite extension of $\mathbf{Q}_\ell$ with valuation ring $\mathcal{O}$, uniformizer λ, and residue field $\mathbf{F}$. For a continuous homomorphism $\rho : G_F \to \mathrm{GL}_n(\mathcal{O})$, we write $\overline{\rho} : G_F \to \mathrm{GL}_n(\mathbf{F})$ for the mod λ reduction of ρ.

For $n \in \mathbf{Z}_+$, we denote by Mat_n (resp., GL_n) the affine group scheme over $\mathbf{Z}$ of $n \times n$ (resp., invertible) matrices. Given a matrix $\gamma \in \mathrm{Mat}_{2n}$, we will write it as $\gamma = \begin{bmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{bmatrix}$, where the blocks are in Mat_n. Set $\mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} : {}^t g J_n g = \mu_n(g) J_n, \mu_n(g) \in \mathrm{GL}_1\}$, where $J_n = \begin{bmatrix} 0_n & -1_n \\ 1_n & 0_n \end{bmatrix}$, where 1_n is the n by n identity matrix, and $\mu_n : \mathrm{GL}_{2n} \to \mathrm{GL}_1$ is the homomorphism defined via the equation given in the definition. Write $\mathrm{GSp}^+_{2n}(\mathbf{R})$ for the subgroup of $\mathrm{GSp}_{2n}(\mathbf{R})$ consisting of elements g with $\mu_n(g) > 0$. We set $\mathrm{Sp}_{2n} = \ker(\mu_n)$ and

$$\Gamma_n = \mathrm{Sp}_{2n}(\mathbf{Z}) = \left\{ g \in \mathrm{GL}_{2n}(\mathbf{Z}) : {}^t g J_n g = J_n \right\}.$$

Note that $\mathrm{Sp}_2 = \mathrm{SL}_2$, the subgroup scheme of GL_2 of matrices of determinant one.

The Siegel upper half-space is given by

$$\mathfrak{h}_n = \{ z = x + iy \in \mathrm{Mat}_n(\mathbf{C}) : x, y \in \mathrm{Mat}_n(\mathbf{R}),\ {}^t z = z, y > 0\},$$

where we write $y > 0$ to indicate that y is positive definite. The group $\mathrm{GSp}^+_{2n}(\mathbf{R})$ acts on $\mathfrak{h}_n$ via $\gamma z = (a_\gamma z + b_\gamma)(c_\gamma z + d_\gamma)^{-1}$.

For a function $f : \mathfrak{h}_n \to \mathbf{C}$ set $(f|_\kappa \gamma)(z) = \mu_n(\gamma)^{nk/2} j(\gamma, z)^{-k} f(\gamma z)$ for $\gamma \in \mathrm{GSp}_{2n}^+(\mathbf{R})$ and $z \in \mathfrak{h}_n$, where $j(\gamma, z) = \det(c_\gamma z + d_\gamma)$. A Siegel modular form of weight k and level Γ_n is a holomorphic function $f : \mathfrak{h}_n \to \mathbf{C}$ satisfying $(f|_k \gamma)(z) = f(z)$ for all $\gamma \in \Gamma_n$. If $n = 1$, we also require the standard growth condition at the cusp. We denote the $\mathbf{C}$-vector space of Siegel modular forms of weight k and level Γ_n as $M_k(\Gamma_n)$. Any $f \in M_k(\Gamma_n)$ has a Fourier expansion of the form

$$f(z) = \sum_{T \in \Lambda_n} a(T; f) e(\mathrm{Tr}(Tz)),$$

where Λ_n is defined to be the set of n by n half-integral (diagonal entries are in $\mathbf{Z}$, off diagonal are allowed to lie in $\frac{1}{2}\mathbf{Z}$) positive semi-definite symmetric matrices and $e(w) := e^{2\pi i w}$. Given a ring $A \subset \mathbf{C}$, we write $f \in M_k(\Gamma_n; A)$ if $a(T; f) \in A$ for all $T \in \Lambda_n$. Define the subspace $S_k(\Gamma_n) = \ker \Phi \subset M_k(\Gamma_n)$ of *cusp forms*, where $\Phi(f)(z) = \lim_{t \to \infty} f\left(\begin{bmatrix} z & 0 \\ 0 & it \end{bmatrix}\right)$.

We will now introduce certain Eisenstein series, which will play a prominent role in this article. For $n \geq 1$ and $0 \leq r \leq n$ define the parabolic subgroup

$$P_{n,r} = \left\{ \begin{bmatrix} a_1 & 0 & b_1 & * \\ * & u & * & * \\ c_1 & 0 & d_1 & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{bmatrix} \in \Gamma_n : \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in \Gamma_r, u \in \mathrm{GL}_{n-r}(\mathbf{Z}) \right\}.$$

We define projections $\star : \mathfrak{h}_n \to \mathfrak{h}_r$, $z = \begin{bmatrix} z^\star & * \\ * & * \end{bmatrix} \mapsto z^\star$ and $\star : P_{n,r} \to \Gamma_r$, $\gamma \mapsto \gamma^\star = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$.

Let $\phi \in S_k(\Gamma_1)$. The Klingen Eisenstein series attached to ϕ is the series

$$E_\phi^{2,1}(z) = \sum_{\gamma \in P_{2,1} \backslash \Gamma_2} \phi((\gamma z)^\star) j(\gamma, z)^{-k},$$

where $z \in \mathfrak{h}_2$. The Eisenstein series converges for $k \geq 12$ (see [33, Theorem 1, p. 67] for example). Note that [33, Theorem 1, p. 67] gives $\Phi(E_\phi^{2,1}) = \phi$.

Given two Siegel modular forms $f_1, f_2 \in M_k(\Gamma_n)$ with at least one a cusp form, set

$$\langle f_1, f_2 \rangle = \int_{\Gamma_n \backslash \mathfrak{h}_n} f_1(z) \overline{f_2(z)} (\det y)^k d\mu z,$$

where $z = x + iy$ with $x = (x_{\alpha,\beta})$, $y = (y_{\alpha,\beta}) \in \mathrm{Mat}_n(\mathbf{R})$, $d\mu z = (\det y)^{-(n+1)} \prod_{\alpha \leq \beta} dx_{\alpha,\beta} \prod_{\alpha \leq \beta} dy_{\alpha,\beta}$ with $dx_{\alpha,\beta}$ and $dy_{\alpha,\beta}$ the usual Lebesgue measure on $\mathbf{R}$.

Given $\gamma \in \mathrm{GSp}_{2n}^+(\mathbf{Q})$, we write $T(\gamma)$ to denote the double coset $\Gamma_n \gamma \Gamma_n$ and set $T(\gamma) f = \sum_i f|_k \gamma_i$, where the γ_i are given by the finite decomposition $\Gamma_n \gamma \Gamma_n = \coprod_i \Gamma_n \gamma_i$ and $f \in M_k(\Gamma_n)$. Let $m > 1$. We define $T^{(n)}(m)$ via

$$T^{(n)}(m) = \sum_{\substack{d_1 e_1 = \cdots = e_n d_n = m \\ d_1 | d_2 | \cdots | d_n | e_n | e_{n-1} | \cdots | e_1}} T(\mathrm{diag}(d_1, \ldots, d_n, e_1, \ldots, e_n)).$$

In particular, for p a prime, we have

$$T^{(n)}(p) = T(\mathrm{diag}(1_n, p1_n)).$$

We also define

$$T_i^{(n)}(p^2) = T(\mathrm{diag}(1_{n-i}, p1_i, p^2 1_{n-i}, p1_i)), \quad 1 \le i \le n.$$

The spaces $M_k(\Gamma_n)$ and $S_k(\Gamma_n)$ are both stable under the action of $T^{(n)}(p)$ and $T_i^{(n)}(p^2)$ for $1 \le i \le n$ and all p. We say a nonzero $f \in M_k(\Gamma_n)$ is an eigenform if it is an eigenvector of $T^{(n)}(p)$ and $T_i^{(n)}(p^2)$ for all p and all $1 \le i \le n$. As we will be focused on the case $n = 2$, we specialize to that case. We let $\mathbf{T}'$ denote the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{C}}(S_k(\Gamma_2))$ generated by the Hecke operators $T^{(2)}(p)$ and $T_1^{(2)}(p^2)$ for all primes p.

Recall that $E/\mathbf{Q}_\ell$ denotes a finite extension with valuation ring $\mathcal{O}$ and uniformizer λ. Given eigenforms $f_1, f_2 \in M_k(\Gamma_n; \mathcal{O})$, following the notation in [61] we write $f_1 \equiv_{\mathrm{ev}} f_2 \pmod{\lambda}$ if $\lambda_{f_1}(T) \equiv \lambda_{f_2}(T) \pmod{\lambda}$ for all $T \in \mathbf{T}'$, where $Tf_i = \lambda_{f_i}(T) f_i$.

For an eigenform $\phi \in S_k(\Gamma_1)$, we set

$$L(s, \phi) := \prod_p (1 - \lambda_\phi(p) p^{-s} + p^{k-1-2s})^{-1},$$

$$L(s, Sym^2\phi) = \prod_p \left[(1 - \alpha_p^2 p^{-s})(1 - \alpha_p \beta_p p^{-s})(1 - \beta_p^2 p^{-s})\right]^{-1},$$

where $\lambda_\phi(p)$ is the eigenvalue of $T(p) := T^{(1)}(p)$ corresponding to ϕ and α_p, β_p denote the roots of $X^2 - \lambda_\phi(p) X + p^{k-1}$. The symmetric square L-function converges in the right half-plane $\Re(s) > k$, satisfies a functional equation, and has analytic continuation to the entire complex plane.

For an eigenform $f \in S_k(\Gamma_2)$, we define

$$\begin{aligned} L_p(X, f, \mathrm{spin}) = (1 - \lambda_f(p) X + (\lambda_f(p)^2 - \lambda_f(p^2) - p^{2k-4}) X^2 \\ - \lambda_f(p) p^{2k-3} X^3 + p^{4k-6} X^4), \end{aligned}$$

where we write $\lambda_f(p)$ is the eigenvalue of $T^{(2)}(p)$ corresponding to f and $\lambda_f(p^2)$ for the eigenvalue $T^{(2)}(p^2)$ corresponding to f.

Theorem 2.1 [59, Theorem 1] *Let $f \in S_k(\Gamma_2)$ be an eigenform. For a sufficiently large finite extension $F/\mathbf{Q}_\ell$, one has $L_p(X, f, \mathrm{spin}) \in F[X]$ for all primes $p \ne \ell$ and there is a semisimple continuous representation $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_4(F)$, which is unramified outside of ℓ so that for $p \ne \ell$, one has $L_p(X, f; \mathrm{spin}) = \det(1 - \rho_f(\mathrm{Frob}_p) X)$.*

3 Congruence

We keep the notation of Section 2. Throughout this section, we fix an even weight $k \ge 12$ and an odd prime ℓ and make the following assumption.

Assumption 3.1 *Given an even weight $k \ge 12$ and prime ℓ, assume that $E/\mathbf{Q}_\ell$ is sufficiently large to contain the fields F from Theorem 2.1 for all forms $f \in S_k(\Gamma_2)$. We also assume that for every eigenform $\phi \in S_k(\Gamma_1)$, the field E contains all the Hecke eigenvalues* *of ϕ as well as the value $L_{\mathrm{alg}}(2k-2,\mathrm{Sym}^2\phi)$ (see (3.1) for the definition). In addition, we suppose that E contains a primitive cube root of unity.*

Recall that we denote the valuation ring of E by $\mathcal{O}$. Let $\phi \in S_k(\Gamma_1)$ be a normalized eigenform and consider the Klingen Eisenstein series $E_\phi^{2,1}$. In this section, we show under certain conditions that $E_\phi^{2,1}$ is eigenvalue-congruent to a cuspidal Siegel modular form with irreducible Galois representation.

Write

$$E_\phi^{2,1}(z) = \sum_{T\in\Lambda_2} a(T;E_\phi^{2,1})e(\mathrm{Tr}(Tz)).$$

For T that are singular, i.e., $\det T = 0$, one has T is unimodularly equivalent to $\begin{bmatrix} n & 0 \\ 0 & 0\end{bmatrix}$ for some $n \in \mathbf{Z}_{\geq 0}$. For such T, one has $a(T;E_\phi^{2,1}) = a(n;\phi)$, where $\phi(z) = \sum_{n>0} a(n;\phi)e(nz)$.

We use the following result to prove our congruence.

Corollary 3.2 [61, Corollary 2.3] *Assume $\ell \geq 7$. Let g be a Hecke eigenform in $M_k(\Gamma_2;\mathcal{O})$ with Fourier expansion $g(z) = \sum_{T\in\Lambda_2} a(T;g)e(\mathrm{Tr}(Tz))$. Assume that $\lambda \mid a(T;g)$ for all T with $\det T = 0$ and that there exists at least one $T > 0$ with $a(T;g) \in \mathcal{O}^\times$. Then, there exists a Hecke eigenform $f \in S_k(\Gamma_2;\mathcal{O})$ so that $g \equiv_{\mathrm{ev}} f \not\equiv_{\mathrm{ev}} 0 \pmod{\lambda}$.*

For $T = \begin{bmatrix} m & r/2 \\ r/2 & n\end{bmatrix}$, we say T is primitive if $\gcd(m,n,r) = 1$. We set $\det(2T) = \Delta(T)\mathfrak{f}^2$ for a positive integer $\mathfrak{f}$ and where $-\Delta(T)$ is the discriminant of the quadratic field $\mathbf{Q}(\sqrt{-\det(2T)})$. We set $\chi_T = \left(\frac{-\Delta(T)}{\cdot}\right)$, the quadratic character associated with the field $\mathbf{Q}(\sqrt{-\det(2T)})$.

Define $\vartheta_T(z) = \sum_{a,b\in\mathbf{Z}^2} e(z(ma^2+rab+nb^2)) = \sum_{n\geq 0} b(n;\vartheta_T)e(nz)$. Given $\nu \in \mathbf{Z}_{\geq 1}$, set

$$\vartheta_T^{(\nu)}(z) = \sum_{n\geq 0} b(\nu^2 n;\vartheta_T)e(nz).$$

One can check that $\vartheta_T^{(\nu)} \in M_1(\Gamma(4\det T))$, where $\Gamma(N) = \ker(\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}))$ and $M_k(\Gamma(N))$ denotes the modular forms of weight k and level $\Gamma(N)$. Set

$$D(s,\phi,\vartheta_T^{(\nu)}) = \sum_{n\geq 1} a(n;\phi)b(\nu^2 n;\vartheta_T)n^{-s}.$$

We have that $D(s,\phi,\vartheta_T^{(\nu)})$ converges in a right half-plane with meromorphic continuation to the entire complex plane [47]. Set

$$L_{\mathrm{alg}}(2k-2,\mathrm{Sym}^2\phi) := \frac{L(2k-2,\mathrm{Sym}^2\phi)}{\pi^{3k-3}\langle\phi,\phi\rangle}, \tag{3.1}$$

$$L_{\mathrm{alg}}(k-1,\chi_T) = \frac{\Delta(T)^{k-3/2}L(k-1,\chi_T)}{\pi^{k-1}},$$

and

$$D_{\mathrm{alg}}(k-1,\phi,\vartheta_T^{(\nu)}) = \frac{D(k-1,\phi,\vartheta_T^{(\nu)})}{\pi^{k-1}\langle\phi,\phi\rangle}.$$

We have each of these terms is algebraic (see [47, 51, 62]. Moreover, we have via [62, Equation (22)] that if $\ell > k-1$, then $L_{\mathrm{alg}}(k-1,\chi_T)$ is ℓ-integral.

Theorem 3.3 [38] *Let $\phi \in S_k(\Gamma_1)$ be a normalized eigenform with a Fourier expansion as above. Let $T > 0$ be primitive. We have*

$$a(T;E_\phi^{2,1}) = (-1)^{k/2}\frac{(k-1)!}{(2k-2)!}2^{k-1}\frac{L_{\mathrm{alg}}(k-1,\chi_T)}{L_{\mathrm{alg}}(2k-2,\mathrm{Sym}^2\phi)} \cdot \sum_{\substack{m|\mathfrak{f}\\ m>0}} M_T(\mathfrak{f}m^{-1})\sum_{\substack{t|m\\ t>0}}\mu(t)D_{\mathrm{alg}}(k-1,\phi,\vartheta_T^{(m/t)}),$$

where

$$M_T(a) = \sum_{\substack{d|a\\ d>0}}\mu(d)\chi_T(d)d^{k-2}\sigma_{2k-3}(ad^{-1}) \text{ and } \sigma_s(d) = \sum_{\substack{g|d\\ g>0}} g^s.$$

Note that while this theorem is only stated for Fourier coefficients indexed by primitive T, we have that Fourier coefficients indexed by non-primitive T are an integral linear combination of Fourier coefficients indexed by primitive T by [38, Equation (1.3)] so we only need to consider the primitive T to guarantee the hypotheses of Corollary 3.2 are satisfied.

