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INVARIANT MEASURES OF ULTIMATELY BOUNDED STOCHASTIC PROCESSES

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The author discussed in [4] the ultimate boundedness of a system which is governed by a stochastic differential equation

$$dX(t) = f(t, X(t))dt + G(t, X(t))dW(t) , \qquad t \geqq 0 . \tag{1}$$

In this paper we investigate the invariant measure of an ultimately bounded process assuming stationarity: namely we are interested in a process governed by

$$dX(t) = f(X(t))dt + G(X(t))dW(t) , \qquad t \geqq 0 . \tag{2}$$

where $X(t)$ and $f(x)$ are n-vectors, $G(x)$ is an $n \times m$-matrix, and $W(t)$ is an m-dimensional Wiener process. We assume that $f(x)$ and $G(x)$ satisfy Lipschitz continuity.

Let $X(t)$ be a conservative Feller process defined on the state space $\boldsymbol{R}^n$. The corresponding semi-group $\{T_t\}$ of $X(t)$ is the set of operators T_t on the space $C = \{f(x);$ bounded continuous function on $\boldsymbol{R}^n\}$ and is defined by

$$T_t f(x) = \int_{R^n} f(y)P(t, x, dy) \qquad \text{for } f \in C , \tag{3}$$

where $P(t, x, B)$ is the transition function of $X(t)$.

A process $X(t)$ is said to be p-th ultimately bounded ($p > 0$) if there exists a constant K such that $\overline{\lim}_{t\to\infty} M_x |X(t)|^p \leqq K$ for any x, where M_x means the conditional expectation under the condition $X(0) = x$. An invariant measure μ of a process $X(t)$ means that μ is a positive regular measure and satisfies $\int_{R^n} P(t, x, B)d\mu(x) = \mu(B)$ for any $t > 0$ and Borel set B.

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THEOREM 1. *If a conservative Feller process $X(t)$ is $p(>0)$-th ultimately bounded, then there exists a finite invariant measure of $X(t)$.*

Proof. Fix a point $x \in \boldsymbol{R}^n$ and put

$$\Phi_N(f) = \frac{1}{N}\int_0^N T_t f(x) dt , \qquad N = 1, 2, \cdots . \tag{4}$$

Then $\Phi_N(f)$ is a linear functional on C which satisfies

i) $\Phi_N(f) \geqq 0$, if $f \geqq 0$,

ii) $\Phi_N(1) = 1$.

Therefore Φ_N defines a probability measure on $\boldsymbol{R}^n$, which we will denote by the same notation $\Phi(\cdot)$.

We will prove that the family $\{\Phi_N\}$, $N = 1, 2, \cdots$, is tight, that is

$$\lim_{k\to\infty} \inf_N \Phi_N(S_k) = 1 , \tag{5}$$

where $S_k = \{x ; |x| \leqq k\}$, $k = 1, 2, \cdots$. From the assumption of p-th ultimate boundedness of $X(t)$, there exist two constants K and t_0 such that $M_x|X(t)|^p \leqq K$ for $t \geqq t_0$. Using Tchebychev inequality to this inequality, we have

$$P(t, x, S_k^c) \leqq \frac{K}{k^p} \qquad \text{for } t \geqq t_0 , \tag{6}$$

where S_k^c is the complement of S_k. This inequality is equivalent to

$$P(t, x, S_k) \geqq 1 - \frac{K}{k^p} \qquad \text{for } t \geqq t_0 . \tag{7}$$

Therefore we get the following inequality;

$$\Phi_N(S_k) \geqq \frac{1}{N}\int_{t_0}^N P(t, x, S_k) dt \geqq \frac{1}{N}(N - t_0)\left(1 - \frac{K}{k^p}\right) \qquad \text{for } N > t_0 . \tag{8}$$

From this inequality we know that when any positive number ε is given, there are two constant $k_0(\varepsilon)$ and $N_0(\varepsilon)$ such that

$$\Phi_N(S_k) \geqq 1 - \varepsilon \qquad \text{for } k \geqq k_0 \text{ and } N \geqq N_0 . \tag{9}$$

It is valid that there is a constant $k_1(\varepsilon)$ such that

$$\Phi_N(S_k) \geqq 1 - \varepsilon \qquad \text{for } k \geqq k_1 \text{ and } N = 1, 2, \cdots, N_0 , \tag{10}$$

because $\Phi_N(\cdot)$ is a probability measure. Two inequalities (9) and (10) prove (5), that is, the family $\{\Phi_N\}$, $N = 1, 2, \cdots$, is tight.

