Can. J. Math., Vol. XXIV, No. 6, 1972, pp. 1198–1216

ON THE RANGES OF CERTAIN FRACTIONAL INTEGRALS

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1. Introduction. Suppose $1 \leqq p < \infty$, μ is real, and denote by $L_{\mu,p}$ the collection of functions f, measurable on $(0, \infty)$, and which satisfy

$$(1.1) \qquad ||f||_{\mu,p} = \left[\int_0^\infty x^{\mu-1}|f(x)|^p dx \right]^{1/p} < \infty.$$

Also denote by $[X]$ the collection of bounded operators from a Banach space X to itself. For $\nu > 0$, $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, let

$$(1.2) \qquad (I_{\nu,\alpha,\xi}f)(x) = \frac{\nu x^{-\nu(\xi+\alpha-1)}}{\Gamma(\alpha)} \int_0^x (x^\nu - t^\nu)^{\alpha-1} t^{\nu\xi-1} f(t)dt,$$

and

$$(1.3) \qquad (J_{\nu,\beta,\eta}f)(x) = \frac{\nu x^{\nu\eta}}{\Gamma(\beta)} \int_x^\infty (t^\nu - x^\nu)^{\beta-1} t^{-\nu(\beta+\eta-1)-1} f(t)dt,$$

where ξ and η are complex numbers. $I_{\nu,\alpha,\xi}$ and $J_{\nu,\beta,\eta}$ are generalizations of the Riemann-Liouville and Weyl fractional integrals respectively, and consequently we shall refer to them as fractional integrals. There is a vast literature of these fractional integrals, particularly for $\nu = 1$ and $\nu = 2$; see [**6**] for an excellent summary, and [**2**] for many applications.

In particular, it is essentially known, and we shall prove below, that if $\mu/p\nu < \operatorname{Re}\xi$, $I_{\nu,\alpha,\xi} \in [L_{\mu,p}]$, and that if $\mu/p\nu > -\operatorname{Re}\eta$, $J_{\nu,\beta,\eta} \in [L_{\mu,p}]$. Hence if $-\operatorname{Re}\eta < \mu/p\nu < \operatorname{Re}\xi$, one can ask, for what values of the parameters appearing is it true that $J_{\nu,\beta,\eta}(L_{\mu,p}) \supseteq I_{\nu,\alpha,\xi}(L_{\mu,p})$, or $I_{\nu,\alpha,\xi}(L_{\mu,p}) \supseteq J_{\nu,\beta,\eta}(L_{\mu,p})$. If it transpires that there are parameter values for which either or both of these inclusions are true, it is then natural to ask whether the operators $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}$ and $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}$ belong to $[L_{\mu,p}]$, for these parameter values, and to investigate their general characters.

The objective of this paper is to answer the questions posed in the previous paragraph. Kober has answered them in a number of special cases. Specifically in [**6**] he answered them for $\mu = \nu = 1$, $p = 2$, while in [**7**] he answered them for $\nu = 1$, $\xi = 1$, $\eta = 0$, $\mu = 1$ and $\mu = p - 1$. Although his restrictions to $\nu = 1$ and a particular value of μ are inessential and can be removed by changes of variable, the other restrictions cannot be so removed. Further, our methods will be quite different from Kober's.

Received November 19, 1971. This research was supported by the National Research Council of Canada, grant no. A4048.

https://doi.org/10.4153/CJM-1972-130-9 Published online by Cambridge University Press

We shall apply our results, as Kober did in [**6**], to the study of the product of two Hankel transformations of different orders and considerably extend the known results. In particular, we obtain a substantial extension of certain of the results of Muckenhoupt and Stein in [**8**], and also an extension of the classical results on the boundedness of the Hilbert transformation of odd functions. Yet another application of the results is to the study of the range of the Hankel transformation.

In section two we prove some lemmas which we will need later about the spaces $L_{\mu,p}$, while in section three we prove a number of results outlining the basic properties of the fractional integrals, and in section four we give a definition of the Mellin transformation and compute its effect on the fractional integrals. The results of sections three and four are essentially known; indeed almost all of them are in [**6**] for $\mu = \nu = 1$. However we include them both for completeness and for later reference, and give very brief proofs.

To answer the questions posed earlier we must introduce an auxiliary operator. This we do in section five, and using our main tool, multiplier operator theory, we find its boundedness and other properties. Section six yields the answers to our questions; for example, using the auxiliary operator defined in the previous section, we show that if

$$-\operatorname{Re}\eta < \mu/p\nu < \operatorname{Re}\xi \quad \text{and} \quad \operatorname{Re}\beta \leqq \operatorname{Re}\alpha,$$

then

$$J_{\nu,\beta,\eta}(L_{\mu,p}) \supseteq I_{\nu,\alpha,\xi}(L_{\mu,p}) \quad \text{and} \quad (J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi} \in [L_{\mu,p}].$$

In section seven we show that the range of validity of these results can be extended by extending the operators in question, and we find explicit formulas for the extended operators. Sections eight and nine are devoted to the applications mentioned earlier, while in section ten we indicate some further problems to be studied.

2. Properties of the spaces $L_{\mu,p}$. In this section we find some properties of the spaces $L_{\mu,p}$ which will simplify a great deal of our later work.

Definition 2.1. If $f \in L_{\mu,p}$, and $-\infty < t < \infty$, let

$$(C_{\mu,p}f)(t) = e^{\mu t/p}f(e^t). \tag{2.1}$$

LEMMA 2.1. *$C_{\mu,p}$ is an isometric isomorphism of $L_{\mu,p}$ onto $L_p(-\infty, \infty)$.*

Proof. $C_{\mu,p}$ is obviously an algebraic isomorphism, and

$$\|C_{\mu,p}f\|_p = \left[\int_{-\infty}^{\infty} |(C_{\mu,p}f)(t)|^p dt\right]^{1/p} = \left[\int_{-\infty}^{\infty} e^{\mu t}|f(e^t)|^p dt\right]^{1/p}$$
$$= \left[\int_0^{\infty} x^{\mu-1}|f(x)|^p dx\right]^{1/p} = \|f\|_{\mu,p}.$$

Definition 2.2. Denote by C_0 the collection of functions continuous on $(0, \infty)$ and vanishing outside some closed interval $[a, b]$ where $0 < a < b < \infty$.