Lemma 3.4 *Assume $\ell > 4k-7$. Let $f \in S_k(\Gamma_2;\mathcal{O})$ be an eigenform. If there exists a normalized eigenform $\phi \in S_k(\Gamma_1;\mathcal{O})$ so that $f \equiv_{\mathrm{ev}} E_\phi^{2,1} \pmod{\lambda}$ and that $\overline{\rho}_\phi$ is irreducible, then ρ_f is irreducible.*

Proof We know via [59] that if ρ_f is reducible, then the automorphic representation associated with f is either CAP or a weak endoscopic lift. Moreover, by [42, Corollary 4.5] since $f \in S_k(\Gamma_2)$ and $k > 2$, the automorphic representation attached to f can be CAP only with respect to the Siegel parabolic, i.e., f is a classical Saito–Kurokawa lift. Suppose that f is a Saito–Kurokawa lift of $\psi \in S_{2k-2}(\Gamma_1)$. Then, we have $\overline{\rho}_f^{\mathrm{ss}} = \overline{\rho}_\psi \oplus \overline{\varepsilon}^{k-1} \oplus \overline{\varepsilon}^{k-2}$. Using the fact that $f \equiv_{\mathrm{ev}} E_\phi^{2,1} \pmod{\lambda}$ and that the eigenvalues of $E_\phi^{2,1}$ are given by $\lambda(p;E_\phi^{2,1}) = a(p;\phi) + p^{k-2}a(p;\phi)$, the Brauer–Nesbitt and Chebotarev Theorems give that $\overline{\rho}_f^{\mathrm{ss}} = \overline{\rho}_\phi \oplus \overline{\rho}_\phi(k-2)$, where recall that we write $\overline{\rho}_\phi(k-2)$ for $\overline{\rho}_\phi \otimes \overline{\varepsilon}^{k-2}$. This is a contradiction if $\overline{\rho}_\phi$ is irreducible. Thus, f cannot be a Saito–Kurokawa lift. It remains to show that the automorphic representation associated with f is not a weak endoscopic lift. The possible decompositions of ρ_f are given in [48, Theorem 3.2.1] under the assumption that $\ell > 4k-7$. Of these, the only case remaining to check is Case B(v), which states if $\rho_f = \sigma \oplus \sigma'$ with σ and σ' both two-dimensional, then $\det(\sigma) = \det(\sigma')$. In our case, this would require $\det(\rho_\phi) = \det(\rho_\phi(k-2))$, i.e., $\overline{\varepsilon}^{k-1} = \overline{\varepsilon}^{2k-3}$, which is impossible by our assumption that $\ell > 4k-7$. Thus, ρ_f is irreducible. ∎

Theorem 3.5 *Assume that $\ell > 4k-7$. Let $\phi \in S_k(\Gamma_1;\mathcal{O})$ be a normalized eigenform. Suppose that $\lambda \mid L_{\mathrm{alg}}(2k-2,\mathrm{Sym}^2\phi)$. Furthermore, assume there exists $T_0 > 0$ so that*

$$\mathrm{val}_\lambda\left(L_{\mathrm{alg}}(2k-2,\mathrm{Sym}^2\phi)a(T_0,E_\phi^{2,1})\right) \leq 0.$$

Then, there exists an eigenform $f \in S_k(\Gamma_2;\mathcal{O})$ so that

$$E_\phi^{2,1} \equiv_{\mathrm{ev}} f \pmod{\lambda}.$$

If in addition $\overline{\rho}_\phi$ is irreducible, then ρ_f is irreducible.

Proof Set $H_\phi^{2,1}(z) = L_{\mathrm{alg}}(2k-2,\mathrm{Sym}^2\phi)E_\phi^{2,1}(z)$. For $T \geq 0$, define $c(T) = \mathrm{val}_\lambda(a(T;H_\phi^{2,1}))$. Let $c = \min_{T\geq 0} c(T)$. Since $H_\phi^{2,1} \in M_k(\Gamma_2)$, the Fourier coefficients $a(T;H_\phi^{2,1})$ have bounded denominators so c is well-defined [46]. Moreover, our assumption that there is a $T_0 > 0$ with $\mathrm{val}_\lambda(a(T_0;H_\phi^{2,1})) = \mathrm{val}_\lambda\left(L_{\mathrm{alg}}(2k-2,\mathrm{Sym}^2\phi)a(T_0,E_\phi^{2,1})\right) \leq 0$ gives that $c \leq 0$. Set

$$G_\phi^{2,1}(z) = \lambda^{-c}H_\phi^{2,1}(z).$$

We have $a(T;G_\phi^{2,1}) \in \mathcal{O}$ for all $T \geq 0$ since $c(T) - c \geq 0$ for all $T \geq 0$. Observe that for T with $\det T = 0$, we have $a(T;G_\phi^{2,1}) = \lambda^{-c}L_{\mathrm{alg}}(2k-2,\mathrm{Sym}^2\phi)a(n;\phi)$ for some $n \in \mathbf{Z}_{\geq 0}$. Since $a(n;\phi) \in \mathcal{O}$ by assumption and $-c \geq 0$, this gives $\lambda \mid a(T;G_\phi^{2,1})$ for all T with $\det T = 0$, i.e., all the Fourier coefficients indexed by singular T vanish modulo λ. Moreover, since $c = c(\widetilde{T})$ for some $\widetilde{T}$, we have $a(\widetilde{T};G_\phi^{2,1}) \in \mathcal{O}^\times$ for some $\widetilde{T}$. Since $c \leq 0$ and $\lambda \mid a(T;G_\phi^{2,1})$ for all singular T, we have $\widetilde{T} > 0$. Thus, Corollary 3.2 and the fact that $G_\phi^{2,1}$ and $E_\phi^{2,1}$ have the same eigenvalues gives an eigenform $f \in S_k(\Gamma_2;\mathcal{O})$ so that $E_\phi^{2,1} \equiv_{\mathrm{ev}} f \not\equiv 0 \pmod{\lambda}$. By Lemma 3.4, we get that ρ_f is irreducible. ■

Example 3.6 Consider the space $M_{26}(\Gamma_2)$. This space has dimension seven and is spanned by $E^{2,0}$ (Siegel Eisenstein series), $E_\phi^{2,1}$ (Klingen Eisenstein series), three Saito–Kurokawa lifts, and two non-lift forms Υ_1 and Υ_2, where here $\phi \in S_{26}(\Gamma_1)$ is the unique newform given by

$$\phi(z) = e(z) - 48e(2z) - 195804e(3z) + \cdots.$$

We have via [21] that

$$\begin{aligned} &L_{\mathrm{alg}}(50,\mathrm{Sym}^2\phi) \\ &\quad = \frac{2^{41}\cdot 163\cdot 187273}{3^{26}\cdot 5^{10}\cdot 7^7\cdot 11^4\cdot 13^2\cdot 17^2\cdot 19\cdot 23^2\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 657931}.\end{aligned}$$

We consider $\ell \in \{163, 187273\}$ and show that both primes produce an example for Theorem 3.5.

The Klingen Eisenstein series associated with ϕ is given in the beta version of LMFDB. By considering the Fourier coefficients indexed by $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}$, one can see that the Klingen Eisenstein series given there, say E_ϕ^{LMFDB}, is given by

$$E_\phi^{2,1}(z) = -\frac{E_\phi^{\mathrm{LMFDB}}(z)}{2^6 \cdot 3^3 \cdot 11 \cdot 19 \cdot 163 \cdot 187273}.$$

We have from LMFDB that

$$a\left(\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}; E_\phi^{2,1}\right) = \frac{2^2 \cdot 5 \cdot 43}{11 \cdot 19 \cdot 163 \cdot 187273}.$$

Consider $G_\phi^{2,1}(z) = L_{\mathrm{alg}}(50, \mathrm{Sym}^2 \phi) E_\phi^{2,1}(z)$. We have for ℓ as above that $\ell \mid a(T; G_\phi^{2,1})$ for all T with $\det T = 0$ and $a\left(\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}; G_\phi^{2,1}\right) \not\equiv 0 \pmod{\ell}$. Thus, by Theorem 3.5, there exists a non-trivial Hecke eigenform $f \in S_k(\Gamma_2; \mathbf{Z}_\ell)$ with $E_\phi^{2,1} \equiv_{\mathrm{ev}} f \pmod{\ell}$.

Consider first the prime $\ell = 163$ and suppose that $\overline{\rho}_{\phi,163}^{\mathrm{ss}} = \psi_1 \oplus \psi_2$ for some characters ψ_1, ψ_2. Since $\overline{\rho}_\phi$ is unramified for all $p \neq \ell$, we see that ψ_1 and ψ_2 are each an integer power of $\overline{\varepsilon}$ (see the proof of Lemma 5.3). As $163 \nmid a(163; \phi)$, we know ϕ is ordinary at 163 and we get $\overline{\rho}_{\phi,163}^{\mathrm{ss}} = \overline{\varepsilon}^{25} \oplus 1$. By [45, Proposition 2.1] we can find a lattice such that

$$\overline{\rho}_{\phi,163} = \begin{bmatrix} 1 & * \\ 0 & \overline{\varepsilon}^{25} \end{bmatrix} \not\cong 1 \oplus \overline{\varepsilon}^{25}.$$

One can use ordinarity of ϕ to show that $*$ gives an unramified 163-extension of $\mathbf{Q}(\zeta_{163})$ (see, e.g., the proof of Theorem 4.28 in [10]). By Herbrand's Theorem, this implies that $163 \mid B_{26}$. However, one can check this is not true, so we must have that $\overline{\rho}_{\phi,163}$ is irreducible and so $E_\phi^{2,1}$ must be congruent (modulo 163) to a cusp form f that is not a Saito–Kurokawa lift, i.e., ρ_f is irreducible by Theorem 3.5. One uses LMFDB to check that $f = \Upsilon_2$.

Now consider the case that $\ell = 187273$. In this case, it is less practical to calculate $a(187273; \phi)$, so we directly eliminate the possibility that $E_\phi^{2,1}$ is congruent to a Saito–Kurokawa lift modulo 187273. The space to consider is $S_{50}(\Gamma_1)$. This space has one Galois conjugacy class of newforms consisting of three newforms, call them ψ_1, ψ_2, and ψ_3. Each newform has a field of definition K_{ψ_i} generated by a root α_i of

$$c(x) = x^3 + 24225168x^2 - 566746931810304x - 13634883228742736412672.$$

One has that $\lambda(2, E_\phi^{2,1}) = -805306416$ and that $\lambda(2, \psi_i) = 2^{49} + 2^{48} + \alpha_i$. One uses SAGE to check that $\lambda(2, E_\phi^{2,1}) \not\equiv \lambda(2, \psi_i) \pmod{187273}$, so $E_\phi^{2,1}$ must be congruent to a cusp form that is not a Saito–Kurokawa lift. One uses LMFDB to see that $E_\phi^{2,1} \equiv_{\mathrm{ev}} \Upsilon_1 \pmod{187273}$.

4 Extensions of Fontaine–Laffaille modules

In this section, we gather various facts (in particular, Propositions 4.8 and 4.20) about extensions of Fontaine–Laffaille modules, which we use in this article but which to the best of our knowledge have not been published elsewhere.

4.1 Definitions

We keep our assumption that ℓ is an odd prime. We fix integers a, b such that $0 \leq b - a \leq \ell - 2$. In this section, let E be an arbitrary finite extension of $\mathbf{Q}_\ell$ with ring of integers $\mathcal{O}$, uniformizer λ, and residue field $\mathbf{F}$. Write $\mathrm{LCA}_\mathcal{O}$ (respectively, $\mathrm{LCN}_\mathcal{O}$) for the category of local complete Artinian (respectively, Noetherian) $\mathcal{O}$-algebras with residue field $\mathbf{F}$. For a category $\mathcal{C}$, we will write $X \in \mathcal{C}$ to mean that X is an object of $\mathcal{C}$.

Definition 4.1 [31, Definition 2.3]/[13, Definition 4.1]

1. A Fontaine–Laffaille module is a finitely generated $\mathbf{Z}_\ell$-module M together with a decreasing filtration by $\mathbf{Z}_\ell$-module direct summands M^i for $i \in \mathbf{Z}$ such that there exists $k \leq l$ with $M^i = M$ for $i \leq k$ and $M^{i+1} = 0$ for $i \geq l$, and a collection of $\mathbf{Z}_\ell$-linear maps $\phi^i_M : M^i \to M$ such that $\phi^i_M|_{M^{i+1}} = \ell \phi^{i+1}_M$ for all i and $M = \sum_i \phi^i_M(M^i)$. The category of all Fontaine–Laffaille modules is denoted $MF^f_{\mathbf{Z}_\ell}$. Morphisms in this category are $\mathbf{Z}_\ell$-linear maps $f : M \to N$ satisfying $f(M^i) \subset N^i$ and $f \circ \phi^i_M = \phi^i_N \circ f|_{M^i}$ for all i. We will write $MF^f_{\mathrm{tor},\mathbf{Z}_\ell}$ for the full subcategory whose objects are of finite length as $\mathbf{Z}_\ell$-modules.
2. For a fixed interval $[k, l]$, we denote the full subcategory of $MF^f_{?,\mathbf{Z}_\ell}$ whose objects M have a filtration satisfying $M^k = M$ and $M^{l+1} = 0$ by $MF^{f,[k,l]}_{?,\mathbf{Z}_\ell}$ for $? \in \{\varnothing, \mathrm{tor}\}$.
3. For any $A \in \mathrm{LCA}_\mathcal{O}$, a Fontaine–Laffaille module over A consists of an object $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$ together with a map $\theta : A \to \mathrm{End}_{MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}}(M)$ that makes M into a free finitely generated module over A in such a way that M^i is an A-direct summand of M for each i. A morphism between two such objects is required to additionally preserve the A-structure. We will denote this category of Fontaine–Laffaille modules over A as $MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$.
4. For $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$, any integer i for which $M^i/M^{i+1} \neq 0$ is called a Fontaine–Laffaille weight for M. The set of Fontaine–Laffaille weights for M will be denoted by $\mathrm{FL}(M)$.

Remark 4.2 We impose the stronger restriction on the length of the filtration as in [12, Section 4] and [18, Section 2.4.1] compared to that in Section 1.1.2 of [20] or [31, Definition 2.3] (which allow the length to be $\ell - 1$).

Definition 4.3 We introduce the following examples of Fontaine–Laffaille modules:

1. If $0 \in [a, b]$, we write $\mathbf{1} \in MF^{f,[a,b]}_{\mathbf{Z}_\ell}$ for the Fontaine–Laffaille module defined by $\mathbf{1}^i = \mathbf{Z}_\ell$ for $i \leq 0$ and $\mathbf{1}^i = 0$ for $i > 0$. We set $\phi^i : \mathbf{1}^i \to \mathbf{1}$ to be given by $x \mapsto \ell^{-i}x$ for $i \leq 0$.
2. For any $A \in \mathrm{LCA}_\mathcal{O}$, we define $M_{n,A} \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ to be the free rank one A-module equipped with the filtration $M^i_{n,A} = A$ for $i \leq n$, $M^{n+1}_{n,A} = 0$ and $\phi^i : M^i_{n,A} \to M_{n,A}$ given by $x \mapsto \ell^{n-i}x$ for $i \leq n$. We put $\mathbf{1}_A = M_{0,A}$.

Definition 4.4 [13, Definition 4.9] For $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$ and $s \in \mathbf{Z}$ define $M(s)$ to be the same underlying $\mathbf{Z}_\ell$-module, but change the filtration to $M(s)^i = M^{i-s}$ for any $i \in \mathbf{Z}$. This means that $M(s) \in MF^{f,[a+s,b+s]}_{\mathrm{tor},\mathbf{Z}_\ell}$.

4.2 Extensions

To ease notation in the rest of this section, we put $\mathcal{C}_A^I = MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ for $A \in \mathrm{LCA}_{\mathcal{O}}$. Here, $I = [a, b]$.

Definition 4.5 (Definition/Lemma) Given $M, N \in \mathcal{C}_A^I$ define a filtration on the A-module $\mathrm{Hom}_A(M, N)$ by

$$\mathrm{Hom}_A(M,N)^i = \{f \in \mathrm{Hom}_A(M,N) \mid f(M^j) \subset N^{j+i} \text{ for all } j \in \mathbf{Z}\}$$

and $\mathbf{Z}_\ell$-linear maps $\phi^i : \mathrm{Hom}_A(M,N)^i \to \mathrm{Hom}_A(M,N)$ by

$$\phi^i(f)(\phi^j_M(m)) = \phi^{i+j}_N(f(m))$$

(note that $M = \sum \phi^j_M(M^j)$) for $f \in \mathrm{Hom}_A(M,N)^i$ and all $m \in M^j$ and $j \in \mathbf{Z}$. We claim this defines a Fontaine–Laffaille structure and that $\mathrm{Hom}_A(M,N) \in MF^{f,[a-b,b-a]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$.

Proof First note that there exists a canonical A-module homomorphism $\psi : M^\vee \otimes_A N \to \mathrm{Hom}_A(M,N)$, where $M^\vee = \mathrm{Hom}_A(M,A)$. Definition 4.19 in [13] defines a Fontaine–Laffaille structure on $M^\vee$ (and Lemmas 4.20 and 4.21 prove that this structure is well-defined and so we get an object in $MF^{f,[-b,-a]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$). Definition 4.17 in [13] then gives us the Fontaine–Laffaille structure on $M^\vee \otimes_A N$.