From the tightness of $\{\Phi_N\}$, $N=1,2,\cdots$, we can conclude that there are a probability measure $\Phi(\cdot)$ and a subsequence $\{\Phi_{N_m}\}$, $m=1,2,\cdots$, such that

$$\lim_{m\to\infty}\Phi_{N_m}(f)=\Phi(f) \qquad \text{for } f\in C\,. \tag{11}$$

Using this equality and the boundedness of $f\in C$ and $T_t f\in C$, we have

$$\begin{aligned}\Phi(T_t f)&=\lim_{m\to\infty}\frac{1}{N_m}\int_0^{N_m}T_{t+s}f(x)ds\\&=\lim_{m\to\infty}\frac{1}{N_m}\int_0^{N_m}T_sf(x)ds+\frac{1}{N_m}\left(\int_{N_m}^{N_m+t}T_sf(x)ds-\int_0^tT_sf(x)ds\right)\\&=\Phi(f)\qquad\text{for } f\in C \text{ and } t\geqq 0\,.\end{aligned}\tag{12}$$

This equality stands for that $\Phi(\cdot)$ is an invariant measure of $X(t)$.

(Q.E.D.)

Remark 1. We know from the proof of Theorem 1 that the invariant measure of $X(t)$ is not unique. Every starting point x determines an invariant measure.

COROLLARY 1. *If the system* (2) *is* $p(>0)$*-th ultimately bounded, then it has a finite invariant measure.*

Proof. It is well-known that the solution of (2) is a conservative Feller process. (cf. [2]). (Q.E.D.)

COROLLARY 2. *If the system* (2) *is non-degenerate and* $p(>0)$*-th ultimately bounded, then it is positive recurrent.*

Proof. It is proved by W. M. Wonham [6, Appendix] that the system (2) is a diffusion process in the sense of R. Z. Khas'minskii [3] if it is non-degenerate. And R. Z. Khas'miskii proved that a diffusion process is positive recurrent if and only if it is recurrent and has a finite invariant measure ([3], Theorem 3.3 and Lemma 5.3). We already knows that an ultimately bounded process is recurrent ([4], § 5) and Corollary 1 assures the existence of a finite invariant measure. (Q.E.D.)

Remark 2. We know that a $p(>0)$-th ultimately bounded process is weakly recurrent and that an exponentially $p(>1)$-th ultimately bounded process is weakly positive recurrent ([4], § 5). But it is not known whether a $p(>0)$-th ultimately bounded process is weakly positive recurrent or not. Corollary 2 gives us a partial answer of this problem.

THEOREM 2. *Let $X(t)$ be a p-th ultimately bounded Markov process with a finite invariant measure ν. Then it satisfies*

$$\int_{R^n} |x|^p \nu(dx) < \infty .$$

Proof. Put $f(x) = |x|^p$ and $f_n(x) = \chi_{[0,n]}(f(x))$, where χ is a characteristic function. We note that $f_n(x) \in L^1(\boldsymbol{R}^n, \nu)$. From the assumption of p-th ultimate boundedness, there is a constant K' such that

$$\overline{\lim_{t\to\infty}} M_x f(X_t) \leqq K' \qquad \text{for any } x .$$

By the use of Ergodic theorem for Markov process with invariant measure (cf. [5] pp. 388), there exists the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} T_k f_n(x) = f_n^*(x) \qquad (\nu\text{-a.e.}) \tag{13}$$

and

$$E_\nu f_n^*(x) = E_\nu f_n(x) , \tag{14}$$

where $T_k f_n(x) = \int_{R^n} f(y)P(k, x, dy)$ and $E_\nu f_n(x) = \int_{R^n} f_x(x)d\nu(x)$. From the inequality $f_n(x) \leq f(x)$ and the assumption of p-th ultimate boundedness, we have

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{k=1}^{N} T_k f_n(x) \leqq \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{k=1}^{N} T_k f(x) \leqq K' \qquad \text{for any } x \in \boldsymbol{R}^n . \tag{15}$$

From (13) and (15) we obtain $f_n^*(x) \leqq K'$ (ν-a.e.), and from this inequality we have

$$E_\nu f_n^*(x) \leqq K' . \tag{16}$$

The formulas (14), (16) and the fact $f_n(x) \uparrow f(x)$ $(n \to \infty)$ imply that

$$E_\nu f(x) = \lim_{n\to\infty} E_\nu f_n(x) = \lim_{n\to\infty} E_\nu f_n^*(x) \leqq K' . \qquad \text{(Q.E.D.)}$$

COROLLARY. *If $X(t)$ is ∞-th ultimately baunded, then any finite invariant measure ν of $X(t)$ satisfies $E_\nu |x|^p < \infty$ for any $p > 0$.*

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