LEMMA 2.2. *C_0 is dense in $L_{\mu,p}$ for any μ and any p satisfying $1 \leqq p < \infty$.*

Proof. Let $f \in L_{\mu,p}$ and $\epsilon > 0$. Since the continuous functions with compact support are dense in $L_p(-\infty, \infty)$, there is a continuous function G and a number $R > 0$ so that $G(x) = 0$ if $|x| > R$, and $||C_{\mu,p}f - G||_p < \epsilon$. Let $g = C_{\mu,p}{}^{-1}G$. Then g is clearly continuous, g vanishes outside (e^{-R}, e^R), and

$$||f - g||_{\mu,p} = ||C_{\mu,p}f - G|| < \epsilon.$$

LEMMA 2.3. *Suppose $f \in L_{\mu_1,p_1} \cap L_{\mu_2,p_2}$, where $1 \leqq p_i < \infty$, $i = 1, 2$ and $\epsilon > 0$. Then there is a $g \in C_0$ so that*

$$||f - g||_{\mu_i,p_i} < \epsilon, \quad i = 1, 2.$$

Proof. Choose $R > 1$ so that

$$\left[\left(\int_0^{R^{-1}} + \int_R^\infty\right) x^{\mu_i - 1} |f(x)|^{p_i} dx\right]^{1/p_i} < \frac{\epsilon}{2}, \quad i = 1, 2.$$

Let $m = \max(\sup x^{\mu_1 - 1}, \sup x^{\mu_2 - 1})$, where the sup is taken over (R^{-1}, R). We may suppose $p_1 \leqq p_2$, and we choose g, continuous and vanishing outside (R^{-1}, R), so that

$$\left[\int_{R^{-1}}^R |f(x) - g(x)|^{p_2} dx\right]^{1/p_2} < K$$

where $K = \min(\epsilon/(2m^{1/p_2}), \epsilon/(2m^{1/p_1}(R - R^{-1})^{(p_2 - p_1)/p_1 p_2}))$. Then

$$\begin{aligned} ||f - g||_{\mu_2,p_2} &< \frac{\epsilon}{2} + \left[\int_{R^{-1}}^R x^{\mu_2 - 1} |f(x) - g(x)|^{p_2} dx\right]^{1/p_2} \\ &< \frac{\epsilon}{2} + m^{1/p_2}\left[\int_{R^{-1}}^R |f(x) - g(x)|^{p_2} dx\right]^{1/p_2} \\ &< \frac{\epsilon}{2} + m^{1/p_2} K \\ &\leqq \epsilon, \end{aligned}$$

and using Hölder's inequality if $p_1 < p_2$,

$$\begin{aligned} ||f - g||_{\mu_1,p_1} &< \frac{\epsilon}{2} + m^{1/p_1}\left[\int_{R^{-1}}^R |f(x) - g(x)|^{p_1} dx\right]^{1/p_1} \\ &\leqq \frac{\epsilon}{2} + m^{1/p_1}\left[\int_{R^{-1}}^R |f(x) - g(x)|^{p_2} dx\right]^{1/p_2} (R - R^{-1})^{(p_2 - p_1)/p_1 p_2} \\ &< \frac{\epsilon}{2} + m^{1/p_1} K (R - R^{-1})^{(p_2 - p_1)/p_1 p_2} \\ &\leqq \epsilon, \end{aligned}$$

as was to be shown.

https://doi.org/10.4153/CJM-1972-130-9 Published online by Cambridge University Press

3. Properties of the fractional integrals. In this section we derive the basic boundedness properties of $I_{\nu,\alpha,\xi}$ and $J_{\nu,\beta,\eta}$, and derive some of their other properties. The boundedness comes as a corollary to the following lemma.

LEMMA 3.1. *Suppose k is measurable on* $(0, \infty)$ *and*

$$A = \int_0^\infty x^{(\mu/p)-1}|k(x)|dx < \infty.$$

For $f \in L_{\mu,p}$, let

$$(Kf)(x) = \int_0^\infty k(x/t)f(t)dt/t.$$

Then $(Kf)(x)$ exists for almost all x, $K \in [L_{\mu,p}]$, and $||K|| \leqq A$.

Proof. This follows by elementary changes of variable from a well-known theorem on convolutions; see [**11**; p. 97, Lemma β].

COROLLARY 3.1. *If $\mu/p\nu < \mathrm{Re}\,\xi$, $I_{\nu,\alpha,\xi} \in [L_{\mu,p}]$. If $\mu/p\nu > -\mathrm{Re}\,\eta$, $J_{\nu,\beta,\eta} \in [L_{\mu,p}]$.*

Proof. For $I_{\nu,\alpha,\xi}$, take

$$k(x) = \begin{cases} 0, & 0 < x < 1, \\ \nu x^{-\nu(\xi+\alpha-1)}(x^\nu - 1)^{\alpha-1}/\Gamma(\alpha), & x > 1, \end{cases}$$

and for $J_{\nu,\beta,\eta}$ take

$$k(x) = \begin{cases} \nu x^{\nu\eta}(1 - x^\nu)^{\alpha-1}/\Gamma(\beta), & 0 < x < 1, \\ 0, & x > 1, \end{cases}$$

in Lemma 3.1.

Our next results establish the main elementary properties of the fractional integrals.

LEMMA 3.2. *If $f \in L_{\mu,p}$, $g(x) = f(x^{1/\nu})$, $h(x) = f(x^{-1})$, then*

$$(I_{\nu,\alpha,\xi}f)(x^{1/\nu}) = (I_{1,\alpha,\xi}g)(x), \text{ if } \mu/p\nu < \mathrm{Re}\,\xi, \tag{3.1}$$

and

$$(J_{\nu,\beta,\eta}f)(x^{1/\nu}) = (J_{1,\beta,\eta}g)(x), \text{ if } \mu/p\nu > -\mathrm{Re}\,\eta. \tag{3.2}$$

Further, if $\mu/p\nu < \mathrm{Re}\,\xi$, then

$$(I_{\nu,\alpha,\xi}f)(x^{-1}) = (J_{\nu,\alpha,\xi}h)(x). \tag{3.3}$$

Proof. The proof follows by elementary changes of variables.

LEMMA 3.3. *If $f \in L_{\mu,p}$, $g \in L_{\mu,p'}$, and $\mu/p\nu < \mathrm{Re}\,\xi$, then*

$$\int_0^\infty (I_{\nu,\alpha,\xi}f)(x)g(x)x^{\mu-1}dx = \int_0^\infty f(x)(J_{\nu,\alpha,\eta}g)(x)x^{\mu-1}dx,$$

where $\eta = \xi - \mu/\nu$.

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Proof. The result is clear by an elementary use of Fubini's theorem.

LEMMA 3.4. *$I_{\nu,\alpha,\xi}$ is one-to-one on $L_{\mu,p}$ if $\mu/p\nu < \operatorname{Re}\xi$ and $J_{\nu,\beta,\eta}$ is one-to-one on $L_{\mu,p}$ if $\mu/p\nu > -\operatorname{Re}\eta$.*

Proof. From (3.1), (3.2), and (3.3) it suffices to prove the result for $I_{1,\alpha,\xi}$, and this follows from a well-known result on the Laplace transformation; see [**12**, Chapter 2, Theorems 12.1 and 6.2].

4. Mellin transformation. In this section we give a definition of the Mellin transformation, state a lemma giving its principal properties, and then we deduce the action of the Mellin Transformation on the fractional integrals.

Definition 4.1. Let $f \in L_{\mu,p}$, where $1 \leqq p \leqq 2$, and let

$$(\mathcal{M}f)((\mu/p) + it) = (C_{\mu,p}f)^{\wedge}(t). \tag{4.1}$$

The transformation $\mathcal{M}$ will be called the Mellin transformation. (Here $\hat{F}$ is the Fourier transform of F, defined by

$$\hat{F}(t) = \int_{-\infty}^{\infty} e^{itu}F(u)du \text{ if } F \in L_1 \cap L_p,$$

and by continuity on $L_p(-\infty, \infty)$ when $1 \leqq p \leqq 2$.)