We claim that transferring this structure on $M^\vee \otimes_A N$ via ψ to $\mathrm{Hom}_A(M,N)$ matches our definition. Recall from [13] that $(M^\vee)^i = \{f \in \mathrm{Hom}_A(M,A) | f(M^k) \subset \mathbf{1}_A^{i+k}$ for all $k \in \mathbf{Z}\}$ and $(M^\vee \otimes N)^n = \sum_{i+j=n} (M^\vee)^i \otimes_A N^j$. We will first show that $\psi((M^\vee \otimes N)^n) \subset \mathrm{Hom}_A(M,N)^n$. Let $f_i \otimes n_j \in (M^\vee)^i \otimes_A N^j$. Then, $\psi(f_i \otimes n_j) : m \in M^k \mapsto f_i(m)n_j \in N^j$. In fact, the image lies in N^{n+k}. This is clear for $j \geq n+k$. If $j < n+k$ (and hence $0 < i+k$) it follows since $f_i(m) \in \mathbf{1}_A^{i+k} = 0$. To show the reverse inclusion $\psi((M^\vee \otimes N)^n) \supset \mathrm{Hom}_A(M,N)^n$ consider $f \in \mathrm{Hom}_A(M,N)^n$ and let j be maximal among integers l such that $f(M) \subset N^l$. To satisfy $f(M^k) \subset N^{k+n}$ for all integers k, we need $f(M^k) = 0$ for $k+n > j$ by maximality of j. This means that we need f to factor through M/M^{1-i} for $i := n-j$. By [13, Lemma 4.20] we have $(M^\vee)^i = \mathrm{Hom}_A(M/M^{1-i}, A)$ so we get

$$(M^\vee)^i \otimes N^j = \mathrm{Hom}_A(M/M^{1-i}, A) \otimes N^j \overset{\psi}{\cong} \mathrm{Hom}_A(M/M^{1-i}, N^j).$$

We conclude that $f \in \psi^{-1}((M^\vee)^i \otimes N^j) \subset \psi^{-1}((M^\vee \otimes N)^n)$.

Now, we check the $\mathbf{Z}_\ell$-linear maps: Recall from [13] that for $f \in M^\vee$, we have $\phi^i_{M^\vee}(f)(\phi^j_M(m)) = \phi^{i+j}(f(m))$ for all $m \in M^j$ and $j \in \mathbf{Z}$. We also have $\phi^n_{M^\vee \otimes_A N} = \sum_{i+j=n} \phi^i_{M^\vee} \otimes \phi^j_N$. We claim that $\phi^n_{\mathrm{Hom}_A(M,N)} \circ \psi = \psi \circ \phi^n_{M^\vee \otimes_A N} : (M^\vee \otimes N)^n \to \mathrm{Hom}_A(M,N)$. For this, one calculates that both sides map $f \otimes n \in (M^\vee)^i \otimes N^{n-i}$ to the homomorphism, for which

$$\phi^k_M(m) \mapsto \begin{cases} 0 & \text{if } i+k \geq 0 \\ \phi^{n+k}_N(f(m)x) & \text{if } i+k \leq 0 \end{cases}$$

for any $m \in M^k$ (for $\psi \circ \phi^n_{M^\vee \otimes_A N}$ this uses $\phi^{n+k}_N|_{N^{n-i}} = \ell^{k+i}\phi^{n-i}_N$ for $i+k \leq 0$). This claim, combined with the results in [13] shows that the definition of $\phi^n_{\mathrm{Hom}_A(M,N)}$ is well-defined and satisfies the requirements for $\mathrm{Hom}_A(M,N)$ to be a Fontaine–Laffaille module in $MF^{f,[a-b,b-a]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$. ■

For $M, N \in \mathcal{C}^I_A$ consider the map $\phi - 1 : \mathrm{Hom}_A(M,N)^0 \to \mathrm{Hom}_A(M,N)$, which takes f to the homomorphism that sends $m = \sum_j \phi^j_M(m_j)$ to

$$\sum_j \phi^j_N(f(m_j)) - f(m) = \sum_j \left(\phi^j_N(f(m_j)) - f(\phi^j_M(m_j))\right).$$

Note that $\ker(\phi - 1) = \mathrm{Hom}_{\mathcal{C}^I_A}(M,N)$.

Proposition 4.6 [18, Lemma 2.4.2] and [31, Proposition 2.17] *Given $M, N \in \mathcal{C}^I_A$, we have an exact sequence of A-modules (note that $\mathrm{Hom}_{\mathrm{Fil},A}(M,N)$ in [31] equals $\mathrm{Hom}_A(M,N)^0$)*

$$0 \to \mathrm{Hom}_{\mathcal{C}^I_A}(M,N) \to \mathrm{Hom}_A(M,N)^0 \xrightarrow{\phi-1} \mathrm{Hom}_A(M,N) \to \mathrm{Ext}^1_{\mathcal{C}^I_A}(M,N) \to 0.$$

Given $M, N \in \mathcal{C}^I_A$, we write $\mathrm{FL}(M) > \mathrm{FL}(N)$ if there is an integer j such that all elements of $\mathrm{FL}(M)$ are greater than or equal to j, and all elements of $\mathrm{FL}(N)$ are strictly less than j.

Proposition 4.7 *The extension group $\mathrm{Ext}^1_{\mathcal{C}^I_A}(M,N)$ is a finitely generated A-module. Furthermore, one has:*

1. *If $\mathrm{FL}(M) > \mathrm{FL}(N)$ then $\mathrm{Ext}^1_{\mathcal{C}^I_A}(M,N) \cong \mathrm{Hom}_A(M,N)$, in particular, it is a free A-module and $\mathrm{rk}_A(\mathrm{Ext}^1_{\mathcal{C}^I_A}(M,N)) = \mathrm{rk}_A(M)\mathrm{rk}_A(N)$.*
2. *If $\mathrm{FL}(M) < \mathrm{FL}(N)$ then $\mathrm{Ext}^1_{\mathcal{C}^I_A}(M,N) = 0$.*

Proof This follows from Proposition 4.6. In particular, $\mathrm{Ext}^1_{\mathcal{C}^I_A}(M,N)$ is a quotient of the finitely generated A-module $\mathrm{Hom}_A(M,N)$. The calculation on [31, p. 238] ("two notable cases") is carried out for $MF^{f,[0,\ell-1]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$, but applies verbatim to $\mathcal{C}^I_A$. If $\mathrm{FL}(M) > \mathrm{FL}(N)$ then this calculation shows that $\mathrm{Hom}_A(M,N)^0 = 0$, while if $\mathrm{FL}(M) < \mathrm{FL}(N)$ then one gets $\mathrm{Hom}_A(M,N)^0 = \mathrm{Hom}_A(M,N)$. ■

Proposition 4.8 (Hom-tensor adjunction) *Let $M, N \in \mathcal{C}^I_A$. Assume that $\mathrm{Hom}_A(M,N)$ equipped with the filtration as in Definition 4.5 is an object in $\mathcal{C}^I_A$ and that $0 \in I$. Then, there exists a canonical isomorphism of A-modules:*

$$\mathrm{Ext}^1_{\mathcal{C}^I_A}(M,N) \cong \mathrm{Ext}^1_{\mathcal{C}^I_A}(\mathbf{1}_A, \mathrm{Hom}_A(M,N)).$$

Proof The statement follows from the existence of the following commutative diagram with exact columns:

(4.1)
$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}_{\mathcal{C}_A^I}(M,N) \arrow[d] & \mathrm{Hom}_{\mathcal{C}_A^I}(\mathbf{1}_A, \mathrm{Hom}_A(M.N)) \arrow[d] \\
\mathrm{Hom}_A(M,N)^0 \arrow[r, "\psi'"] \arrow[d, "\phi-1"'] & \mathrm{Hom}_A(\mathbf{1}_A, \mathrm{Hom}_A(M,N))^0 \arrow[d, "\phi-1"] \\
\mathrm{Hom}_A(M,N) \arrow[r, "\psi"] \arrow[d, "\alpha"'] & \mathrm{Hom}_A(A, \mathrm{Hom}_A(M,N)) \arrow[d] \\
\mathrm{Ext}^1_{\mathcal{C}_A^I}(M,N) \arrow[r, "\tilde{\psi}"] \arrow[d] & \mathrm{Ext}^1_{\mathcal{C}_A^I}(\mathbf{1}_A, \mathrm{Hom}_A(M,N)) \arrow[d] \\
0 & 0
\end{tikzcd}$$

The exactness of both columns follows from Proposition 4.6. The second horizontal arrow is the usual isomorphism ψ of A-modules given by $f \mapsto (a \mapsto af)$ (recall that the underlying module of the object $\mathbf{1}_A$ is A) with the inverse map sending g to $g(1)$, where 1 is the multiplicative identity of A. The map $\tilde{\psi}$ is defined by lifting an element of $\mathrm{Ext}^1_{\mathcal{C}_A^I}(M,N)$ to $\mathrm{Hom}_A(M,N)$ and using ψ. The exactness of the first column ensures that such a map is well-defined.

The first horizontal arrow is the restriction ψ' of ψ to $\mathrm{Hom}_A(M,N)^0$ (note that $\mathrm{Hom}_A(M,N)^0$ is a subgroup of $\mathrm{Hom}_A(M,N)$ even though $\phi - 1$ is not necessarily injective). We need to check that ψ' lands in $\mathrm{Hom}_A(\mathbf{1}_A, \mathrm{Hom}_A(M,N))^0$. By its definition, we need to check if $f(\mathbf{1}_A^j) \subset \mathrm{Hom}_A(M,N)^j$. If $j > 0$ there is nothing to check as then $\mathbf{1}_A^j = 0$, so assume that $j \leq 0$. Then, $\mathbf{1}_A^j = A$ and $\mathrm{Hom}_A(M,N)^j \supset \mathrm{Hom}_A(M,N)^0$. So, it is enough to show that if $f \in \mathrm{Hom}_A(M,N)^0$ then $\psi'(f)(A) \subset \mathrm{Hom}_A(M,N)^0$. Let $a \in A$. Then, $\psi'(f)(a) = af$, which clearly lies in $\mathrm{Hom}_A(M,N)^0$ as $\mathrm{Hom}_A(M,N)^0$ is an A-module.

Now, let $g \in \mathrm{Hom}_A(\mathbf{1}_A, \mathrm{Hom}_A(M,N))^0$. We need to show that $\psi^{-1}(g)$ lands in $\mathrm{Hom}_A(M,N)^0$. Again we need to consider $\psi^{-1}(g)(\mathbf{1}_A^j)$. If $j > 0$, then $g = 0$, hence we are done. Assume that $j \leq 0$. Then, $\mathbf{1}_A^j = A$ and $\psi^{-1}(g) = g(1)$. As $1 \in \mathbf{1}_A^0$ and $g \in \mathrm{Hom}_A(\mathbf{1}_A, \mathrm{Hom}_A(M,N))^0$ we must have that $g(1) \in \mathrm{Hom}_A(M,N)^0$. So, we are done again.

This shows that ψ' is a bijection, hence an isomorphism. Hence, by the second Four Lemma, $\tilde{\psi}$ is injective, and since it is clearly surjective, it is an isomorphism. ∎

4.3 Fontaine–Laffaille Galois representations

Fix an interval $I = [a,b]$ with $a, b \in \mathbf{Z}$ and $b - a \leq \ell - 2$. In this section, we introduce certain categories of $G_{\mathbf{Q}_\ell}$-representations and define a covariant version V_I of the functor in [25] from the categories of Fontaine–Laffaille modules defined in Section 4.1 to these categories of Galois representations.

Let A_{cris} and B_{cris} denote the usual Fontaine's ℓ-adic period rings (see Definitions 7.3 and 7.7 in [26] and [24]). We recall that a $\mathbf{Q}_\ell[G_{\mathbf{Q}_\ell}]$-module V is called crystalline if $\dim_{\mathbf{Q}_\ell} V = \dim_{\mathbf{Q}_\ell} H^0(\mathbf{Q}_\ell, V \otimes_{\mathbf{Q}_\ell} B_{\mathrm{cris}})$. Our convention is that the Hodge–Tate weight of the cyclotomic character is +1.

Definition 4.9 Let $A \in \mathrm{LCA}_{\mathcal{O}}$. We introduce the following categories:

(i) $\mathrm{Rep}^f_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$, the category of $\mathbf{Z}_\ell[G_{\mathbf{Q}_\ell}]$-modules that are finitely generated as $\mathbf{Z}_\ell$-modules.

(ii) $\mathrm{Rep}^f_{\mathrm{tor},\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$, the full subcategory of $\mathrm{Rep}^f_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ whose objects are required to be of finite length as $\mathbf{Z}_\ell[G_{\mathbf{Q}_\ell}]$-modules.

(iii) $\mathrm{Rep}^{\mathrm{cris},I}_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$, the full subcategory of $\mathrm{Rep}^f_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ whose objects are isomorphic to T/T', where T and T' are $G_{\mathbf{Q}_\ell}$-stable finitely generated submodules of a crystalline $\mathbf{Q}_\ell$-representation with Hodge–Tate weights in I.

(iv) $\mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{tor},\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$, the full subcategory of $\mathrm{Rep}^f_{\mathrm{tor},\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ whose objects are isomorphic to T/T', where T and T' are $G_{\mathbf{Q}_\ell}$-stable lattices in a crystalline $\mathbf{Q}_\ell$-representation with Hodge–Tate weights in I.

(v) $\mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$, the category of free finite rank A-modules M with an A-linear $G_{\mathbf{Q}_\ell}$-action, for which there exists a crystalline representation of $G_{\mathbf{Q}_\ell}$ defined over E with Hodge–Tate weights in I containing $G_{\mathbf{Q}_\ell}$-stable $\mathcal{O}$-lattices $T' \subset T$, and an $\mathcal{O}$-algebra map $A \to \mathrm{End}_{\mathcal{O}}(T/T')$ such that M is isomorphic as an $A[G_{\mathbf{Q}_\ell}]$-module to T/T'. We will call objects of this category *Fontaine–Laffaille A-representations (with weights in I)*.

Remark 4.10 Definition 4.9(v) matches Definition 2.1 in [31] .

Definition 4.11 [12, p. 363] and [13, Definitions 4.7 and 4.9] Similar to [13] we define the following two functors:

1. A covariant functor $T_{\mathrm{cris}} : MF^{f,[2-\ell,0]}_{\mathbf{Z}_\ell} \to \mathrm{Rep}^f_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ defined via

$$T_{\mathrm{cris}}(M) := \ker\left(1 - \phi^0_{A_{\mathrm{cris}}\otimes_{\mathbf{Z}_\ell} M} : \mathrm{Fil}^0(A_{\mathrm{cris}} \otimes_{\mathbf{Z}_\ell} M) \to A_{\mathrm{cris}} \otimes_{\mathbf{Z}_\ell} M\right).$$

2. A covariant functor $V_I : MF^{f,[a,b]}_{\mathbf{Z}_\ell} \to \mathrm{Rep}^f_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$, defined via

$$V_I(M) = T_{\mathrm{cris}}(M(-b))(-b). \tag{4.2}$$

Recall that $M(-b)$ was defined in Definition 4.4, while $(-b)$ on the outside denotes the Tate twist as defined in Section 2.

Remark 4.12 We note that for $? \in \{\varnothing, \mathrm{tor}\}$, the category $MF^{f,[a,b]}_{?,\mathbf{Z}_\ell}$ is a full subcategory of $MF^{f,[a,a+\ell-2]}_{?,\mathbf{Z}_\ell}$, since they are both full subcategories of $MF^f_{?,\mathbf{Z}_\ell}$ (cf. Definition 4.1), so in particular (4.2) makes sense.

Remark 4.13 Note that V_I extends T_{cris} to general I (in particular, $V_{[2-\ell,0]} = T_{\mathrm{cris}}$). Also observe that for $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$, we have $M(-b) \in MF^{f,[2-\ell,0]}_{\mathrm{tor},\mathbf{Z}_\ell}$ since $M(-b)^1 = M^{b+1} = 0$ and $M(-b)^{2-\ell} = M^{2-\ell+b} = M$ as $b + 2 - \ell \leq a$. In particular, the definition of V_I makes sense.

Compared to [13] we work with the more restrictive interval $[2-\ell, 0]$ for T_{cris} and correct a sign error in the Galois twist in [13, Definition 4.9]

Theorem 4.14 [12, Theorem 4.3] [41, Section 2] [20, Section 1.1.2] [27, Section 2.2] [13, Fact 4.10] and [31, Theorem 2.10] *We have:*

(i) *The covariant functor* $V_{[a,b]} : MF^{f,[a,b]}_{\mathbf{Z}_\ell} \to \mathrm{Rep}^f_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ *is well-defined, exact, and fully faithful.*

(ii) *For* $M \in MF^{f,[a,b]}_{\mathbf{Z}_\ell}$, *one has* $V_{[a,b]}(M) = \varprojlim_n V_{[a,b]}(M/\ell^n)$.

(iii) *The essential image of* $V_{[a,b]}$ *is closed under formation of sub-objects, quotients, and finite direct sums. It is given by the subcategory* $\mathrm{Rep}^{\mathrm{cris},[-b,-a]}_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$. *For* $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$, *the lengths of* M *and* $V_I(M)$ *as* $\mathbf{Z}_\ell$*-modules agree; in particular, the essential image of* $MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$ *under* $V_{[a,b]}$ *is* $\mathrm{Rep}^{\mathrm{cris},[-b,-a]}_{\mathrm{tor},\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$.

(iv) *For* $A \in \mathrm{LCA}_{\mathcal{O}}$, *the functor* $V_{[a,b]}$ *induces a functor from* $MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ *to the category of free finite rank* A*-modules with an* A*-linear* $G_{\mathbf{Q}_\ell}$*-action, which we will also denote by* $V_{[a,b]}$. *Its essential image is given by* $\mathrm{Rep}^{\mathrm{cris},[-b,-a]}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$. *In fact,* $V_{[a,b]}$ *gives an equivalence of categories between* $MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$ *and* $\mathrm{Rep}^{\mathrm{cris},[-b,-a]}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$.

Remark 4.15

(1) Note that for $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}$, we have $V_{[a+s,b+s]}(M(s)) = V_{[a,b]}(M)(-s)$.

(2) For $I = [a,b] = [0, \ell-2]$, the functor V_I agrees with that of the functor $\mathbb{V}$ in [20, p. 670] by [14, Proposition 3.2.1.7]

(3) For $M \in MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$, the Hodge–Tate weights of $V_I(M)$ (in the sense of Definition 4.9(3)) equal the negatives of the Fontaine–Laffaille weights of M, defined in Definition 4.1(3), due to our convention that the Hodge–Tate weight of the cyclotomic character is +1.

As an immediate consequence of the equivalence of categories in Theorem 4.14(iv), we obtain the following corollary.

Corollary 4.16 *For any* $M, N \in MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$, *there is an isomorphism of* A*-modules*

$$\mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A}(M,N) \cong \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},-I}_A(G_{\mathbf{Q}_\ell})}(V_I(M), V_I(N)). \tag{4.3}$$

4.4 Local Selmer groups

Let $I = [a,b]$ be an interval as in the previous section (so $0 \le b-a \le \ell-2$) but we now also require that $0 \in I$ (so that $\mathbf{1} \in MF^{f,I}_{\mathbf{Z}_\ell}$, see Definition 4.3).

For an extension between two objects M, N in $\mathrm{Rep}_A(G_{\mathbf{Q}_\ell})0 \to M \to E \to N \to 0$, we define the n-th Tate twist of the extension to be the extension $0 \to M(n) \to E(n) \to N(n) \to 0$. For a subgroup G of $\mathrm{Ext}^1_{\mathrm{Rep}_A(G_{\mathbf{Q}_\ell})}(M,N)$, we define $G(n)$ to consist of extensions which are the n-th Tate twists of the elements of G.

Given an extension $\mathcal{E} \in \mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A}(M_3, M_1)$ represented by an exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

we will write $V_I(\mathcal{E})$ for the extension in $\mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},-I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})}(V_I(M_3), V_I(M_1))$ represented by

$$0 \to V_I(M_1) \to V_I(M_2) \to V_I(M_3) \to 0.$$

This uses the exactness of the functor V_I (cf. Theorem 4.14(i)). Since we defined $V_I(M) = T_{\mathrm{cris}}(M(-b))(-b)$ (see Equation (4.2)), we conclude the following lemma.

Lemma 4.17 *For $A \in \mathrm{LCA}_{\mathcal{O}}$ and $M \in MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$, we have*

$$\begin{aligned} V_I(\mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A}(\mathbf{1}_A, M)) &= \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},-I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})}(T_{\mathrm{cris}}(M_{-b,A})(-b), T_{\mathrm{cris}}(M(-b))(-b)) \\ &\cong \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},[0,\ell-2]}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})}(A(b), T_{\mathrm{cris}}(M(-b)))(-b). \end{aligned}$$

Note that the latter is naturally isomorphic to $\mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},[0,\ell-2]}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})}(A(b), T_{\mathrm{cris}}(M(-b)))$ and they give rise to the same subgroup of $H^1(\mathbf{Q}_\ell, V_I(M))$, see Definition 4.18.

Definition 4.18 For $M \in MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A$, let $H^1_{f,I}(\mathbf{Q}_\ell, V_I(M)) = V_I(\mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} A}(\mathbf{1}_A, M)) \subset H^1(\mathbf{Q}_\ell, V_I(M))$.

Remark 4.19 This is a more precise version of the definition made in [6, Section 5.2.1] (where the prime ℓ was denoted by p). In [6] we worked (implicitly) with $I = [0, p-2]$, but the results in [6, Section 5] (in particular, Corollary 5.4 and Proposition 5.8 restated below) carry over to $H^1_{f,I}$ defined here for general I.

T.B. and K.K. would like to clarify how certain definitions and results in some of our papers fit in with this more precise description of the groups $H^1_{f,I}$: In [8] the relevant interval I is $I = [1-k, k-1]$ for Section 5, and p should satisfy $p - 1 > 2k - 2$. The examples in Section 6 of [loc. cit] satisfy this stronger condition. Similarly, in [9] one has $I = [3-2k, 2k-3]$ $(p - 1 > 4k - 6)$. In [6, Section 6] the suitable interval I is such that $\mathrm{Hom}_{\mathcal{O}}(\tilde{\rho}_2, \tilde{\rho}_1)$ has Hodge–Tate weights in I. For $i, j \in \{1, 2\}$, the local condition at $v \mid p$ for the Selmer groups $H^1_\Sigma(F, \mathrm{Hom}_{\mathbf{F}}(\rho_i, \rho_j))$ is $H^1_{f,I}(F_v, \mathrm{Hom}_{\mathbf{F}}(\tilde{\rho}_i, \tilde{\rho}_j))$. In [loc.cit.] Section 9, one has $I = [-1, 1]$ $(p - 1 > 2)$, in Section 10, $I = [1-k, k-1]$ $(p - 1 > 2k - 2)$. In [7, Sections 7 and 8] the same comment applies as for [6, Section 9]

In J.B.'s paper [16] the argument in Sections 8 and 9 to show the splitting at ℓ of $\begin{pmatrix} \overline{\varepsilon}^{k-2} & * \\ 0 & \overline{\varepsilon}^{k-1} \end{pmatrix}$ by relating it to $H^1_f(\mathbf{Q}_\ell, \mathbf{F}(-1)) = 0$ requires an interval I containing -1 and $2k - 3$, so would need $p - 1 > 2k - 2$. However, one could instead not twist and invoke Proposition 4.7.

Similar comments apply to other results in the literature, e.g., in [20, Corollary 2.3] the expression $H^1_f(\mathbf{Q}_\ell, \mathrm{ad}^0_\kappa \overline{L})$ is only indirectly defined by $H^1_f(\mathbf{Q}_\ell, \mathrm{ad}_\kappa \overline{L}) = H^1_f(\mathbf{Q}_\ell, \mathrm{ad}^0_\kappa \overline{L}) \oplus H^1_f(\mathbf{Q}_\ell, \kappa)$. To define the Selmer group for the trace zero endomorphisms and prove this identity requires ad^0_κ to lie in the essential image of the

Fontaine–Laffaille functor, and therefore $I = [1-k, k-1]$ should be specified, rather than $I = [0, \ell - 2]$ as in [20, Section 1.1.2]

If $M, N \in \mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$, then $M \oplus N \in \mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$ and it is clear that

$$H^1_{f,I}(\mathbf{Q}_\ell, M \oplus N) = H^1_{f,I}(\mathbf{Q}_\ell, M) \oplus H^1_{f,I}(\mathbf{Q}_\ell, N) \tag{4.4}$$

because the extension groups as well as the functor V_I commute with direct sums.

Proposition 4.20 *For any* $n \in [2-\ell, \ell-2]$ *such that* $0, -n \in I$, *the group* $H^1_{f,I}(\mathbf{Q}_\ell, V_I(M_{-n,\mathbf{F}}))$ *is independent of I. In fact, we have*

$$H^1_{f,I}(\mathbf{Q}_\ell, \mathbf{F}(n)) = \begin{cases} 0 & n < 0 \\ H^1_{\mathrm{un}}(\mathbf{Q}_\ell, \mathbf{F}) & n = 0 \\ H^1_{\mathrm{fl}}(\mathbf{Q}_\ell, \mu_\ell) & n = 1 \\ H^1(\mathbf{Q}_\ell, \mathbf{F}(n)) & n > 1, \end{cases}$$

where

$$H^1_{\mathrm{un}}(\mathbf{Q}_\ell, \mathbf{F}) := \ker(H^1(\mathbf{Q}_\ell, \mathbf{F}) \to H^1(I_\ell, \mathbf{F})) \cong \mathrm{Hom}(G_{\mathbf{Q}_\ell}/I_\ell, \mathbf{F})$$

and $H^1_{\mathrm{fl}}(\mathbf{Q}_\ell, \mu_\ell)$ *denotes the peu ramifiée classes, namely, those classes corresponding to* $\mathbf{Z}_\ell^\times/(\mathbf{Z}_\ell^\times)^\ell \subset \mathbf{Q}_\ell^\times/(\mathbf{Q}_\ell^\times)^\ell \cong H^1(\mathbf{Q}_\ell, \mathbf{F}(1))$. *For* $n \geq 0$, *we note that* $\dim_{\mathbf{F}} H^1_{f,I}(\mathbf{Q}_\ell, \mathbf{F}(n)) = 1$.

Remark 4.21

(1) Proposition 4.20 justifies writing $H^1_\Sigma(\mathbf{Q}_\ell, V_I(M_n))$ as we did in [8], without specifying the interval I, as long as I contains $-n$. Under the conditions of Proposition 4.23 (see comment after Proposition 5.1), once we have fixed a suitable interval I, we will also drop the subscript I in this article.

(2) Note that the definition of $H^1_{f,I}(\mathbf{Q}_\ell, V_I(M_n))$ depends on $n \in \mathbf{Z}$, even though the coefficients $V_I(M_n) = \mathbf{F}(n)$ only depend on n mod $\ell - 1$.

(3) [41, Section 9.3] states a version of this result for the local crystalline cohomology of unramified extensions of $\mathbf{Q}_\ell$ and with $\mathbf{Z}_\ell/\ell^m(n)$ coefficients for $m \in \mathbf{Z}_{>0}$.

Proof We first note that $H^1(\mathbf{Q}_\ell, \mathbf{F}(n))$ is one-dimensional for $n \neq 0, 1$, which follows from local Tate duality and the Euler characteristic formula (see, e.g., [58, Theorem 1 and Proposition 3]

For $n = 0$, we refer the reader to [18, Corollary 2.4.4] for identifying $H^1_{f,I}(\mathbf{Q}_\ell, \mathbf{F}(n))$ with $H^1_{\mathrm{un}}(\mathbf{Q}_\ell, \mathbf{F})$. That $H^1_{\mathrm{un}}(\mathbf{Q}_\ell, \mathbf{F})$ is one-dimensional follows since $\#H^1(G_{\mathbf{Q}_\ell}/I_\ell, \mathbf{F}) = \#H^0(\mathbf{Q}_\ell, \mathbf{F})$. Recall that

$$H^1_{f,I}(\mathbf{Q}_\ell, \mathbf{F}(n)) = H^1_{f,I}(\mathbf{Q}_\ell, V_I(M_{-n,\mathbf{F}})) = V_I(\mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} \mathbf{F}}(M_{0,\mathbf{F}}, M_{-n,\mathbf{F}})).$$

If $n < 0$ then by Proposition 4.7(ii) $\mathrm{Ext}^1_{MF^{f,I}_{\mathrm{tor},\mathbf{Z}_\ell} \otimes_{\mathbf{Z}_\ell} \mathbf{F}}(M_{0,\mathbf{F}}, M_{-n,\mathbf{F}}) = 0$ since the Fontaine–Laffaille weights satisfy the inequality $-n > 0$.

On the other hand, if $n > 0$ then $H^1_{f,I}(\mathbf{Q}_\ell, V_I(M_{-n}))$ is one-dimensional by Proposition 4.7(i). For $n > 1$, this equals $H^1(\mathbf{Q}_\ell, \mathbf{F}(n))$ by our observation at the start of the proof.

For $n = 1$, we have $H^1(\mathbf{Q}_\ell, \mathbf{F}(1)) \cong \mathbf{Q}_\ell^\times/(\mathbf{Q}_\ell^\times)^\ell$ is two-dimensional, and one can identify the Fontaine–Laffaille extensions with the peu ramifiée classes (see, e.g., [15, Lemma 8.1.3] ■

Remark 4.22 Note that $[2-\ell, 0]$ contains both 0 and $2-\ell$ (and is the only interval of this length that contains both). Then, since $\mathbf{F}(-1) = \mathbf{F}(\ell-2) = V_{[2-\ell,0]}(M_{2-\ell})$, we get

$$\begin{aligned} H^1_{f,[2-\ell,0]}(\mathbf{Q}_\ell, \mathbf{F}(-1)) &= H^1_{f,[2-\ell,0]}(\mathbf{Q}_\ell, \mathbf{F}(\ell-2)) \\ &= H^1_{f,[2-\ell,0]}(\mathbf{Q}_\ell, V_{[2-\ell,0]}(M_{2-\ell,\mathbf{F}})) \\ &\neq 0, \end{aligned}$$

corresponding to the crystalline non-split extension $\begin{pmatrix} \overline{\varepsilon}^{\ell-2} & * \\ 0 & 1 \end{pmatrix}$. Note that $1 \notin [2-\ell, 0]$.

However, for all other intervals $I \subset [2-\ell, \ell-2]$ of length $\ell - 2$, we have $1 \in I$ and so

$$\begin{aligned} H^1_{f,I}(\mathbf{Q}_\ell, \mathbf{F}(-1)) &= V_I(\mathrm{Ext}^1_{MF^{f,[a,b]}_{\mathrm{tor},\mathbf{Z}_\ell}}(M_{0,\mathbf{F}}, M_{1,\mathbf{F}})) \\ &= T_{\mathrm{cris}}(\mathrm{Ext}^1_{MF^{f,[2-\ell,0]}_{\mathrm{tor},\mathbf{Z}_\ell}}(M_{-b,\mathbf{F}}, M_{1-b,\mathbf{F}}))(-b) \\ &= 0 \end{aligned}$$

by Proposition 4.20. This demonstrates that $H^1_{f,I}(\mathbf{Q}_\ell, \mathbf{F}(n))$ is only independent of I for I containing $-n$.

Following [12] for a $\mathbf{Q}_\ell[G_{\mathbf{Q}_\ell}]$-module V define $H^1_f(\mathbf{Q}_\ell, V) = \ker\left(H^1(\mathbf{Q}_\ell, V) \to H^1(\mathbf{Q}_\ell, V \otimes_{\mathbf{Q}_\ell} B_{\mathrm{cris}})\right)$. Let V be a finite-dimensional E-vector space and $T \subset V$ be a $G_{\mathbf{Q}_\ell}$-stable $\mathcal{O}$-lattice, i.e., T is a free $\mathcal{O}$-submodule of V that spans V as a vector space over E. We set $W = V/T$ and $W[\lambda^m] = \{w \in W : \lambda^m w = 0\} \cong T/\lambda^m T$ for any $m \in \mathbf{Z}_{>0}$. Note that $W[\lambda^m]$ lies in $\mathrm{Rep}^{\mathrm{cris},-I}_{\mathcal{O}/\lambda^m}(G_{\mathbf{Q}_\ell})$ if V is crystalline with Hodge–Tate weights in $-I$. We let $H^1_f(\mathbf{Q}_\ell, W)$ be the image of $H^1_f(\mathbf{Q}_\ell, V)$ under the natural map $H^1(\mathbf{Q}_\ell, V) \to H^1(\mathbf{Q}_\ell, W)$.