The reason for denoting the variable of $\mathcal{M}f$ by $(\mu/p) + it$ is that often, for example if $f \in C_0$, the integral

$$\int_0^{\infty} x^{s-1}f(x)dx, \quad s = \sigma + it$$

exists for $\sigma = \mu/p$, and in that case it equals $(\mathcal{M}f)(\mu/p + it)$ a.e. Indeed the integral often exists for an interval of σ-values including $\sigma = \mu/p$, and in that case the integral is a holomorphic function of s for $\operatorname{Re} s$ in the interior of the interval. The main properties of the Mellin transformation are summed up in the following lemma.

LEMMA 4.1. *If $1 \leqq p \leqq 2$, $\mathcal{M}$ is a bounded linear transformation of $L_{\mu,p}$ into $L_{p'}(-\infty, \infty)$. If $p = 2$, $\mathcal{M}$ is unitary if $L_2(-\infty, \infty)$ has measure $dt/2\pi$. If k satisfies the hypotheses of k of Lemma 3.1, and K is as defined in that Lemma, then*

$$\mathcal{M}(Kf) = \mathcal{M}(k)\mathcal{M}(f).$$

Proof. The first two statements follow by elementary changes of variables from well-known results about the Fourier transformation, and the third by changes of variable in the result $(F * G)^{\wedge} = \hat{F}\hat{G}$; see [**11**, 2.1.9].

COROLLARY 4.1. (i) *If $f \in L_{\mu,p}$, $1 \leqq p \leqq 2$, $\mu/p\nu < \operatorname{Re}\xi$, then*

$$\mathcal{M}(I_{\nu,\alpha,\xi}f)(s) = (\Gamma(\xi - (s/\nu))/\Gamma(\xi + \alpha - (s/\nu)))(\mathcal{M}f)(s), \operatorname{Re} s = \mu/p;$$

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(ii) *If* $f \in L_{\mu,p}$, $1 \leqq p \leqq 2$, $\mu/p\nu > -\operatorname{Re}\eta$, *then*

$$\mathcal{M}(J_{\nu,\beta,\eta}f)(s) = (\Gamma(\eta + (s/\nu))/\Gamma(\eta + \beta + (s/\nu)))(\mathcal{M}f)(s), \operatorname{Re} s = \mu/p.$$

Proof. This follows from Lemma 4.1, using the k's of Corollary 3.1.

5. An auxiliary operator. In this section we develop the theory of an auxiliary operator needed to prove our main results. For this we make use of the multiplier operator theory of Fourier transforms. We first introduce and study the function we will use as a multiplier.

Definition 5.1. We define $m_\nu(a, b, c, d)$ (m_ν for short) by

$$m_\nu(a, b, c, d)(s) = \{\Gamma(a + (s/\nu))\Gamma(b - (s/\nu))\}/\{\Gamma(c + (s/\nu))/\Gamma(d - (s/\nu))\},$$

where $s = \sigma + it$ is a complex number.

LEMMA 5.1.

$$|m_\nu(\sigma + it)| \sim |t|^{\operatorname{Re}((a+b)-(c+d))}, \tag{5.1}$$

as $|t| \to \infty$, *uniformly in* σ *for* σ *in any bounded interval, and*

$$(d/dt)m_\nu(\sigma + it) = m_\nu(\sigma + it)\{(\operatorname{Re}((a+b) - (c+d)))/t + O(t^{-2})\}/\nu, \tag{5.2}$$

as $|t| \to \infty$.

Proof. From [**4**, 1.18(6)]

$$|\Gamma(x + iy)| \sim (2\pi)^{1/2}|y|^{x-1/2}e^{-\pi|y|/2}, \text{ as } |y| \to \infty,$$

uniformly in x for x in any bounded interval, and (5.1) follows. Also

$$(d/dt)m_\nu(\sigma + it) = im_\nu(\sigma + it)\{\psi(a + (\sigma + it)/\nu) - \psi(b - (\sigma + it)/\nu) \\ -\psi(c + (\sigma + it)/\nu) + \psi(d - (\sigma + it)/\nu)\}/\nu,$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. But from [**4**, 1.18(7)],

$$\psi(z) = \log z - (2z)^{-1} + O(|z|^{-2}) \text{ as } |z| \to \infty \text{ in } |\arg z| \leqq \pi - \delta.$$

Hence if x and y are real

$$\begin{aligned}\psi(x + iy) &= \log(x + iy) - (2(x + iy))^{-1} + O(|x + iy|^{-2}) \\ &= \log iy - i(x - 1/2)/y + O(y^{-2}) \text{ as } |y| \to \infty,\end{aligned}$$

and thus (5.2) follows.

THEOREM 5.1. *If* $\operatorname{Re}(a + b) \leqq \operatorname{Re}(c + d)$, *and neither* $\operatorname{Re} a + (\sigma/\nu)$ *nor* $\operatorname{Re} b - (\sigma/\nu)$ *is zero or a negative integer, then for* $1 < p < \infty$, $m_\nu(\sigma + it)$ *is an* L_p *multiplier.*

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Proof. Since m_ν is analytic and $\operatorname{Re}(a + b) \leqq \operatorname{Re}(c + d)$, it follows from (5.1) that $m_\nu(\sigma + it)$ is bounded on $-\infty < t < \infty$. Also from (5.2), $(d/dt)m_\nu(\sigma + it) = O(|t|^{-1})$ as $|t| \to \infty$, and hence from [**9**, Chapter 4, Theorem 3], $m_\nu(\sigma + it)$ is an L_p multiplier for $1 < p < \infty$.

Definition 5.2. We denote the multiplier operator generated by $m_\nu(\sigma + it)$ by $T_\sigma(a, b, c, d; \nu)$ (T_σ for short), where a and b satisfy the hypotheses of Theorem 5.1. We define $H_{\mu,p}(a, b, c, d; \nu)$ ($H_{\mu,p}$ for short) by $H_{\mu,p} = (C_{\mu,p})^{-1}T_{\mu/p}C_{\mu,p}$.

LEMMA 5.2. *If a, b, c, and d satisfy the hypotheses of Theorem* 5.1, *then*
(i) $T_\sigma \in [L_p(-\infty, \infty)]$, $1 < p < \infty$, *and*
(ii) *for* $F \in L_p(-\infty, \infty)$, $1 < p \leqq 2$,

$$(T_\sigma F)\hat{\ }(t) = m_\nu(\sigma + it)\hat{F}(t). \tag{5.3}$$

If in addition, $\operatorname{Re}(a + b) = \operatorname{Re}(c + d)$, *and neither* $\operatorname{Re} c + (\sigma/\nu)$ *nor* $\operatorname{Re} d - (\sigma/\nu)$ *is zero or a negative integer, then* T_σ *is a one-to-one mapping of* $L_p(-\infty, \infty)$ *onto itself, and*

$$(T_\sigma(a, b, c, d; \nu))^{-1} = (T_\sigma(c, d, b, a; \nu)). \tag{5.4}$$

Proof. (i) is immediate from the definition of T_σ. (ii) is immediate for $p = 2$ from [**9**, Chapter IV, 3.1] and then follows immediately for $1 < p < 2$ from the fact that both sides represent a bounded transformation of $L_p(-\infty, \infty)$ into $L_{p'}(-\infty, \infty)$, coinciding on a dense subset of $L_p(-\infty, \infty)$.
From (5.3), we have for $p = 2$ that

$$T_\sigma(c, d, b, a; \nu)T_\sigma(a, b, c, d; \nu) = I,$$

and thus by continuity for $1 < p < \infty$, so that, under the hypotheses, T_σ is a one-to-one mapping of $L_p(-\infty, \infty)$ onto itself.

COROLLARY 5.2. *If a, b, c, and d satisfy the hypotheses of Theorem* 5.1 *with* $\sigma = \mu/p$, *then*
(i) $H_{\mu,p} \in [L_{\mu,p}]$, $1 < p < \infty$,
(ii) *for* $f \in L_{\mu,p}$, $1 < p \leqq 2$,

$$(\mathscr{M} H_{\mu,p}f)(s) = m_\nu(s)(\mathscr{M}f)(s), \operatorname{Re} s = \mu/p. \tag{5.5}$$

If in addition $\operatorname{Re}(a + b) = \operatorname{Re}(c + d)$, *and neither* $\operatorname{Re} c + (\mu/p\nu)$ *nor* $\operatorname{Re} d - (\mu/p\nu)$ *is zero or a negative integer, then* $H_{\mu,p}$ *is a one-to-one mapping of* $L_{\mu,p}$ *onto itself, and*

$$(H_{\mu,p}(a, b, c, d; \nu))^{-1} = H_{\mu,p}(c, d, a, b; \nu). \tag{5.6}$$

$H_{\mu,p}$ depends explicitly on μ and p, but actually this dependence is not as essential as it appears, as the following theorem shows.

THEOREM 5.2. *Suppose $m_\nu(s)$ has no poles in the strip $\sigma_1 < \operatorname{Re} s < \sigma_2$, and suppose $f \in L_{\mu_i,p_i}$, $i = 1, 2$, where $1 < p_i < \infty$ and $\sigma_1 < \mu_i/p_i < \sigma_2$. Then*

$$H_{\mu_1,p_1}f = H_{\mu_2,p_2}f \quad a.e.$$

Proof. Suppose first that $f \in C_0$, and let

$$F(s) = \int_0^\infty x^{s-1}f(x)dx.$$

Clearly F is entire. Now

$$C_{\mu_1,p_1}H_{\mu_1,p_1}f = T_{\mu_1/p_1}C_{\mu_1,p_1}f.$$

Clearly $C_{\mu_1,p_1}f \in L_2(-\infty, \infty)$, and hence from (5.3)

$$(T_{\mu_1/p_1}C_{\mu_1,p_1}f)^\wedge(t) = m_\nu(\mu/p + it)(C_{\mu_1,p_1}f)^\wedge(t).$$

But $C_{\mu_1,p_1}f$ is clearly also in $L_1(-\infty, \infty)$ and hence

$$(C_{\mu_1,p_1}f)^\wedge(t) = \int_{-\infty}^\infty e^{itu}(C_{\mu_1,p_1}f)(u)du = \int_{-\infty}^\infty e^{(\mu_1 u/p_1)+itu}f(e^u)du$$
$$= \int_0^\infty x^{(\mu_1/p_1)+it-1}f(x)dx = F((\mu_1/p_1) + it).$$

Hence from [**11**, Theorem 48],

$$(C_{\mu_1,p_1}H_{\mu_1,p_1}f)(u) = \lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} e^{-iut}m_\nu((\mu_1/p_1) + it)F((\mu_1/p_1) + it)dt,$$

the limit being in the topology of $L_2(-\infty, \infty)$. But then there is a sequence $\{R_j\}$, with $\lim R_j = \infty$, such that

$$(C_{\mu_1,p_1}H_{\mu_1,p_1}f)(u) = \lim_{j\to\infty} \frac{1}{2\pi}\int_{-R_j}^{R_j} e^{-iut}m_\nu((\mu_1/p_1) + it)F((\mu_1/p_1) + it)dt$$

almost everywhere on $(-\infty, \infty)$, or

$$(H_{\mu_1,p_1}f)(x) = \lim_{j\to\infty} \frac{1}{2\pi}\int_{-R_j}^{R_j} x^{-(\mu_1/p_1)-it}m_\nu((\mu_1/p_1) + it)F((\mu_1/p_1) + it)dt$$
$$= \lim_{j\to\infty} \frac{1}{2\pi i}\int_{(\mu_1/p_1)-iR_j}^{(\mu_1/p_1)+iR_j} x^{-s}m_\nu(s)F(s)ds,$$

almost everywhere on $(0, \infty)$.

Similarly

$$(C_{\mu_2,p_2}H_{\mu_2,p_2}f)(u) = \lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} e^{-iut}m_\nu((\mu_2/p_2) + it)F((\mu_2/p_2) + it)dt$$
$$= \lim_{j\to\infty} \frac{1}{2\pi}\int_{-R_j}^{R_j} e^{-iut}m_\nu((\mu_2/p_2) + it)F((\mu_2/p_2) + it)dt,$$

the limits being in the topology of $L_2(-\infty, \infty)$. But then there is a subsequence $\{S_j\}$ of $\{R_j\}$ such that

$$(C_{\mu_2,p_2}H_{\mu_2,p_2}f)(u) = \lim_{j\to\infty} \frac{1}{2\pi}\int_{-S_j}^{S_j} e^{-iut}m_\nu((\mu_2/p_2)+it)F((\mu_2/p_2)+it)dt$$

almost everywhere on $(-\infty, \infty)$, or

$$(H_{\mu_2,p_2}f)(x) = \lim_{j\to\infty} \frac{1}{2\pi}\int_{-S_j}^{S_j} x^{-(\mu_2/p_2)-it}m_\nu((\mu_2/p_2)+it)F((\mu_2/p_2)+it)dt$$

$$= \lim_{j\to\infty} \frac{1}{2\pi i}\int_{(\mu_2/p_2)-iS_j}^{(\mu_2/p_2)+iS_j} x^{-s}m_\nu(s)F(s)ds,$$

almost everywhere on $(0, \infty)$.

Hence, since $\{S_j\}$ is a subsequence of $\{R_j\}$, we have for almost all x,

$$(5.7)\quad (H_{\mu_1,p_1}f)(x) - (H_{\mu_2,p_2}f)(x) = \lim_{j\to\infty} \frac{1}{2\pi i}\left[\int_{(\mu_1/p_1)-iS_j}^{(\mu_1/p_1)+iS_j} - \int_{(\mu_2/p_2)-iS_j}^{(\mu_2/p_2)+iS_j}\right]x^{-s}m(s)ds.$$

If $\mu_1/p_1 = \mu_2/p_2$, the right hand side of this equation is zero, and $(H_{\mu_1,p_1}f)(x) = (H_{\mu_2,p_2}f)(x)$ a.e. If $\mu_1/p_1 \neq \mu_2/p_2$, let γ be the rectangle with vertices $(\mu_1/p_1) \pm iS_j$ and $(\mu_2/p_2) \pm iS_j$. Then since γ is contained in the strip $\sigma_1 < \operatorname{Re} s < \sigma_2$, $m_\nu(s)$ is holomorphic in this strip, and F is entire,

$$\int_\gamma x^{-s}m_\nu(s)F(s)ds = 0,$$

from which (5.7) can be written

$$(H_{\mu_1,p_1}f)(x) - (H_{\mu_2,p_2}f)(x) = \lim_{j\to\infty}\left[\frac{1}{2\pi i}\int_{\mu_1/p_1}^{\mu_2/p_2} x^{-\sigma-iS_j}m_\nu(\sigma+iS_j)F(\sigma+iS_j)d\sigma - \int_{\mu_1/p_1}^{\mu_2/p_2} x^{-\sigma+iS_j}m_\nu(\sigma-iS_j)F(\sigma-iS_j)d\sigma\right].$$