Proposition 4.23 [20, Proposition 2.2] *Assume V is a crystalline $E[G_{\mathbf{Q}_\ell}]$-module as above with Hodge–Tate weights in $-I = [-b, -a]$ (and $0 \in I$). For $T \subset V$ and $W = V/T$ as above, we then have $H^1_f(\mathbf{Q}_\ell, W) = \varinjlim_m H^1_{f,I}(\mathbf{Q}_\ell, W[\lambda^m])$.*

Proof We note that the proof of [20, Proposition 2.2] carries over from $[0, \ell-2]$ to general I (in particular, one has Proposition 4.6) and apply the argument with (in their notation) V_1 the trivial $G_{\mathbf{Q}_\ell}$-representation and $V_2 = V$. ■

Corollary 4.24 [20, (33)] and [6, Corollary 5.4] *For every $m \in \mathbf{Z}_{>0}$, we have an exact sequence of $\mathcal{O}$-modules*

$$0 \to H^0(\mathbf{Q}_\ell, W)/\lambda^m \to H^1_{f,I}(\mathbf{Q}_\ell, W[\lambda^m]) \to H^1_f(\mathbf{Q}_\ell, W)[\lambda^m] \to 0.$$

Corollary 4.25 *For $n \in \mathbf{Z}$ with $0, n \in I \subset [2-\ell, \ell-2]$ and $n \neq 0$, we have*

$$H^1_{f,I}(\mathbf{Q}_\ell, V_I(M_{-n,\mathbf{F}})) = H^1_f(\mathbf{Q}_\ell, E/\mathcal{O}(n))[\lambda].$$

Proof Note that $H^0(\mathbf{Q}_\ell, E/\mathcal{O}(n)[\lambda]) = 0$ since $n \not\equiv 0 \mod \ell - 1$. This implies $H^0(\mathbf{Q}_\ell, E/\mathcal{O}(n)) = 0$, hence, we are done by Corollary 4.24. ∎

5 Selmer groups

5.1 Definitions

For M a topological $\mathbf{Z}_\ell[G_\mathbf{Q}]$-module set

$$H^1_{\mathrm{un}}(\mathbf{Q}_p, M) := \ker\left(H^1(\mathbf{Q}_p, M) \to H^1(I_p, M)\right)$$

for every prime p. Let $E/\mathbf{Q}_\ell$ be a finite extension with valuation ring $\mathcal{O}$, uniformizer λ, and residue field $\mathbf{F}$. Let V be a finite-dimensional E-vector space on which one has a continuous E-linear $G_\mathbf{Q}$ action. For finite primes p with $p \neq \ell$, we set

$$H^1_f(\mathbf{Q}_p, V) = H^1_{\mathrm{un}}(\mathbf{Q}_p, V).$$

For $p = \ell$, we recall from Section 4 that

$$H^1_f(\mathbf{Q}_\ell, V) = \ker\left(H^1(\mathbf{Q}_\ell, V) \to H^1(\mathbf{Q}_\ell, V \otimes_{\mathbf{Q}_\ell} B_{\mathrm{cris}})\right).$$

Let $T \subset V$ be a $G_\mathbf{Q}$-stable $\mathcal{O}$-lattice. We set $W = V/T$ and $W[\lambda^n] = \{w \in W : \lambda^n w = 0\} \cong T/\lambda^n T$. For every p, we let $H^1_f(\mathbf{Q}_p, W)$ be the image of $H^1_f(\mathbf{Q}_p, V)$ under the natural map $H^1(\mathbf{Q}_p, V) \to H^1(\mathbf{Q}_p, W)$. We have $H^1_f(\mathbf{Q}_p, W) = H^1_{\mathrm{un}}(\mathbf{Q}_p, W)$ for all $p \neq \ell$, as long as V is unramified at p, which for us will always be the case.

We define the global Selmer group of W as

$$H^1_f(\mathbf{Q}, W) = \ker\left\{H^1(\mathbf{Q}, W) \to \bigoplus_p \frac{H^1(\mathbf{Q}_p, W)}{H^1_f(\mathbf{Q}_p, W)}\right\}.$$

We note that as $H^1_f(\mathbf{Q}_\ell, W)$ commutes with direct sums and so clearly does $H^1_{\mathrm{un}}(\mathbf{Q}_\ell, W)$, we get that $H^1_f(\mathbf{Q}, W)$ does as well.

Let $I = [a, b]$ with $a, b \in \mathbf{Z}$ and $b - a \leq \ell - 2$ and assume that $0 \in I$. If V is crystalline with Hodge–Tate weights in $-I$, we define

$$\begin{aligned} &H^1_{f,I}(\mathbf{Q}, W[\lambda^n]) \\ &\quad = \ker\left\{H^1(\mathbf{Q}, W[\lambda^n]) \to \bigoplus_{p \neq \ell} \frac{H^1(\mathbf{Q}_p, W[\lambda^n])}{H^1_{\mathrm{un}}(\mathbf{Q}_p, W[\lambda^n])} \oplus \frac{H^1(\mathbf{Q}_\ell, W[\lambda^n])}{H^1_{f,I}(\mathbf{Q}_\ell, W[\lambda^n])}\right\}. \end{aligned}$$

As noted in (4.4), $H^1_f(\mathbf{Q}_\ell, W[\lambda^n])$ also commutes with direct sums and so we get that $H^1_{f,I}(\mathbf{Q}, W[\lambda^n])$ does as well.

Proposition 5.1 *Assume that the interval $I = [a, b]$ contains 0 and V is $E[G_\mathbf{Q}]$-module, which is finite-dimensional as an E-vector space and a crystalline $G_{\mathbf{Q}_\ell}$-module with*

Hodge–Tate weights in $-I$. If $H^0(\mathbf{Q}, W[\lambda]) = 0$ then we have

$$H^1_f(\mathbf{Q}, W)[\lambda^n] \cong H^1_{f,I}(\mathbf{Q}, W[\lambda^n]).$$

Proof [6, Proposition 5.8] proves the claim under the assumption $H^0(\mathbf{Q}, W) = 0$.

Suppose we have $\alpha \in H^0(\mathbf{Q}, W)$. We know every element of W is annihilated by some power of λ, so if $\alpha \neq 0$ there is an integer m so that $\lambda^m \alpha = 0$ but $\lambda^n \alpha \neq 0$ for all $0 < n < m$. However, this gives $\lambda^{m-1}\alpha \in H^0(\mathbf{Q}, W[\lambda]) = 0$, so it must be that $\alpha = 0$. Thus, $H^0(\mathbf{Q}, W) = 0$ as desired. ∎

After a suitable interval, I has been fixed, we will therefore also drop the subscript I and write $H^1_f(\mathbf{Q}, W[\lambda^n])$.

Let G be a group, R a commutative ring with identity, and M_i finitely generated free R-modules with R-linear action given by $\rho_i : G \to \mathrm{GL}_R(M_i)$ for $i = 1, 2$. The action of G on $\mathrm{Hom}_R(\rho_2, \rho_1)$ is given by $(g \cdot \varphi)(v) = \rho_1(v)\varphi(\rho_2(g^{-1})v)$. In particular, if $\rho_1 = \rho_2 = \rho$, we define the adjoint representation of ρ to be the $R[G]$-module $\mathrm{ad}\,\rho = \mathrm{Hom}_R(\rho, \rho)$. We write $\mathrm{ad}^0\rho$ for the $R[G]$-submodule of $\mathrm{ad}\,\rho$ consisting of endomorphisms of trace zero.

If ρ is of rank n and $2n \in R^\times$ then we have an isomorphism of $R[G]$-modules

$$\mathrm{ad}\,\rho \cong \mathrm{ad}^0\rho \oplus R. \tag{5.1}$$

5.2 Non-vanishing of a Selmer group

In this section, we explain how the congruence of a Siegel cusp form to the Klingen Eisenstein series in Section 3 leads to a non-zero element of $H^1_f(\mathbf{Q}, \mathrm{ad}^0(\rho_{\phi,\lambda})(2-k) \otimes E/\mathcal{O})$.

From now on, we fix the weight $k \geq 12$ even and the prime ℓ satisfying $\ell > 4k - 5$ and impose Assumption 3.1 on the field $E/\mathbf{Q}_\ell$. Let $\phi \in S_k(\Gamma_1)$ be a normalized eigenform. Let ρ_ϕ be the λ-adic Galois representation associated with ϕ and assume $\overline{\rho}_\phi$ is irreducible. Let $f \in S_k(\Gamma_2)$ be an eigenform with irreducible Galois representation ρ_f so that f is eigenvalue congruent to $E^{2,1}_\phi$ modulo λ.

The following result shows we can choose a lattice so that the residual Galois representation gives rise to a non-split extension.

Lemma 5.2 *There exists a $G_{\mathbf{Q}}$-stable lattice in the space of ρ_f such that with respect to this lattice*

$$\overline{\rho}_f = \begin{bmatrix} \overline{\rho}_\phi & * \\ & \overline{\rho}_\phi(k-2) \end{bmatrix} \not\cong \overline{\rho}_\phi \oplus \overline{\rho}_\phi(k-2).$$

Proof Using the compactness of $G_{\mathbf{Q}}$, one can show that there exists a $G_{\mathbf{Q}}$-stable lattice Λ' in the space of ρ_f. One uses Brauer–Nesbitt Theorem together with the Chebotarev Density Theorem to conclude that $\overline{\rho}^{ss}_{f,\Lambda'} = \overline{\rho}_\phi \oplus \overline{\rho}_\phi(k-2)$. Now, the existence of the desired lattice which gives the non-split extension follows from Theorem 4.1 in [9]. ∎

From now on, whenever we write ρ_f, we assume we have made a choice of lattice as in Lemma 5.2, so we consider ρ_f as a map from $G_{\mathbf{Q}}$ to $\mathrm{GL}_4(\mathcal{O})$.

We now choose the interval $I = [3-2k, 2k-3]$ so that it contains all the Hodge–Tate weights of ρ_f, ρ_ϕ, $\rho_\phi(k-2)$, $\mathrm{ad}\,\rho_\phi(2-k)$, and $\mathrm{ad}\,\rho_\phi(k-2)$. Note that $-I = I$. We assume that $\ell - 2 \geq 4k-6$. When we write H^1_f from now on, this refers to $H^1_{f,I}$ as defined in Section 5.1.

Let ρ be any of the representations above and write V for the representation space of ρ. We choose a $G_{\mathbf{Q}}$-stable lattice $T \subset V$ and recall that the isomorphism class of the semi-simplification of the $\mathbf{F}[G_{\mathbf{Q}}]$-representation $T/\lambda T$ is independent of the choice of T. It is well-known that if $T/\lambda T$ is irreducible then the $\mathcal{O}$-length of $H^1_f(\mathbf{Q}, W)$ is independent of T, where as before $W = V/T$. By Proposition 5.1, we then conclude that also the $\mathcal{O}$-length of $H^1_f(\mathbf{Q}, W[\lambda^n])$ is independent of the choice of T as long as $H^0(\mathbf{Q}, W) = 0$.

Lemma 5.3 *Under our assumptions (in particular, $\overline{\rho}_\phi$ irreducible and $\ell > 4k-5$), the modulo λ reduction of $\mathrm{ad}^0\,\rho_\phi$ is irreducible.*

Proof Assume the three-dimensional representation $\mathrm{ad}^0\,\overline{\rho}_\phi$ is reducible. Then, it either has a one-dimensional $G_{\mathbf{Q}}$-stable subspace or quotient. Since $\mathrm{ad}\,\rho_\phi$ and $\mathbf{1}$ are self-dual, so is $\mathrm{ad}^0\,\overline{\rho}_\phi$. Hence, we can assume without loss of generality that $\mathrm{ad}^0\,\overline{\rho}_\phi$ has a $G_{\mathbf{Q}}$-stable line. Write ψ for the character by which $G_{\mathbf{Q}}$ acts on the line.

As $\overline{\rho}_\phi$ is unramified away from ℓ and the order of ψ is prime to ℓ, we have $\psi = \overline{\varepsilon}^a$ for some integer $a \in I$. This would require $H^0(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi(-a)) \neq 0$. Note that $H^0(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(-a)) = \mathrm{Hom}_{G_{\mathbf{Q}}}(\overline{\rho}_\phi(a), \overline{\rho}_\phi)$. If $a \equiv 0 \pmod{(\ell-1)}$, then this space is one-dimensional by Schur's Lemma since $\overline{\rho}_\phi$ is irreducible. So, $H^0(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi) = 0$, contradiction.

If $a \not\equiv 0 \pmod{(\ell-1)}$, then $H^0(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(-a)) = H^0(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi(-a)) \neq 0$. This means that $\overline{\rho}_\phi$ is isomorphic to $\overline{\rho}_\phi(a)$. Considering the determinant, $\overline{\varepsilon}^a$ must be the trivial character or the quadratic character $\overline{\varepsilon}^{(\ell-1)/2}$. Both are ruled out since $a \in I = [3-2k, 2k-3]$ by our assumption that $\ell > 4k-5$. ∎

Remark 5.4 From Lemma 5.3, we conclude that when $\rho \in \{\rho_\phi, \rho_\phi(k-2), \mathrm{ad}^0\,\rho_\phi(2-k), \mathrm{ad}^0\,\rho_\phi(k-2)\}$, the $\mathcal{O}$-lengths of $H^1_f(\mathbf{Q}, W)$ and $H^1_f(\mathbf{Q}, W[\lambda^n])$ are independent of the choice of T. As we will ever only be interested in the order of these groups, the choice of T is immaterial and we will simply assume that such a choice was made. So, for example, we will use the notation $H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho_{\phi,\lambda}(k-2) \otimes E/\mathcal{O})$, thus assuming that when we write $\mathrm{ad}^0\,\rho_{\phi,\lambda}(k-2)$, we have made a choice of a lattice for this representation. Likewise any one-dimensional representation ρ is irreducible, so the $\mathcal{O}$-length of $H^1_f(\mathbf{Q}, \rho \otimes E/\mathcal{O})$ is independent of the choice of T.

For the representation $\mathrm{ad}\,\rho(m)$, $m \in \{k-2, 2-k\}$ (which is reducible), we choose a lattice which is a direct sum of a lattice inside $\mathrm{ad}^0\,\rho(m)$ and a lattice inside $E(m)$. So, from now on, whenever we write $\mathrm{ad}\,\rho(m)$ we mean such a lattice. Since the formation of Selmer groups commutes with direct sums, we then get

$$H^1_f(\mathbf{Q}, \mathrm{ad}\,\rho_\phi(m) \otimes E/\mathcal{O}) = H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho_\phi(m) \otimes E/\mathcal{O}) \oplus H^1_f(\mathbf{Q}, E/\mathcal{O}(m)) \tag{5.2}$$

for $m \in \{k-2, 2-k\}$. Note that the $\mathcal{O}$-length (and in particular, the non-triviality) of $H^1_f(\mathbf{Q}, \operatorname{ad}\rho(m) \otimes E/\mathcal{O})$ is independent of the choice of a lattice inside $\operatorname{ad}\rho_\phi(m)$ as long as it is the direct sum of lattices in $\operatorname{ad}^0 \rho_\phi(m)$ and $E(m)$.

Theorem 5.5 *With the set-up as above, we have* $H^1_f(\mathbf{Q}, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}) \neq 0$.

Proof We have via Lemma 5.2 that there is a lattice $T_f \subset V_f$ so that the residual representation $\overline{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_4(\mathbf{F})$ has the form

$$\overline{\rho}_f = \begin{bmatrix} \overline{\rho}_\phi & \psi \\ 0 & \overline{\rho}_\phi(k-2) \end{bmatrix} \tag{5.3}$$

and is not semisimple. The fact that ψ as in (5.3) gives a non-trivial class $[\psi]$ in $H^1(\mathbf{Q}, \operatorname{Hom}_{\mathbf{F}}(\overline{\rho}_2, \overline{\rho}_1)) = H^1(\mathbf{Q}, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda])$ is clear. We need to show that $[\psi]$ lies in $H^1_f(\mathbf{Q}, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda])$ and that the latter group injects into $H^1_f(\mathbf{Q}, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O})$.

We first show that $[\psi]$ satisfies the conditions to be in $H^1_f(\mathbf{Q}, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda])$. We have that ρ_f is unramified at all primes $p \neq \ell$, so the local conditions are satisfied for all primes $p \neq \ell$.

Since f has level one and weight k, $\rho_f|_{D_\ell}$ is crystalline with Hodge–Tate weights in $[0, 2k-3] \subset I = -I$. Hence, $\overline{\rho}_f$ (considered as a $G_{\mathbf{Q}_\ell}$-module) belongs to $\mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{free},\mathbf{F}}(G_{\mathbf{Q}_\ell})$ and gives rise to an element of $\operatorname{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{free},\mathbf{F}}(G_{\mathbf{Q}_\ell})}(\overline{\rho}_\phi(k-2), \overline{\rho}_\phi) \subset \operatorname{Ext}^1_{\mathbf{F}[G_{\mathbf{Q}_\ell}]}(\rho_\phi(k-2) \otimes E/\mathcal{O}[\lambda], \rho_\phi \otimes E/\mathcal{O}[\lambda])$. By our choice of I, we can use (4.3) and Proposition 4.8 to get a non-zero element in

$$\operatorname{Ext}^1_{\mathrm{Rep}^{\mathrm{cris},I}_{\mathrm{free},\mathbf{F}}(G_{\mathbf{Q}_\ell})}(\mathbf{F}, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda]) \subset \operatorname{Ext}^1_{\mathbf{F}[G_{\mathbf{Q}_\ell}]}(\mathbf{F}, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda]).$$

As this extension maps to $[\psi|_{G_{\mathbf{Q}_\ell}}]$ in $H^1(\mathbf{Q}_\ell, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda])$ under the canonical isomorphism $\operatorname{Ext}^1_{\mathbf{F}[G_{\mathbf{Q}_\ell}]}(\mathbf{F}, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda]) \cong H^1(\mathbf{Q}_\ell, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda])$, we conclude that

$$[\psi|_{G_{\mathbf{Q}_\ell}}] \in H^1_f(\mathbf{Q}_\ell, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda]) \subset H^1(\mathbf{Q}_\ell, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda]).$$

Therefore, we have established that $[\psi] \in H^1_f(\mathbf{Q}, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda])$. By Proposition 5.1, this group is isomorphic to $H^1_f(\mathbf{Q}, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O})[\lambda]$ if $H^0(\mathbf{Q}, \operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda]) = 0$. The latter holds since

$$\operatorname{ad}\rho_\phi(2-k) \otimes E/\mathcal{O}[\lambda]^{G_{\mathbf{Q}}} = \operatorname{Hom}_{G_{\mathbf{Q}}}(\overline{\rho}_\phi(k-2), \overline{\rho}_\phi) = 0 \tag{5.4}$$

as $\overline{\rho}_\phi$ and $\overline{\rho}_\phi(k-2)$ are absolutely irreducible (by assumption) and non-isomorphic since $k-2 \not\equiv 0, \frac{\ell-1}{2} \pmod{\ell-1}$ as $\ell > 4k-5$ and $k \neq 2$ (cf. the proof of Lemma 5.3). ∎

Lemma 5.6 *Let n be an even integer satisfying $3-2k < n \leq 0$. Assuming $\ell \nmid \#\mathrm{Cl}^{\overline{\varepsilon}^n}_{\mathbf{Q}(\zeta_\ell)^+}$, one has $H^1_f(\mathbf{Q}, \mathbf{F}(n)) = 0$ and, if additionally $n \neq 0$, $H^1_f(\mathbf{Q}, E/\mathcal{O}(n)) = 0$.*

Proof We see from Proposition 4.20 that any cohomology class in $H^1_f(\mathbf{Q}, \mathbf{F}(n))$ must vanish when restricted to I_ℓ. As all classes in $H^1_f(\mathbf{Q}, \mathbf{F}(n))$ are unramified away from ℓ, we get that they are unramified everywhere. Using inflation-restriction sequence where $H = \mathrm{Gal}(\mathbf{Q}(\zeta_\ell)^+/\mathbf{Q})$, we see that

$$H^1(\mathbf{Q}, \mathbf{F}(n)) \cong H^1(\mathbf{Q}(\zeta_\ell)^+, \mathbf{F}(n))^H = \mathrm{Hom}_H(G_{\mathbf{Q}(\zeta_\ell)^+}, \mathbf{F}(n)).$$

Note that everywhere unramified classes map to homomorphisms that kill all the inertia groups. Hence, the image of $H^1_f(\mathbf{Q}, \mathbf{F}(n))$ lands inside $\mathrm{Hom}\left(\mathrm{Cl}^{\overline{\varepsilon}^n}_{\mathbf{Q}(\zeta_\ell)^+}, \mathbf{F}\right) = 0$.

Note that a torsion $\mathcal{O}$-module M is zero if and only if $M[\lambda] = 0$. Therefore, the vanishing of $H^1_f(\mathbf{Q}, E/\mathcal{O}(n))$ follows from Proposition 5.1, which tells us that $H^1_f(\mathbf{Q}, E/\mathcal{O}(n))[\lambda] = H^1_f(\mathbf{Q}, \mathbf{F}(n))$ if $H^0(\mathbf{Q}, E/\mathcal{O}(n)) = 0$. We know that $H^0(\mathbf{Q}_\ell, E/\mathcal{O}(n)[\lambda]) = H^0(\mathbf{Q}, \mathbf{F}(n)) = 0$ for $n \neq 0$ since $n \not\equiv 0 \pmod{\ell - 1}$ under our assumption $\ell > 4k - 5$. ∎

Corollary 5.7 *Let $\phi \in S_k(\Gamma_1)$ be as in Theorem 3.5 and assume the hypotheses of Theorem 3.5 are satisfied. Assuming $\ell \nmid \#\mathrm{Cl}^{\overline{\varepsilon}^{2-k}}_{\mathbf{Q}(\zeta_\ell)^+}$, one has $H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho_\phi(2-k) \otimes E/\mathcal{O}) \neq 0$.*

Proof This follows from Theorem 5.5, Lemma 5.6, and isomorphism (5.2). ∎

Remark 5.8 If we assume Vandiver's conjecture for the prime ℓ, this gives that $\ell \nmid \#\mathrm{Cl}^{\overline{\varepsilon}^{2-k}}_{\mathbf{Q}(\zeta_\ell)^+}$.

6 Modularity

We begin with the following commutative algebra result that will be useful in this section.

Lemma 6.1 *If J is an ideal of $\mathbf{F}[[X_1, \dots, X_n]]$ that is strictly contained in the maximal ideal, then $\mathbf{F}[[X_1, \dots, X_n]]/J$ admits an $\mathbf{F}$-algebra surjection to $\mathbf{F}[T]/T^2$.*

Proof For a positive integer k, let I_k be the ideal of $\mathbf{F}[[X_1, \dots, X_k]]$ generated by all the monomials of degree at least 2. Set $S_k := \mathbf{F}[[X_1, \dots, X_k]]/I_k$ and write $\phi_k : \mathbf{F}[[X_1, \dots, X_k]] \to S_k$ for the canonical $\mathbf{F}$-algebra surjection. If $\phi_n(J) = 0$, then composing ϕ_n with the map $S_n \to \mathbf{F}[[T]]/T^2$ sending X_1 to T and X_i for $i > 1$ to zero gives the desired surjection.

Now suppose $\phi_n(J) \neq 0$. Without loss of generality (renumbering the variables if necessary), we may assume then that J contains an element of the form $u := X_n + f(X_1, \dots, X_{n-1}) + g(X_1, \dots, X_n)$, where f is homogeneous of degree one and all the terms in g have degree at least 2. Note that we can assume without loss of generality that some power of X_n appears in g. (Indeed, if g contains no X_n then we replace u by $u + u^2 \in J$.) By Theorem 7.16(a) in [23] there is a unique $\mathbf{F}$-algebra map from $\mathbf{F}[[X_1, \dots, X_n]]$ to itself sending X_n to $-f - g$ and X_i to itself for $i < n$. In other words, for any power series $h(X_1, \dots, X_n)$, the element $h(X_1, \dots, X_{n-1}, -f - g)$ also lives in $\mathbf{F}[[X_1, \dots, X_n]]$ and we denote it by $h'(X_1, \dots, X_n)$. Clearly, $h - h' \in J$.

Thus, for any power series h, where the smallest total degree of any term containing X_n is s, we have

$$h \equiv h' \pmod{J}$$

for some power series h' with the smallest total degree of any term containing X_n equal to $s' > s$. By the same process, we get an h'' such that $h' \equiv h''$ mod J and the smallest total degree of any term X_n in h'' is strictly greater than s'. This way, we can construct a sequence of power series h_s where for every s, we have the smallest total degree of any term containing X_n being greater than or equal to s and such that $h - h_s \in J$ for every s. We note that h_s is a Cauchy sequence with respect to the $(X_1, \dots, X_n)$-adic topology (indeed, for $t, u > s$, we see that $h_t - h_u$ lies in $(X_1, \dots, X_n)^s$). Set $h_0 = \lim_{s\to\infty} h_s$. As J is a closed ideal, we get that $h_0 - h \in J$. For every s, we have

$$h_0 \equiv h_s \equiv w_s \mod X_n^s,$$

for some $w_s \in \mathbf{F}[[X_1, \dots, X_{n-1}]]$. Note that the w_s also form a Cauchy sequence since h_s does. Set $w := \lim_{s\to\infty} w_s \in \mathbf{F}[[X_1, \dots, X_{n-1}]]$. Thus, $h_0 \equiv w$ modulo $\bigcap_s (X_n^s) \subset \bigcap_s (X_1, \dots, X_n)^s = 0$, so $h_0 \in \mathbf{F}[[X_1, \dots, X_{n-1}]]$.

Hence, the natural $\mathbf{F}$-algebra map $\psi_{n-1} : \mathbf{F}[[X_1, \dots, X_{n-1}]] \to \mathbf{F}[[X_1, \dots, X_n]]/J$ given by $h_0 \mapsto h_0 + J$ is surjective. Thus, we get an $\mathbf{F}$-algebra isomorphism $\mathbf{F}[[X_1, \dots, X_n]]/J \to \mathbf{F}[[X_1, \dots, X_{n-1}]]/J_{n-1}$, where $J_{n-1} = \ker \psi_{n-1}$.

If $\phi_{n-1}(J_{n-1}) \neq 0$, continue this way obtaining a sequence of ideals $J_{n-2}, J_{n-3}, \dots$. If at any stage $(1 \le r \le n-2)$, we get $\phi_{n-r}(J_{n-r}) = 0$, then we are done. Otherwise, we can eliminate all but one variable and get $\mathbf{F}[[X_1, \dots, X_n]]/J \cong \mathbf{F}[[X_1]]/J_1$ and now we must have $\phi_1(J_1) = 0$ as otherwise J_1 and hence J is maximal. ∎

Recall that in the earlier sections we fixed the weight $k \ge 12$ even and prime $\ell > 4k - 5$ and imposed Assumption 3.1 on the field $E/\mathbf{Q}_\ell$. We also fixed the Fontaine–Laffaille interval $I = [3 - 2k, 2k - 3]$. Let $\phi \in S_k(\Gamma_1)$ be a newform such that $\overline{\rho}_\phi$ is irreducible. The goal of this section is to prove a modularity theorem under the following assumption.

Assumption 6.2 *For k and ϕ as above, we assume that:*

(i) *there exists $f \in S_k(\Gamma_2)$ such that $f \equiv_{\mathrm{ev}} E_\phi^{2,1}$ (mod λ), and*
(ii) *$\# H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho_\phi(2-k) \otimes_{\mathcal{O}} E/\mathcal{O}) = \# \mathcal{O}/\lambda$ (recall that the left-hand side is independent of the choice of lattice, see Remark 5.4), and*
(iii) *$H^1_f(\mathbf{Q}, \mathrm{ad}^0 \overline{\rho}_\phi) = 0$.*

Remark 6.3 Assumption 6.2 (i) is satisfied under the assumptions of Theorem 3.5, and so is one inequality in Assumption 6.2 (ii) under the assumptions of Corollary 5.7.

We impose Assumption 6.2 and fix f as in Assumption 6.2 in what follows. We will write $G_{\{\ell\}}$ for the Galois group of the maximal Galois extension of $\mathbf{Q}$ unramified away from ℓ. Let $\rho_f : G_{\{\ell\}} \to \mathrm{GL}_4(E)$ be as in Theorem 2.1. Lemma 3.4 gives that ρ_f is irreducible. We will use Mazur's deformation theory and refer the reader to standard references, such as [19, 43] for the definitions and basic properties.

Definition 6.4 For $B \in \mathrm{LCN}_{\mathcal{O}}$, we say that a representation $\rho : G_{\mathbf{Q}_\ell} \to \mathrm{GL}_n(B)$ is *Fontaine–Laffaille* (with Hodge–Tate weights in $-I$) if $\rho \otimes_B A$ lies in $\mathrm{Rep}^{\mathrm{cris},-I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$ (see Definition 4.9(v)) for every Artinian quotient A of B. By Theorem 4.14(iv), this is equivalent to requiring $\rho \otimes_B A$ to lie in the essential image of the Fontaine–Laffaille functor.

Remark 6.5 We know that any choice of $\mathcal{O}$-lattice ρ_L in ρ_ϕ or ρ_f is Fontaine–Laffaille in this sense, since their restrictions to $G_{\mathbf{Q}_\ell}$ lie in $\mathrm{Rep}^{\mathrm{cris},-I}_{\mathbf{Z}_\ell}(G_{\mathbf{Q}_\ell})$ and therefore in the essential image of the Fontaine–Laffaille functor by Theorem 4.14(iii). Since they are also free $\mathcal{O}$-modules this implies by Theorem 4.14 (iii) and (iv) that $\rho_L \otimes B$ lies in $\mathrm{Rep}^{\mathrm{cris},-I}_{\mathrm{free},A}(G_{\mathbf{Q}_\ell})$ for every Artinian quotient B of $\mathcal{O}$.

For any local complete Noetherian $\mathcal{O}$-algebra A with residue field $\mathbf{F}$ by a deformation of a residual Galois representation $\tau : G_{\{\ell\}} \to \mathrm{GL}_n(\mathbf{F})$, we will mean a strict equivalence class of lifts $\tilde{\tau} : G_{\{\ell\}} \to \mathrm{GL}_n(A)$ of τ that are Fontaine–Laffaille at ℓ. This deformation condition is introduced in [6, Section 5.3] and [18, p. 35]

As is customary, we will denote a strict equivalence class of deformations by any of its members. If τ has scalar centralizer then this deformation problem is representable by a local complete Noetherian $\mathcal{O}$-algebra which we will denote by R_τ [44]. In particular, the identity map in $\mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R_\tau, R_\tau)$ furnishes what is called the universal deformation $\tau^{\mathrm{univ}} : G_{\{\ell\}} \to \mathrm{GL}_n(R_\tau)$.

Lemma 6.6 *One has $R_{\overline{\rho}_\phi} \cong R_{\overline{\rho}_\phi(k-2)} \cong \mathcal{O}$. Furthermore, ρ_ϕ (resp., $\rho_\phi(k-2)$) is the unique deformation of $\overline{\rho}_\phi$ (resp., $\overline{\rho}_\phi(k-2)$) to $\mathrm{GL}_2(\mathcal{O})$.*

Proof We have

$$\# \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(R_{\overline{\rho}_\phi}, \mathbf{F}[X]/X^2) = \# H^1_f(\mathbf{Q}, \mathrm{ad}\, \overline{\rho}_\phi) = 0, \tag{6.1}$$

where the first equality follows from the fact that our deformation condition is the property of being Fontaine–Laffaille (see, e.g., [18, Section 2.4.1]), and the second one holds since we have $H^1_f(\mathbf{Q}, \mathrm{ad}\, \overline{\rho}_\phi) = H^1_f(\mathbf{Q}, \mathrm{ad}^0\, \overline{\rho}_\phi) \oplus H^1_f(\mathbf{Q}, \mathbf{F}) = 0$ and $H^1_f(\mathbf{Q}, \mathbf{F}) = 0$ by Lemma 5.6 as we have imposed Assumption 6.2(iii).

By Theorem 7.16 in [23] we know that any local complete Noetherian $\mathcal{O}$-algebra with residue field $\mathbf{F}$ is a quotient of $\mathcal{O}[[X_1, \ldots, X_n]]$ for some positive integer n. Hence, $S := R_{\overline{\rho}_\phi}/(\lambda R_{\overline{\rho}_\phi}) \cong \mathbf{F}[[X_1, \ldots, X_n]]/J$ for some ideal J. Suppose first that J is not maximal. Then, by Lemma 6.1, we know that S admits a surjection φ to $\mathbf{F}[T]/T^2$. This contradicts (6.1), hence $S = \mathbf{F}$. We now use the complete version of Nakayama's Lemma to conclude that the structure map $\mathcal{O} \to R_{\overline{\rho}_\phi}$ is surjective (cf. [23, Exercise 7.2] or [37, Theorem 8.4]). Let us briefly explain why this version applies here. As $R_{\overline{\rho}_\phi} \otimes_{\mathcal{O}} \mathbf{F} \neq 0$, we see that $\lambda \in \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of $R_{\overline{\rho}_\phi}$. Hence,

$$\bigcap_n \lambda^n R_{\overline{\rho}_\phi} \subset \bigcap_n \mathfrak{m}^n. \tag{6.2}$$

The latter intersection is zero, since $R_{\overline{\rho}_\phi}$ is complete, so separated with respect to $\mathfrak{m}$. Hence, (6.2) implies that $R_{\overline{\rho}_\phi}$ is separated with respect to $\lambda R_{\overline{\rho}_\phi}$ allowing for the application of the complete version of Nakayama's Lemma.

As ρ_ϕ is a deformation to $\mathcal{O}$, we conclude that $R_{\overline{\rho}_\phi} = \mathcal{O}$. This implies that if $\rho : G_{\{\ell\}} \to \mathrm{GL}_2(\mathcal{O})$ is any deformation of $\overline{\rho}_\phi$, one has $\rho \cong \rho_\phi$. Similarly, if $\rho : G_{\{\ell\}} \to \mathrm{GL}_2(\mathcal{O})$ is a deformation of $\overline{\rho}_\phi(k-2)$ then $\rho(2-k)$ is a deformation of $\overline{\rho}_\phi$. Note that our choice of $I = [3-2k, 2k-3]$ means that this twisting stays inside our category of Fontaine–Laffaille representations. Hence, we get that $\rho(2-k) \cong \rho_\phi$, and so we are done. ∎

Remark 6.7 Note that the determinant of our deformations is automatically fixed as $H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi) = H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi)$ under our assumptions. This means that all deformations ρ of $\overline{\rho}_\phi$ (respectively, $\overline{\rho}_\phi(k-2)$) satisfy $\det\rho = \varepsilon^{k-1}$ (respectively, $\det\rho = \varepsilon^{2k-3}$).

Remark 6.8 Regarding Assumption 6.2(iii), we note that if one additionally assumes that $\overline{\rho}_\phi$ is absolutely irreducible when restricted to $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell})$ then [20, Theorem 3.7] (see also [28, Theorem 5.20] relates $H^1_f(\mathbf{Q}, \mathrm{ad}^0\rho_\phi \otimes E/\mathcal{O})$ (via an $R_{\overline{\rho}_\phi} = \mathbf{T}$ theorem) to a congruence ideal $\eta^{\varnothing}_\phi$. One can use Proposition 5.1 to see that $H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi) = H^1_f(\mathbf{Q}, \mathrm{ad}^0\rho_\phi \otimes E/\mathcal{O})[\lambda] = 0$ if $\eta^{\varnothing}_\phi$ is coprime to ℓ.