But these last two integrals tend to zero as $j\to\infty$. For, from Lemma 5.1, $m(\sigma \pm iS_j)$ is uniformly bounded in σ on the interval of integration, as obviously is $x^{-\sigma}$ since $x > 0$, and $F(\sigma \pm iS_j) \to 0$ as $j \to \infty$ by the Riemann Lebesgue lemma since $f \in C_0$. Hence

$$(H_{\mu_1,p_1}f)(x) = (H_{\mu_2,p_2}f)(x) \quad \text{a.e.}$$

Now if $f \in L_{\mu_1,p_1} \cap L_{\mu_2,p_2}$, then by Lemma 2.3, there is a sequence $\{g_n\}$ of functions in C_0 such that $\|f - g_n\|_{\mu_i,p_i} \to 0$, $i = 1, 2$. Then $\|H_{\mu_1,p_1}f - H_{\mu_1,p_1}g_n\|_{\mu_1,p_1} \to 0$, and hence there is a subsequence $\{n_j\}$ such that $\lim_{j\to\infty} H_{\mu_1,p_1}g_{n_j} = H_{\mu_1,p_1}f$ a.e. But $\|H_{\mu_2,p_2}f - H_{\mu_2,p_2}g_{n_j}\|_{\mu_2,p_2} \to 0$, and hence there is a subsequence $\{n_j'\}$ of $\{n_j\}$ such that $\lim_{j\to\infty} H_{\mu_2,p_2}g_{n_j'} = H_{\mu_2,p_2}f$ a.e.

But since $g_{n_j'} \in C_0$, $H_{\mu_1,p_1}g_{n_j'} = H_{\mu_2,p_2}g_{n_j'}$ a.e. and hence

$$H_{\mu_1,p_1}f = \lim_{j\to\infty} H_{\mu_1,p_1}g_{n_j'} = \lim_{j\to\infty} H_{\mu_2,p_2}g_{n_j'} = H_{\mu_2,p_2}f \quad \text{a.e.}$$

6. Main results. In this section we answer, in the theorem below, the questions posed in the introduction. We first prove a preliminary lemma.

LEMMA 6.1. *Suppose* $-\operatorname{Re}\eta < \mu/p\nu < \operatorname{Re}\xi$, *where* $1 < p < \infty$. *Then* (i) *if* $\operatorname{Re}\beta \leqq \operatorname{Re}\alpha$, *and* $f \in L_{\mu,p}$

$$J_{\nu,\beta,\eta}H_{\mu,p}(\eta + \beta, \xi, \eta, \xi + \alpha; \nu)f = I_{\nu,\alpha,\xi}f, \quad a.e., \tag{6.1}$$

and (ii) *if* $\operatorname{Re}\alpha \leqq \operatorname{Re}\beta$, *and* $f \in L_{\mu,p}$,

$$I_{\nu,\alpha,\xi}H_{\mu,p}(\eta, \xi + \alpha, \eta + \beta, \xi; \nu)f = J_{\nu,\beta,\eta}f, \quad a.e. \tag{6.2}$$

Proof. It suffices to prove (6.1) and (6.2) for $f \in C_0$. Choose μ_0 so that $-\operatorname{Re}\eta < \mu_0/2\nu < \operatorname{Re}\xi$. Then if $\operatorname{Re}\beta \leqq \operatorname{Re}\alpha$,

$$m_\nu(\eta + \beta, \xi, \eta, \xi + \alpha)(s) = \frac{\Gamma(\eta + \beta + (s/\nu))\Gamma(\xi - (s/\nu))}{\Gamma(\eta + (s/\nu))\Gamma(\xi + \alpha - (s/\nu))},$$

so that $m_\nu(\eta + \beta, \xi, \eta, \xi + \alpha)(s)$ is holomorphic in the strip $-\nu \operatorname{Re}\eta < \operatorname{Re} s < \nu \operatorname{Re}\xi$, indeed in the strip $-\nu \operatorname{Re}(\eta + \beta) < \operatorname{Re} s < \nu \operatorname{Re}\xi$, and thus by Theorem 5.2,

$$H_{\mu,p}(\eta + \beta, \xi, \eta, \xi + \alpha; \nu)f = H_{\mu_0,2}(\eta + \beta, \xi, \eta, \xi + \alpha; \nu)f \quad \text{a.e.}$$

Hence to prove (6.1), it suffices to show

$$J_{\nu,\beta,\eta}H_{\mu_0,2}(\eta + \beta, \xi, \eta, \xi + \alpha; \nu)f = I_{\nu,\alpha,\xi}f \quad \text{a.e.}$$

But from Corollaries 4.1 and 5.2, if $\operatorname{Re} s = \mu_0/2$

$$\begin{aligned}\mathcal{M}(J_{\nu,\beta,\eta}&H_{\mu_0,2}(\eta + \beta, \xi, \eta, \xi + \alpha; \nu)f)(s)\\ &= \frac{\Gamma(\eta + (s/\nu))}{\Gamma(\eta + \beta + (s/\nu))} m_\nu(\eta + \beta, \xi, \eta, \xi + \alpha)(s)(\mathcal{M}f)(s)\\ &= \frac{\Gamma(\xi - (s/\nu))}{\Gamma(\xi + \alpha - (s/\nu))}(\mathcal{M}f)(s) = (\mathcal{M} I_{\nu,\alpha,\xi}f)(s),\end{aligned}$$

and (6.1) follows.

Similarly, if $\operatorname{Re}\alpha \leqq \operatorname{Re}\beta$, $m_\nu(\eta, \xi + \alpha, \eta + \beta, \xi)(s)$ is holomorphic in the strip $-\nu \operatorname{Re}\eta < \operatorname{Re} s < \nu \operatorname{Re}\xi$, indeed in the strip $-\nu \operatorname{Re}\eta < \operatorname{Re} s < \nu \operatorname{Re}(\xi + \alpha)$, and thus by Theorem 5.2,

$$H_{\mu,p}(\eta, \xi + \alpha, \eta + \beta, \xi; \nu)f = H_{\mu_0,2}(\eta, \xi + \alpha, \eta + \beta, \xi; \nu)f,$$

and (6.2) follows from this using the Mellin transformation and Corollaries 4.1 and 5.2 as did (6.1).

THEOREM 6.1. *Suppose* $-\operatorname{Re}\eta < \mu/p\nu < \operatorname{Re}\xi$, *where* $1 < p < \infty$. *Then*

(i) *if* $\operatorname{Re}\beta \leqq \operatorname{Re}\alpha$, $J_{\nu,\beta,\eta}(L_{\mu,p}) \supseteq I_{\nu,\alpha,\xi}(L_{\mu,p})$ *and* $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi} \in [L_{\mu,p}]$; *and*

(ii) *if* $\operatorname{Re}\alpha \leqq \operatorname{Re}\beta$, $I_{\nu,\alpha,\xi}(L_{\mu,p}) \supseteq J_{\nu,\beta,\eta}(L_{\mu,p})$ *and* $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta} \in [L_{\mu,p}]$.