Lemma 6.9 *Let G be a group and F be a field. For $i \in \{1,2\}$, let $n_i \in \mathbf{Z}_+$ and $\rho_i : G \to \mathrm{GL}_{n_i}(F)$ be an irreducible representation with $\rho_1 \not\cong \rho_2$. Let $\rho : G \to \mathrm{GL}_{n_1+n_2}(F)$ be a representation such that*

$$\rho = \begin{bmatrix} \rho_1 & a \\ & \rho_2 \end{bmatrix} \not\cong \rho_1 \oplus \rho_2.$$

Then, ρ has scalar centralizer.

Proof This is a simple consequence of Schur's Lemma and the fact that $\tilde{a} : g \to \rho_2(g)^{-1}a(g)$ defines a cocycle from G to $\mathrm{Hom}(\rho_2, \rho_1)$ which is not a coboundary. ■

Fix a lattice in the space of ρ_f as in Lemma 5.2, i.e., such that $\overline{\rho}_f = \begin{bmatrix} \overline{\rho}_\phi & * \\ & \overline{\rho}_\phi(k-2) \end{bmatrix} : G_{\{\ell\}} \to \mathrm{GL}_4(\mathbf{F})$ is non-semisimple. For simplicity, we will write R for the universal deformation ring $R_{\overline{\rho}_f}$ of $\overline{\rho}_f$ and $\rho^{\mathrm{univ}} : G_{\{\ell\}} \to \mathrm{GL}_4(R)$ for the universal deformation. Note that the deformation problem is representable because $\overline{\rho}_f$ is non-semisimple with irreducible, mutually non-isomorphic Jordan–Holder factors, hence by Lemma 6.9, the centralizer of $\overline{\rho}_f$ consists of only scalar matrices. We say that a deformation $\tilde{\rho}$ is *upper-triangular* if $\tilde{\rho}$ is strictly equivalent to a deformation of $\overline{\rho}_f$ of the form $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ with the stars representing 2×2 blocks.

Lemma 6.10 *There do not exist any non-trivial deformations of $\overline{\rho}_f$ into $\mathrm{GL}_4(\mathbf{F}[X]/X^2)$ that are upper-triangular.*

Proof We use Proposition 7.2 in [6] noting that Assumption 6.1(i) in [loc.cit.] is satisfied because we impose the current Assumption 6.2(ii). On the other hand, Assumption 6.1(ii) in [loc.cit.] is satisfied because of Lemma 6.6. ■

Definition 6.11 The smallest ideal I of R such that $\mathrm{tr}\,\rho^{\mathrm{univ}}$ is the sum of two pseudocharacters mod I will be called the *reducibility ideal* of R. We will denote this ideal by I_{re}.

Proposition 6.12 *Let $I \subset R$ be an ideal such that R/I is an Artin ring. Then, $I \supset I_{\mathrm{re}}$ if and only if ρ^{univ} (mod I) is upper-triangular.*

Proof This is proved as Corollary 7.8 in [6]. ■

Corollary 6.13 *The structure map $\mathcal{O} \to R/I_{\mathrm{re}}$ is surjective and descends to an isomorphism $\mathcal{O}/\lambda^s \to R/I_{\mathrm{re}}$ for some $s \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$. In fact, one has*

$$R/I_{\mathrm{re}} \cong \mathcal{O}/\lambda.$$

Proof By Theorem 7.16 in [23] we know that any local complete Noetherian $\mathcal{O}$-algebra with residue field $\mathbf{F}$ is a quotient of $\mathcal{O}[[X_1, \dots, X_n]]$ for some positive integer n. Hence, $S := R/(I_{\mathrm{re}} + \lambda R) \cong \mathbf{F}[[X_1, \dots, X_n]]/J$ for some ideal J. Suppose first that J is not maximal. Then, by Lemma 6.1, we know that S admits a surjection φ to $\mathbf{F}[T]/T^2$. This means that there exists a non-trivial (because the image of φ is not contained in $\mathbf{F}$) deformation of ρ to $\mathbf{F}[T]/T^2$ which is upper-triangular (by Proposition 6.12), which contradicts Lemma 6.10. Thus, indeed, $S = \mathbf{F}$.

Hence, the structure map $\mathcal{O} \to R/I_{\mathrm{re}}$ is surjective by the complete version of Nakayama's Lemma (see the proof of Lemma 6.6). So, $R/I_{\mathrm{re}} \cong \mathcal{O}/\lambda^s$ for some $s \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$.

The composition of ρ^{univ} with the map $R \to R/I_{\mathrm{re}}$ gives rise to a deformation $\rho_{\mathrm{re}} : G_{\{\ell\}} \to \mathrm{GL}_4(R/I_{\mathrm{re}}) = \mathrm{GL}_4(\mathcal{O}/\lambda^s)$. By Proposition 6.12, this deformation is upper triangular, i.e., one has $\rho_{\mathrm{re}} = \begin{bmatrix} *_1 & *_2 \\ & *_3 \end{bmatrix}$. As the property of being Fontaine–Laffaille is preserved by subobjects and quotients, we see that $*_1$ and $*_3$ are Fontaine–Laffaille representations with values in $\mathrm{GL}_2(R/I_{\mathrm{re}}) = \mathrm{GL}_2(\mathcal{O}/\lambda^s)$. Thus, by Lemma 6.6, we can conclude that $*_1 = \rho_\phi$, $*_3 = \rho_\phi(k-2) \bmod \lambda^s$. Hence, by (5.4) and Proposition 5.1, $*_2$ gives rise to a class in $H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho_\phi(2-k) \otimes_{\mathcal{O}} E/\mathcal{O})$ as ρ_{re} is Fontaine–Laffaille. As ρ is non-semi-simple, we conclude that $*_2$ is not annihilated by λ^{s-1}, i.e., the class of $*_2$ gives rise to a subgroup of $H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho_\phi(2-k) \otimes_{\mathcal{O}} E/\mathcal{O})$ isomorphic to $\mathcal{O}/\lambda^s$. Thus, $s \leq 1$ as $\# H^1_f(\mathbf{Q}, \mathrm{ad}^0 \rho_\phi(2-k) \otimes_{\mathcal{O}} E/\mathcal{O}) \leq \# \mathcal{O}/\lambda$ by Assumption 6.2(ii). Finally, $s > 0$ as $\overline{\rho}_f$ itself is reducible. This concludes the proof. ∎

The following proposition does not use Assumption 6.2(ii).

Proposition 6.14 *Assume that* $\dim H^1_f(\mathbf{Q}, \mathrm{ad}\, \overline{\rho}_\phi(k-2)) \leq 1$. *Then, the ideal I_{re} is a principal ideal.*

Proof Since ρ^{univ} is a trace representation in the sense of Section 1.3.3 of [4] Lemma 1.3.7 in [loc.cit.] tells us that we can conjugate ρ^{univ} by a matrix $P \in \mathrm{GL}_2(R)$ (here we use that every finite type projective R-module is free since R is local) to get ρ^{univ} adapted to a data of GMA idempotents for $R[G_{\{\ell\}}]/\ker \rho^{\mathrm{univ}}$. By [4, Lemma 1.3.8] we then get an isomorphism of R-modules

$$R[G_{\{\ell\}}]/\ker \rho^{\mathrm{univ}} \cong \begin{bmatrix} \mathrm{Mat}_2(R) & \mathrm{Mat}_2(B) \\ \mathrm{Mat}_2(C) & \mathrm{Mat}_2(R) \end{bmatrix}$$

for ideals $B, C \subset R$. By [4, Proposition 1.5.1] we further know that $I_{\mathrm{re}} = BC$.

[4, Theorem 1.5.5] proves that there are injections $\mathrm{Hom}_R(B, \mathbf{F}) \hookrightarrow H^1(G_{\{\ell\}}, \mathrm{ad}\, \overline{\rho}_\phi(2-k))$ and $\mathrm{Hom}_R(C, \mathbf{F}) \hookrightarrow H^1(G_{\{\ell\}}, \mathrm{ad}\, \overline{\rho}_\phi(k-2))$. Arguing as in [1, Proposition 4.2] (see also [55, Theorem 4.3.5 and Remark 4.3.6] one sees that the images are contained in the Selmer groups $H^1_f(\mathbf{Q}, \mathrm{ad}\, \overline{\rho}_\phi(2-k))$ and $H^1_f(\mathbf{Q}, \mathrm{ad}\, \overline{\rho}_\phi(k-2))$, respectively. From Assumption 6.2 (ii) and

Proposition 5.1, we see that $H^1(\mathbf{Q}, \operatorname{ad} \overline{\rho}_\phi(2-k)) \cong \mathbf{F}$. Together with the assumption $\dim H^1_f(\mathbf{Q}, \operatorname{ad} \overline{\rho}_\phi(k-2)) \leq 1$, we deduce by Nakayama's Lemma that both B and C, and therefore also I_{re} are principal ideals of R. Note that Nakayama's Lemma applies since B and C are ideals in R, which is Noetherian, hence they are finitely generated over R. ■

Remark 6.15 [1, Proposition 3.10] proves the principality of the reducibility ideal of the reduced Fontaine–Laffaille deformation ring R^{red} for any residual representations with two Jordan-Hölder factors. Our argument (while relying on [1, Proposition 4.2] is slightly more general as it allows us to treat the case of non-reduced deformation rings.

Remark 6.16 By (5.2), we have

$$H^1_f(\mathbf{Q}, \operatorname{ad} \overline{\rho}_\phi(k-2)) = H^1_f(\mathbf{Q}, \operatorname{ad}^0 \overline{\rho}_\phi(k-2)) \oplus H^1_f(\mathbf{Q}, \mathbf{F}(k-2)).$$

However, as opposed to the case of the $(2-k)$-twist of the trivial representation (cf. proof of Lemma 5.6), there is no simple relation between $H^1_f(\mathbf{Q}, \mathbf{F}(k-2))$ and part of a class group except for the case $k = 2$ by Proposition 4.20. By the same proposition for $2 < k \leq \ell$, the group $H^1_f(\mathbf{Q}, \mathbf{F}(k-2))$ requires no ramification condition at ℓ, so equals $H^1(G_{\{\ell\}}, \mathbf{F}(k-2))$.

We have the following results about $H^1(G_{\{\ell\}}, \mathbf{F}(n))$ for $n > 0$.

Proposition 6.17 [8, Proposition 6.5] *Suppose $n \in \mathbf{Z}_{>0}$ and $n \not\equiv 1 \bmod \ell - 1$. Assume that $\ell \nmid \# \mathrm{Cl}^{\overline{\varepsilon}^n}_{\mathbf{Q}(\zeta_\ell)}$. Then, $\dim H^1(G_{\{\ell\}}, \mathbf{F}(n)) \leq 1$.*

Proposition 6.18 *Let $n > 0$ be an even integer. Assume $\ell \nmid B_n$ (the n-th Bernoulli number) and $n \not\equiv 0 \bmod \ell - 1$. Then, $H^1(G_{\{\ell\}}, \mathbf{F}(n)) = 0$.*

Proof Since n is even and $H^0(G_{\{\ell\}}, \mathbf{F}(n)) = 0$ as $n \not\equiv 0 \bmod \ell - 1$ we know $\dim_{\mathbf{F}} H^1(G_{\{\ell\}}, \mathbf{F}(n)) = \dim_{\mathbf{F}} H^2(G_{\{\ell\}}, \mathbf{F}(n))$ by [40, Corollary 8.7.5] (Euler Poincare characteristic). [3, Proposition 1.3] (condition (ii, β)) proves that $H^2(G_{\{\ell\}}, \mathbf{F}(n)) = 0$ if $n \not\equiv 1 \bmod \ell - 1$ (which is automatically satisfied for even n) and $\ell \nmid \# \mathrm{Cl}^{\overline{\varepsilon}^{1-n}}_{\mathbf{Q}(\zeta_\ell)}$. By Herbrand's Theorem (see, e.g., [57, Theorem 6.17] the latter follows from our assumption that $\ell \nmid B_n$ (here we use again $n \not\equiv 0 \bmod \ell - 1$). ■

Remark 6.19 Note that the assumption $\ell \nmid B_n$ is stronger than $\ell \nmid \# \mathrm{Cl}^{\overline{\varepsilon}^n}_{\mathbf{Q}(\zeta_\ell)}$ in [8, Proposition 6.5] As noted in the proof of Proposition 6.18, $\ell \nmid B_n$ implies $\ell \nmid \# \mathrm{Cl}^{\overline{\varepsilon}^{\ell-n}}_{\mathbf{Q}(\zeta_\ell)}$ by Herbrand's Theorem. By the "reflection theorem" [57, Theorem 10.9] this means that also $\ell \nmid \mathrm{Cl}^{\overline{\varepsilon}^n}_{\mathbf{Q}(\zeta_\ell)}$.

This allows us to prove the following modularity theorem.

Theorem 6.20 *Recall that we impose Assumptions 3.1 and 6.2. Furthermore, assume that $\dim H^1_f(\mathbf{Q}, \operatorname{ad} \overline{\rho}_\phi(k-2)) \leq 1$. Then, the structure map $\iota : \mathcal{O} \to R$ is an isomorphism. In particular, if $\tau : G_{\mathbf{Q}} \to \mathrm{GL}_4(E)$ is any continuous irreducible homomorphism* *unramified outside ℓ, crystalline at ℓ with Hodge–Tate weights in $[3-2k, 2k-3]$ and such that*

$$\overline{\tau}^{\mathrm{ss}} = \overline{\rho}_\phi \oplus \overline{\rho}_\phi(k-2),$$

then $\tau \cong \rho^{\mathrm{univ}} \cong \rho_f$, i.e., in particular, τ is modular.

Proof It follows from Corollary 6.13 that I_{re} is a maximal ideal of R. As the deformation ρ_f induces a surjective map $j : R \to \mathcal{O}$, we get the following commutative diagram of $\mathcal{O}$-algebra maps:

$$\begin{tikzcd} \mathcal{O} \arrow[r, "\iota"'] \arrow[d] \arrow[rr, bend left, "\mathrm{id}"] & R \arrow[r, "j"'] \arrow[d] & \mathcal{O} \arrow[d] \\ \mathcal{O}/\lambda \arrow[r, "\overline{\iota}"] \arrow[rr, bend right, "\mathrm{id}"'] & R/I_{\mathrm{re}} \arrow[r, "\overline{j}"] & \mathcal{O}/\lambda. \end{tikzcd} \tag{6.3}$$

As $\overline{\iota}$ is an isomorphism, we get that so is $\overline{j}$. So, using the fact that I_{re} is principal (Proposition 6.14), we can now apply Theorem 6.9 in [5] to the right square to conclude that j is an isomorphism.

Now, let τ be as in the statement of the theorem. Then, τ factors through a representation of $G_{\{\ell\}}$. Using that τ is irreducible, Theorem 4.1 in [9] allows us to find a lattice in the space of τ such that with respect to that lattice, one has

$$\overline{\tau} = \begin{bmatrix} \overline{\rho}_\phi & * \\ & \overline{\rho}_\phi(k-2) \end{bmatrix}$$

that is non-semi-simple. Using Remark 6.5, we see that this lattice is Fontaine–Laffaille, so the star gives rise to a non-zero element in $H^1_f(\mathbf{Q}, \mathrm{ad}^0\rho_\phi(2-k) \otimes_{\mathcal{O}} E/\mathcal{O})$. As the latter group has order $\#\mathcal{O}/\lambda$ by Assumption 6.2(ii), we conclude that $\overline{\tau} \cong \rho$. In particular, τ is a deformation of ρ. Hence, τ gives rise to an $\mathcal{O}$-algebra map $R \to \mathcal{O}$, which must equal j by the first part of the theorem. ∎

Remark 6.21 We return to Example 3.6 and note that Assumption 6.2 (i) holds, as discussed earlier. Since $\ell = 163$ or 187273 do not divide $(2k-1)(2k-3)k!$ for $k = 26$ and $\overline{\rho}_\phi$ is irreducible, [20, Lemma 2.5] proves that $\overline{\rho}_\phi$ stays irreducible when restricted to $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\sqrt{(-1)^{(\ell-1)/2}\ell}))$. Via Remark 6.8, we can therefore check that $H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi) = 0$ as ϕ is the only cusp form of weight 26 and level 1, so in particular, ϕ is not congruent mod ℓ to other forms. Since, in addition, $L_{\mathrm{alg}}(50, \mathrm{Sym}^2\phi)$ has ℓ-valuation 1 for both $\ell = 163$ and 187273, the Bloch–Kato conjecture for $\#H^1_f(\mathbf{Q}, \mathrm{ad}^0\rho_\phi(2-k) \otimes E/\mathcal{O}) = \#\mathcal{O}/\lambda$ (see [22, Conjecture (5.2) and (5)] would imply that Assumption (ii) holds.

We do not know how to check $\dim H^1_f(\mathbf{Q}, \mathrm{ad}\overline{\rho}_\phi(k-2)) \leq 1$, as the corresponding divisible Selmer group is not critical (in the sense of Deligne). Note that $\dim H^1_f(\mathbf{Q}, \mathrm{ad}\overline{\rho}_\phi(k-2)) = \dim H^1_f(\mathbf{Q}, \mathrm{ad}^0\overline{\rho}_\phi(k-2))$ by Proposition 6.18, since neither prime ℓ divides B_{24}.

7 (Non-)principality of Eisenstein ideals

In this section, we formulate conditions when the Eisenstein ideal of the local Hecke algebra acting on $S_k(\Gamma_2)$ is non-principal and $\dim_{\mathbf{F}} H^1_f(\mathbf{Q}, \mathrm{ad}^0 \overline{\rho}_\phi(k-2)) > 1$. In particular, in that case, $R \ncong \mathcal{O}$.

Let $\mathbf{T}'$ be as in Section 2. Let $\mathbf{T}$ denote the $\mathcal{O}$-subalgebra of $\mathbf{T}' \otimes_{\mathbf{Z}} \mathcal{O}$ generated by the operators $T^{(2)}(p)$ and $T_1^{(2)}(p^2)$ for all primes $p \nmid \ell$. Since strong multiplicity, one holds in the level one case, we can choose an orthogonal basis $\mathcal{N}'$ of $S_k(\Gamma_2)$ consisting of eigenforms for all the operators in $\mathbf{T}$.

Each $g \in \mathcal{N}'$ gives rise to $\psi_g \in \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\mathbf{T}, \mathcal{O})$, where $\psi_g(T) = \lambda_g(T)$, with $\lambda_g(T)$ the eigenvalue of the operator T corresponding to g. Thus, we get a map $\Psi : \mathcal{N}' \to \mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\mathbf{T}, \mathcal{O})$ given by $g \mapsto \lambda_g$, which by strong multiplicity, one is an injection.

Lemma 7.1 *The natural $\mathcal{O}$-algebra map*

$$\mathbf{T} \to \prod_{g \in \mathcal{N}'} \mathcal{O} \quad \textit{given by} \quad T \mapsto (\psi_g(T))_g \tag{7.1}$$

is injective and has finite cokernel, i.e., $\mathbf{T}$ can be viewed as a lattice in $\prod_{g \in \mathcal{N}'} \mathcal{O}$.

Proof The injectivity follows from the fact that the elements of $\mathcal{N}'$ form a basis.

We will now show that the map has finite cokernel. Note that the (set) map $\Psi \otimes \overline{\mathbf{Q}}_\ell : \mathcal{N}' \to \mathrm{Hom}_{\overline{\mathbf{Q}}_\ell-\mathrm{alg}}(\mathbf{T} \otimes \overline{\mathbf{Q}}_\ell, \overline{\mathbf{Q}}_\ell) \hookrightarrow \mathrm{Hom}_{\overline{\mathbf{Q}}_\ell}(\mathbf{T} \otimes \overline{\mathbf{Q}}_\ell, \overline{\mathbf{Q}}_\ell)$ given by $g \mapsto \lambda_g \otimes \overline{\mathbf{Q}}_\ell$ is injective (because Ψ is injective), and strong multiplicity one implies that no non-trivial linear relation $\sum_{g \in \mathcal{N}'} c_g \lambda_g = 0$ can hold. Thus, the set $\{\lambda_g \mid g \in \mathcal{N}'\}$ is a linearly independent subset of $\mathrm{Hom}_{\overline{\mathbf{Q}}_\ell}(\mathbf{T} \otimes \overline{\mathbf{Q}}_\ell, \overline{\mathbf{Q}}_\ell)$. Hence,

$$\dim_{\overline{\mathbf{Q}}_\ell} \mathbf{T} \otimes \overline{\mathbf{Q}}_\ell = \dim_{\overline{\mathbf{Q}}_\ell} \mathrm{Hom}_{\overline{\mathbf{Q}}_\ell}(\mathbf{T} \otimes \overline{\mathbf{Q}}_\ell, \overline{\mathbf{Q}}_\ell) \geq \#\mathcal{N}'. \tag{7.2}$$

Tensoring the map (7.1) with $\overline{\mathbf{Q}}_\ell$ we get a corresponding map $\mathbf{T} \otimes \overline{\mathbf{Q}}_\ell \to \prod_{g \in \mathcal{N}'} \overline{\mathbf{Q}}_\ell$, which is injective as (7.1) is. Thus, it must be surjective by (7.2). Hence, the map (7.1) has finite cokernel. ■

We now identify $\mathbf{T}$ with the image of the map (7.1) and note that $\mathbf{T} = \prod_{\mathfrak{m} \in \mathrm{MaxSpec}\mathbf{T}} \mathbf{T}_\mathfrak{m}$, where $\mathbf{T}_\mathfrak{m}$ is the localization of $\mathbf{T}$ at the maximal ideal $\mathfrak{m}$. Let $\mathcal{N}$ be the subset of $\mathcal{N}'$ consisting of all the $g \in \mathcal{N}'$ which satisfy

$$\psi_g(T) \equiv \lambda_{E_\phi^{1,2}}(T) \pmod{\lambda} \quad \text{for all } T \in \mathbf{T}.$$

We write $\mathfrak{m}$ for the corresponding maximal ideal. Set $J \subset \mathbf{T}$ to be the Eisenstein ideal, i.e., J is the ideal of $\mathbf{T}$ generated by the set $\{T^{(2)}(p) - (\mathrm{tr}\, \rho_\phi(\mathrm{Frob}_p) + \mathrm{tr}\, \rho_\phi(k-2)(\mathrm{Frob}_p)) \mid p \neq \ell\}$. Write $J_\mathfrak{m}$ to be the image of J under the canonical map $\mathbf{T} \to \mathbf{T}_\mathfrak{m}$.

Recall that we fixed in Section 5.2 the weight $k \geq 12$ even and prime $\ell > 4k - 5$ and imposed Assumption 3.1 on the field $E/\mathbf{Q}_\ell$. We also fixed the Fontaine–Laffaille interval $I = [3 - 2k, 2k - 3]$. Let $\phi \in S_k(\Gamma_1)$ be a newform such that $\overline{\rho}_\phi$ is irreducible.

For the rest of this section, we also impose Assumption 6.2 and fix the corresponding $f \in S_k(\Gamma_2)$. Then, $f \in \mathcal{N}$, i.e., $\mathbf{T}_\mathfrak{m}/J_\mathfrak{m} \neq 0$. Let $R = R_{\overline{\rho}_f}$ be the universal deformation ring defined in Section 6.

Theorem 7.2 *Recall that we impose Assumptions 3.1 and 6.2. Then, there exists a surjective $\mathcal{O}$-algebra map $\varphi : R \to \mathbf{T}_{\mathfrak{m}}$ such that $\varphi(I_{\mathrm{re}}) = J_{\mathfrak{m}}$ and $J_{\mathfrak{m}}$ is a maximal ideal of $\mathbf{T}_{\mathfrak{m}}$. If, in addition, $\dim_{\mathbf{F}} H^1_f(\mathbf{Q}, \mathrm{ad}\overline{\rho}_\phi(k-2)) \le 1$, then all of the following are true:*

- *the map φ is an isomorphism;*
- *the Hecke ring $\mathbf{T}_{\mathfrak{m}}$ is isomorphic to $\mathcal{O}$;*
- *the Eisenstein ideal $J_{\mathfrak{m}}$ is principal.*

Proof Let $g \in \mathcal{N}$. Then, by Lemma 5.2, there exists a $G_{\mathbf{Q}}$-stable lattice with respect to which one has $\overline{\rho}_g = \begin{bmatrix} \overline{\rho}_\phi & * \\ & \overline{\rho}_\phi(k-2) \end{bmatrix}$ and is not semi-simple. Hence, the $*$ gives rise to an element in $H^1_f(\mathbf{Q}, W[\lambda])$, where $W = \mathrm{ad}^0\rho_\phi(2-k) \otimes_{\mathcal{O}} E/\mathcal{O}$.

By (5.4) and Proposition 5.1, we get $H^1_f(\mathbf{Q}, W[\lambda]) = H^1_f(\mathbf{Q}, W)[\lambda]$. The latter group is cyclic by Assumption 6.2 (ii), so we must have that $\overline{\rho}_g \cong \overline{\rho}_f$, and so after adjusting the basis, if necessary, we get that ρ_g is a deformation of $\overline{\rho}_f$.

This implies that for every $g \in \mathcal{N}$, we get an $\mathcal{O}$-algebra (hence continuous) map $\varphi_g : R \to \mathcal{O}$ with the property that $\mathrm{tr}\,\rho^{\mathrm{univ}}(\mathrm{Frob}_p) \mapsto \lambda_g(T^{(2)}(p))$. This property completely determines φ_g because R is topologically generated by the set $\{\mathrm{tr}\,\rho^{\mathrm{univ}}(\mathrm{Frob}_p) \mid p \neq \ell\}$ by Proposition 7.13 in [6]. Putting these maps together we get an $\mathcal{O}$-algebra map $\varphi : R \to \prod_{g\in\mathcal{N}} \mathcal{O}$ whose image is an $\mathcal{O}$-subalgebra of $\prod_{g\in\mathcal{N}} \mathcal{O}$ generated by $\{T^{(2)}(p) \mid p \neq \ell\}$. Note that $\varphi(R) \subset \mathbf{T}_{\mathfrak{m}}$. To see the opposite inclusion consider the characteristic polynomial $f_p(X) \in R[X]$ of $\rho^{\mathrm{univ}}(\mathrm{Frob}_p)$ for $p \neq \ell$. Combining Theorem 2.1 with the definition of $L_p(X, f; \mathrm{spin})$, we see that the coefficient at X^2 is mapped by φ to $T^{(2)}(p)^2 - T^{(2)}(p^2) - p^{2k-4} \in \prod_{g\in\mathcal{N}} \mathcal{O}$. As $T^{(2)}(p)$ and p^{2k-4} both belong to $\varphi(R)$, so therefore must $T^{(2)}(p^2)$. We now use the fact [2, 3.3.38] and [30, p. 547] that

$$pT_1^{(2)}(p^2) = T^{(2)}(p)^2 - T^{(2)}(p^2) - p(p^2+p+1)T(\mathrm{diag}(p,p,p,p))$$

to conclude that $T_1^{(2)}(p^2) \in \varphi(R)$. Hence, $\varphi(R)$ contains all the Hecke operators away from ℓ, i.e., $\varphi(R) = \mathbf{T}_{\mathfrak{m}}$. We denote the resulting $\mathcal{O}$-algebra epimorphism $R \to \mathbf{T}_{\mathfrak{m}}$ again by φ. We claim that $\varphi(I_{\mathrm{re}}) \subset J_{\mathfrak{m}}$.

Indeed, using the Chebotarev Density Theorem, one sees that

$$\mathrm{tr}\,\rho^{\mathrm{univ}} \equiv \mathrm{tr}\,\rho_\phi + \mathrm{tr}\,\rho_\phi(k-2) \pmod{\varphi^{-1}(J_{\mathfrak{m}})},$$

so $I_{\mathrm{re}} \subset \varphi^{-1}(J_{\mathfrak{m}})$. As φ is a surjection, this implies that $\varphi(I_{\mathrm{re}}) \subset J_{\mathfrak{m}}$. Hence, φ gives rise to a sequence of $\mathcal{O}$-algebra surjections $R/I_{\mathrm{re}} \to \mathbf{T}_{\mathfrak{m}}/\varphi(I_{\mathrm{re}}) \to \mathbf{T}_{\mathfrak{m}}/J_{\mathfrak{m}}$. As $R/I_{\mathrm{re}} = \mathbf{F}$ by Corollary 6.13 we conclude that all these surjections are isomorphisms (note that $\mathbf{T}_{\mathfrak{m}}/J_{\mathfrak{m}} \neq 0$), hence $\varphi(I_{\mathrm{re}}) = J_{\mathfrak{m}}$ and $J_{\mathfrak{m}}$ is maximal. This proves the first claim.

Now assume in addition that $\dim H^1_f(\mathbf{Q}, \mathrm{ad}\overline{\rho}_\phi(k-2)) \le 1$. Then, Theorem 6.20 gives us that $R = \mathcal{O}$, so we get that φ is an isomorphism, and so $R \cong \mathbf{T}_{\mathfrak{m}} \cong \mathcal{O}$. Hence, $J_{\mathfrak{m}}$ is a principal ideal. ∎

Corollary 7.3 *If $J_{\mathfrak{m}}$ is not principal, then $\dim_{\mathbf{F}} H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2)) > 1$. If in addition $\ell \nmid B_{k-2}$ then $\dim_{\mathbf{F}} H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi(k-2)) > 1$.*

Proof The first inequality is just a restatement of one of the claims of Theorem 7.2. The second follows from the first one and Proposition 6.18. ∎

Proposition 7.4 *For each $g \in \mathcal{N}$, write m_g for the largest positive integer m such that $g \equiv E^{\phi}_{2,1} \bmod \lambda^m$. If*

$$\mathrm{val}_\ell(\#\mathbf{T}_\mathfrak{m}/J_\mathfrak{m}) < [\mathbf{F} : \mathbf{F}_\ell] \cdot \sum_{g \in \mathcal{N}} m_g \tag{7.3}$$

then $J_\mathfrak{m}$ is not principal.

Proof Set $A = \prod_{g\in\mathcal{N}} A_g$, where $A_g = \mathcal{O}$ for all $g \in \mathcal{N}$. Let $\phi_g : A \to A_g$ be the canonical projection. Since, by Lemma 7.1, $\mathbf{T}$ is a full rank $\mathcal{O}$-submodule of $\prod_{g\in\mathcal{N}'} \mathcal{O}$, we conclude that the local complete $\mathcal{O}$-subalgebra $\mathbf{T}_\mathfrak{m} \subset A$ is of full rank as an $\mathcal{O}$-submodule and $J_\mathfrak{m} \subset \mathbf{T}_\mathfrak{m}$ is an ideal of finite index. Set $T_g = \phi_g(\mathbf{T}_\mathfrak{m}) = A_g = \mathcal{O}$ and $J_g = \phi_g(J_\mathfrak{m}) = \lambda^{m_g}\mathcal{O}$. Hence, we are in the setup of Section 2 of [11]. Assume $J_\mathfrak{m}$ is principal. Then, Proposition 2.3 in [11] gives us that

$$\#\mathbf{T}_\mathfrak{m}/J_\mathfrak{m} = \prod_{g\in\mathcal{N}} \#T_g/J_g. \tag{7.4}$$

Note that one has

$$\mathrm{val}_\ell\left(\prod_{g\in\mathcal{N}} \#T_g/J_g\right) = [\mathbf{F} : \mathbf{F}_\ell] \cdot \sum_{g\in\mathcal{N}} m_g. \tag{7.5}$$

This equality, together with (7.4), contradicts the inequality (7.3). ∎

Corollary 7.5 *Let m_g be defined as in Proposition 7.4. If $\sum_{g\in\mathcal{N}} m_g > 1$ then $J_\mathfrak{m}$ is not principal and $\dim_\mathbf{F} H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2)) > 1$. If in addition $\ell \nmid B_{k-2}$ then $\dim_\mathbf{F} H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\overline{\rho}_\phi(k-2)) > 1$.*

Proof Note that from the proof of Theorem 7.2, we get that $\mathbf{T}_\mathfrak{m}/J_\mathfrak{m} = \mathbf{F}$, even without assuming $\dim_\mathbf{F} H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2)) \leq 1$. Assume that $J_\mathfrak{m}$ is principal. Then, from (7.4) and (7.5), we conclude that $\sum_{g\in\mathcal{N}} m_g = 1$, which contradicts our assumption. Hence, $J_\mathfrak{m}$ is not principal. The Selmer group inequalities now follow from Corollary 7.3. ∎

Remark 7.6 Corollary 7.3 directly ties the cyclicity of the non-critical Selmer group $H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2))$ with the principality of the Eisenstein ideal $J_\mathfrak{m}$. We note that Assumption 6.2(ii) implies the equality $\mathbf{T}_\mathfrak{m}/J_\mathfrak{m} = \mathbf{F}$. Contrary to what one might think, the existence of several forms $g \equiv E^{\phi}_{2,1} \bmod \lambda$ does not preclude this equality. For example, if there are exactly two linearly independent eigenforms $g_1, g_2 \in \mathcal{N}$ with $m_{g_1} = m_{g_2} = 1$ such that $g_1 \not\equiv g_2 \bmod \lambda^2$ then $\mathbf{T}_\mathfrak{m} \cong \mathcal{O} \times_\mathbf{F} \mathcal{O} = \{(a, b) \in \mathcal{O} \times \mathcal{O} \mid a \equiv b \bmod \lambda\}$ and in this case, $J_\mathfrak{m}$ is the maximal ideal, i.e., $\mathbf{T}_\mathfrak{m}/J_\mathfrak{m} = \mathbf{F}$, so Corollary 7.5 applies and $\dim_\mathbf{F} H^1_f(\mathbf{Q}, \mathrm{ad}\,\overline{\rho}_\phi(k-2)) > 1$.

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