In the respective cases $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}$ *and* $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}$ *are one-to-one. If further* $\mathrm{Re}\,\alpha = \mathrm{Re}\,\beta$, *both* $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}$ *and* $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}$ *map* $L_{\mu,p}$ *onto itself; both are unitary on* $L_{\mu,2}$ *if* $\alpha = \bar{\beta}$ *and* $\xi = \bar{\eta} + (\mu/\nu)$.

Proof. Suppose $\mathrm{Re}\,\beta \leqq \mathrm{Re}\,\alpha$, and let $f \in L_{\mu,p}$. We must show that $I_{\nu,\alpha,\xi}f \in J_{\nu,\beta,\eta}(L_{\mu,p})$. But letting $g = H_{\mu,p}(\eta + \beta, \xi, \eta, \xi + \alpha; \nu)f$, then from Lemma 6.1 and Corollary 5.2, $g \in L_{\mu,p}$, and $I_{\nu,\alpha,\xi}f = J_{\nu,\beta,\eta}g \in J_{\nu,\beta,\eta}(L_{\mu,p})$. Further, from Lemmas 6.1 and 3.4

$$(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi} = H_{\mu,p}(\eta + \beta, \xi, \eta, \xi + \alpha; \nu) \in [L_{\mu,p}].$$

Suppose next $\mathrm{Re}\,\alpha \leqq \mathrm{Re}\,\beta$, and let $f \in L_{\mu,p}$. We must show that $J_{\nu,\beta,\eta}f \in I_{\nu,\alpha,\xi}(L_{\mu,p})$. But letting $h = H_{\mu,p}(\eta, \xi + \alpha, \eta + \beta, \xi; \nu)f$, then from Lemma 6.1 and Corollary 5.2, $h \in L_{\mu,p}$, and $J_{\nu,\beta,\eta}f = I_{\nu,\alpha,\xi}h \in I_{\nu,\alpha,\xi}(L_{\nu,p})$. Further, from Lemmas 6.1 and 3.4

$$(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta} = H_{\mu,p}(\eta, \xi + \alpha, \eta + \beta, \xi; \nu) \in [L_{\mu,p}].$$

That $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}$ and $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}$ are one-to-one follows from Lemma 3.4; that they are onto follows from Corollary 5.2, for as shown above

$$(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi} = H_{\mu,p}(a, b, c, d; \nu)$$

with $a = \eta + \beta, b = \xi, c = \eta, d = \xi + \alpha$, and $\mathrm{Re}(a + b) = \mathrm{Re}(\xi + \eta + \beta) = \mathrm{Re}(\xi + \eta + \alpha) = \mathrm{Re}(c + d)$, and similarly

$$(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta} = H_{\mu,p}(a', b', c', d'; \nu)$$

with

$$a' = \eta, b' = \xi + \alpha, c' = \eta + \beta, d' = \xi,$$

and

$$\mathrm{Re}(a' + b') = \mathrm{Re}(\xi + \eta + \alpha) = \mathrm{Re}(\xi + \eta + \beta) = \mathrm{Re}(c' + d').$$

If $\alpha = \bar{\beta}$ and $\xi = \bar{\eta} + \mu/\nu$,

$$\begin{aligned}|m_\nu(\eta + \beta, \xi, \eta, \xi + \alpha)(\mu/2) + it)| \\ &= \left|\frac{\Gamma(\eta + \beta + ((\mu/2) + it)/\nu))\Gamma(\bar{\eta} + (\mu/\nu) - ((\mu/2) + it)/\nu))}{\Gamma(\eta + ((\mu/2) + it)/\nu))\Gamma(\bar{\eta} + \bar{\beta} + (\mu/\nu) - ((\mu/2) + it)/\nu))}\right| \\ &= 1,\end{aligned}$$

so that from Lemmas 4.1 and 5.2, if $f \in L_{\mu,2}$

$$\begin{aligned}||(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}f||_{\mu,2} &= ||H_{\mu,2}(\eta + \beta, \xi, \eta, \xi + \alpha; \nu)f||_{\mu,2} \\ &= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathscr{M}H_{\mu,p}(\eta + \beta, \xi, \eta, \xi + \alpha; \nu)f)\left(\frac{\mu}{2} + it\right)\right|^2 dt\right]^{1/2} \\ &= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|m_\nu(\eta + \beta, \xi, \eta, \xi + \alpha)\left(\frac{\mu}{2} + it\right)(\mathscr{M}f)\left(\frac{\mu}{2} + it\right)\right|^2 dt\right]^{1/2} \\ &= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|(\mathscr{M}f)\left(\frac{\mu}{2} + it\right)\right|^2 dt\right]^{1/2} = ||f||_{\mu,2},\end{aligned}$$

and $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}$ is unitary; similarly for $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}$.

https://doi.org/10.4153/CJM-1972-130-9 Published online by Cambridge University Press

COROLLARY 6.1. *Suppose* $-\mathrm{Re}\,\eta < \mu/p\nu < \mathrm{Re}\,\xi$, *where* $1 < p < \infty$. *Then if* $f \in L_{\mu,p}$ *and* $\mathrm{Re}\,\beta \leqq \mathrm{Re}\,\alpha$, $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}f = H_{\mu,p}(\eta+\beta, \xi, \eta, \xi+\alpha;\nu)f$; *if* $\mathrm{Re}\,\alpha \geqq \mathrm{Re}\,\beta$, $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}f = H_{\mu,p}(\eta, \xi+\alpha, \eta+\beta, \xi;\nu)f$.

7. Extensions and representations. In this section we will extend $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}$ and $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}$. Since

$$(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi} = H_{\mu,p}(\eta+\beta, \xi, \eta, \xi+\alpha;\nu)$$

on $L_{\mu,p}$ for $-\mathrm{Re}\,\mu < \mu/p\nu < \mathrm{Re}\,\xi$, it seems natural to extend the operator by this equation for all parameter values for which the right hand side makes sense. However, this might lead to inconsistent results; for if $f \in L_{\mu_1,p_1} \cap L_{\mu_2,p_2}$, there is no guarantee that $H_{\mu_1,p_1}f = H_{\mu_2,p_2}f$. But $m_\nu(\eta+\beta, \xi, \eta, \xi+\alpha)(s)$ is holomorphic in the strip $-\nu\,\mathrm{Re}(\eta+\beta) < \mathrm{Re}\,s < \nu\,\mathrm{Re}\,\xi$, and thus by Theorem 5.2, if μ_i/p_i belong to this strip for $i = 1, 2$, $H_{\mu_1,p_1}f = H_{\mu_2,p_2}f$, and thus we can extend $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}$ consistently to $L_{\mu,p}$ for $-\mathrm{Re}(\eta+\beta) < \mu/p\nu < \mathrm{Re}\,\xi$. The following definition covers the case, as well as $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}$.

Definition 7.1. If $-\mathrm{Re}(\beta+\eta) < \mu/p\nu < \mathrm{Re}\,\xi$, we extend $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}$ to $L_{\mu,p}$ by $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi} = H_{\mu,p}(\eta+\beta, \xi, \eta, \xi+\alpha;\nu)$ and if $-\mathrm{Re}\,\eta < \mu/p\nu < \mathrm{Re}(\xi+\alpha)$, we extend $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}$ to $L_{\mu,p}$ by

$$(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi} = H_{\mu,p}(\eta, \xi+\alpha, \eta+\beta, \xi;\nu).$$

The properties of the extended operators are covered in the following theorem.

THEOREM 7.1. (i) *If* $-\mathrm{Re}(\eta+\beta) < \mu/p\nu < \mathrm{Re}\,\xi$, $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi} \in [L_{\mu,p}]$. *It is one-to-one if* $1 < p \leqq 2$, *or if* $-\mathrm{Re}\,\eta < \mu/p\nu < \mathrm{Re}\,\xi$; *if* $-\mathrm{Re}\,\eta < \mu/p\nu < \mathrm{Re}\,\xi$ *and* $\mathrm{Re}\,\alpha = \mathrm{Re}\,\beta$ *it is onto. It is unitary on* $L_{\mu,2}$ *if* $-\mathrm{Re}\,\eta < \mu/2\nu < \mathrm{Re}\,\xi$, $\alpha = \bar{\beta}$ *and* $\xi = \bar{\eta} + (\mu/\nu)$.

(ii) *If* $-\mathrm{Re}\,\eta < \mu/p\nu < \mathrm{Re}(\xi+\alpha)$, $(I_{\nu,\alpha,\xi})^{-1}J_{\beta,\eta} \in [L_{\mu,p}]$. *It is one-to-one if* $1 < p \leqq 2$ *or if* $-\mathrm{Re}\,\eta < \mu/p\nu < \mathrm{Re}\,\xi$; *if* $-\mathrm{Re}\,\eta < \mu/p\nu < \mathrm{Re}\,\xi$ *and* $\mathrm{Re}\,\alpha = \mathrm{Re}\,\beta$ *it is onto. It is unitary on* $L_{\mu,2}$ *if* $-\mathrm{Re}\,\eta < \mu/2\nu < \mathrm{Re}\,\xi$, $\alpha = \bar{\beta}$ *and* $\xi = \bar{\eta} + (\mu/\nu)$.

Proof. All the statements follow from Theorem 6.1 or Corollary 5.2 except the one-to-one-ness when $1 < p \leqq 2$. But from Corollary 5.2, if $f \in L_{\mu,p}$, where $1 < p \leqq 2$ and $-\mathrm{Re}(\eta+\beta) < \mu/p\nu < \mathrm{Re}\,\xi$, then

$$\mathscr{M}((J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}f)(s) = m_\nu(\eta+\beta, \xi, \eta, \xi+\alpha)(s)(\mathscr{M}f)(s),\ \mathrm{Re}\,s = \mu/p,$$

and hence if $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}f = 0$, since the zeros of m_ν are isolated, $(\mathscr{M}f)(\mu/p) + it) = 0$ a.e. and $f = 0$ a.e.; similarly for $(I_{\nu,\alpha,\xi})^{-1}J_{\beta,\eta}$.

We now give a theorem representing $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}$ and $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}$ as integral operators. First we need a definition and preliminary lemma.

Definition 7.2. For $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, ν real, define $k_\nu(\xi, \eta, \alpha, \beta)$, k_ν for short, by

$$(7.1)\quad k_\nu(\xi, \eta, \alpha, \beta)(x) = \nu\Gamma(\xi+\eta+\beta)\begin{cases} x^{\nu(\eta+\beta)}F(\xi+\eta+\beta, \beta+1; \xi+\eta+\alpha+\beta; x^\nu)/ \\ \qquad (\Gamma(\xi+\eta+\alpha+\beta)\Gamma(-\beta)), \quad 0 < x < 1, \\ x^{-\nu\xi}F(\xi+\eta+\beta, 1-\alpha; \xi+\eta; x^{-\nu})/ \\ \qquad (\Gamma(\xi+\eta)\Gamma(\alpha)), \quad x > 1, \end{cases}$$

where $F(a, b; c; z)$ is the Gauss hypergeometric function. Also let

$$(7.2)\qquad l_\nu(\xi, \eta, \alpha, \beta)(x) = k_\nu(\eta, \xi, \beta, \alpha)(x^{-1}).$$

LEMMA 7.1. *If* $-\nu \operatorname{Re}(\eta+\beta) < \operatorname{Re} s < \nu \operatorname{Re}\xi$, *and* $0 < \operatorname{Re}\beta < \operatorname{Re}\alpha$ *then*

$$(7.3)\quad (\mathcal{M} k_\nu)(s) = \{\Gamma(\eta+\beta+(s/\nu))\Gamma(\xi-(s/\nu))\}/\{\Gamma(\eta+(s/\nu))\Gamma(\xi+\alpha-(s/\nu))\}$$

while if $-\nu \operatorname{Re}\eta < \operatorname{Re} s < \nu \operatorname{Re}(\xi+\alpha)$, $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta$, then

$$(7.4)\quad (\mathcal{M} l_\nu)(s) = \{\Gamma(\eta+(s/\nu))\Gamma(\xi+\alpha-(s/\nu)\}/\{\Gamma(\eta+\beta+(s/\nu))\Gamma(\xi-(s/\nu))\}.$$

Proof. The existence of $\mathcal{M} k_\nu$ follows since clearly $k_\nu(x) = O(x^{\nu \operatorname{Re}(\eta+\beta)})$ as $x \to 0+$, $k_\nu(x) = O(x^{-\nu \operatorname{Re}\xi})$ as $x \to \infty$, while from [**4**, 2.1.4(23) and 2.1.3(14)] $k_\nu(x) = O(|1-x|^{\operatorname{Re}(\alpha-\beta)-1})$ as $x \to 1$, if $\operatorname{Re}(\alpha-\beta) < 1$, $k_\nu(x) = 0(|log|1-x||)$ as $x \to 1$ if $\operatorname{Re}(\alpha-\beta) = 1$, and $k_\nu(x) = O(1)$ as $x \to 1$, if $\operatorname{Re}(\alpha-\beta) > 1$.

Now if $-\nu \operatorname{Re}(\eta+\beta) < \operatorname{Re} s < \nu \operatorname{Re}\xi$, using [**4**, 2.4(5)]

$$\begin{aligned}
(\mathcal{M}k_\nu)(s) &= \int_0^\infty x^{s-1}k_\nu(x)dx \\
&= \frac{\nu\Gamma(\xi+\eta+\beta)}{\Gamma(\xi+\eta+\alpha+\beta)\Gamma(-\beta)} \\
&\qquad \times \int_0^1 x^{s+\nu(\eta+\beta)-1}F(\xi+\eta+\beta, \beta+1; \xi+\eta+\alpha+\beta; x^\nu)dx \\
&\qquad + \frac{\nu\Gamma(\xi+\eta+\beta)}{\Gamma(\xi+\eta)\Gamma(\alpha)}\int_1^\infty x^{s-\nu\xi-1}F(\xi+\eta+\beta, 1-\alpha; \xi+\eta; x^{-\nu})dx \\
&= \frac{\Gamma(\xi+\eta+\beta)}{\Gamma(\xi+\eta+\alpha+\beta)\Gamma(-\beta)} \\
&\qquad \times \int_0^1 x^{\eta+\beta+s-1}F(\xi+\eta+\beta, \beta+1; \xi+\eta+\alpha+\beta; x)dx \\
&\qquad + \frac{\Gamma(\xi+\eta+\beta)}{\Gamma(\xi+\eta)\Gamma(\alpha)}\int_0^1 x^{\xi-s-1}F(\xi+\eta+\beta, 1-\alpha; \xi+\eta; x)dx \\
&= \{\Gamma(\eta+(s/\nu))\Gamma(\xi+\alpha-(s/\nu))\}/\{\Gamma(\eta+\beta+(s/\nu))\Gamma(\xi-(s/\nu))\},
\end{aligned}$$

and (7.3) follows. (7.4) follows from (7.3) by change of variables.

THEOREM 7.2. *If* $\mathrm{Re}\,\beta \leqq \mathrm{Re}\,\alpha$ *and* $f \in L_{\mu,p}$, *where* $-\mathrm{Re}(\eta+\beta) < \mu/p\nu < \mathrm{Re}\,\xi$, *then for* $x > 0$

$$(7.5)\quad \nu \int_0^x t^{\nu(\xi+\alpha)-1}((J_{\nu,\beta,\eta})^{-1} I_{\nu,\alpha,\xi} f)(t)dt = x^{\nu(\xi+\alpha)} \int_0^\infty k_\nu(\xi,\eta,\alpha+1,\beta)(x/t)f(t)\frac{dt}{t};$$

if $\mathrm{Re}\,\beta < \mathrm{Re}\,\alpha$,

$$(7.6)\qquad ((J_{\nu,\beta,\eta})^{-1} I_{\alpha,\xi} f)(x) = \int_0^\infty k_\nu(\xi,\eta,\alpha,\beta)(x/t)f(t)\frac{dt}{t}, \text{ a.e.}$$

If $\mathrm{Re}\,\alpha \leqq \mathrm{Re}\,\beta$, *and* $f \in L_{\mu,p}$, *where* $-\mathrm{Re}\,\eta < \mu/p\nu < \mathrm{Re}(\xi+\alpha)$, *then for* $x > 0$

$$(7.7)\quad \nu \int_x^\infty t^{-\nu(\beta+\eta)-1}((I_{\nu,\alpha,\xi})^{-1} J_{\nu,\beta,\eta} f)(t)dt = x^{-\nu(\eta+\beta)} \int_0^\infty l_\nu(\xi,\eta,\alpha,\beta+1)(x/t)f(t)\frac{dt}{t};$$

if $\mathrm{Re}\,\alpha < \mathrm{Re}\,\beta$

$$(7.8)\qquad ((I_{\nu,\alpha,\xi})^{-1} J_{\nu,\beta,\eta} f)(x) = \int_0^\infty l_\nu(\xi,\eta,\alpha,\beta)(x/t)f(t)\frac{dt}{t}, \text{ a.e.}$$

Proof. It suffices to prove (7.5) for $f \in C_0$. For if (7.5) is divided by $x^{\nu(\xi+\alpha)}$, the left hand side is just $I_{\nu,1,\xi+\alpha}((J_{\nu,\beta,\eta})^{-1} I_{\nu,\alpha,\xi})f$ and by Theorem 6.1 and Corollary 3.1, $I_{\nu,1,\xi+\alpha}((J_{\nu,\beta,\eta})^{-1} I_{\nu,\alpha,\xi})$ is a member of $[L_{\mu,p}]$, while the right hand side is a member of $[L_{\mu,p}]$ by Lemmas 3.1 and 7.1. But from Corollaries 4.1 and 5.2, the Mellin transformation of the left hand side of (7.5), after division by $x^{\nu(\xi+\alpha)}$ is

$$\begin{aligned}
&(\Gamma(\xi+\alpha-(s/\nu))/\Gamma(\xi+\alpha+1-(s/\nu))\mathscr{M}((J_{\nu,\beta,\eta})^{-1} I_{\alpha,\xi} f)(s) \\
&\quad = (\Gamma(\xi+\alpha-(s/\nu))/\Gamma(\xi+\alpha+1-(s/\nu)))\{(\Gamma(\eta+\beta \\
&\qquad + (s/\nu))\Gamma(\xi-(s/\nu)))/(\Gamma(\eta+(s/\nu))\Gamma(\xi+\alpha-(s/\nu)))\}(\mathscr{M}f)(s) \\
&\quad = \{(\Gamma(\eta+\beta+(s/\nu))\Gamma(\xi-(s/\nu)))/(\Gamma(\eta \\
&\qquad\qquad + (s/\nu))\Gamma(\xi+\alpha+1-(s/\nu)))\}(\mathscr{M}f)(s),
\end{aligned}$$

where $\mathrm{Re}\,s = \mu_0/2$ for some μ_0, $-\mathrm{Re}(\eta+\beta) < \mu_0/2\nu < \mathrm{Re}\,\xi$. But from Lemmas 4.1 and 7.1 the Mellin transformation of the right hand side of (7.5) is the same, so that (7.5) is true a.e. However the left hand side is clearly continuous and since, as is easy to prove,

$$\int_0^\infty t^{(p'\mu/p)-1}|k_\nu(t)|^{p'}dt < \infty,$$

it follows from Hölder's inequality that so is the right hand side, and (7.5)

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follows. (7.6) follows from (7.5) for it is easy to show that

$$(d/dx)x^{\nu(\xi+\alpha)}k_\nu(\xi, \eta, \alpha + 1, \beta)(x/t) = \nu x^{\nu(\xi+\alpha)-1}k_\nu(\xi, \eta, \alpha, \beta)(x/t) \quad \text{a.e.}$$

from the formulas for the derivative of the hypergeometric function.

Formulas (7.7) and (7.8) are proved similarly.

COROLLARY 7.2. *If* $\operatorname{Re}\beta < \operatorname{Re}\alpha$, *and* $-\operatorname{Re}(\eta + \beta) < \mu/\nu < \operatorname{Re}\xi$, *then* $(J_{\nu,\beta,\eta})^{-1}I_{\nu,\alpha,\xi}$ *can be extended to* $L_{\mu,1}$, *and* (7.6) *remains true. If* $\operatorname{Re}\alpha < \operatorname{Re}\beta$, *and* $-\operatorname{Re}\eta < \mu/\nu < \operatorname{Re}(\xi + \alpha)$, $(I_{\nu,\alpha,\xi})^{-1}J_{\nu,\beta,\eta}$ *can be extended to* $L_{\mu,1}$, *and* (7.8) *remains true.*

Proof. It follows from Lemmas 4.1 and 7.1 that if μ/ν is in the respective ranges above, then the right hand sides of (7.6) and (7.8) represent operators in $[L_{\mu,1}]$, and thus defining the respective extensions by (7.6) and (7.8), our corollary is proved.

8. Application to products of Hankel transformations. The transformation we shall call the Hankel transformation, $\mathscr{H}_{\rho,\lambda}$, is defined for $f \in C_0$ by

$$(8.1) \qquad (\mathscr{H}_{\rho,\lambda}f)(x) = x^{-(\rho-1)}\int_0^\infty t^\rho J_\lambda(xt)f(t)dt,$$

where $\lambda > -1$. For $\rho = \frac{1}{2}$, this is the "Hankel transformation" studied extensively in [**11**, Chapter 8, §§ 4 and 5], and for $\rho = \lambda + 1$, it is the "Hankel transformation" that plays an important role in the Fourier transformation of radial functions. From [**11**, Chapter 8, §§ 4 and 5] it is easily established by simple changes of variables that $\mathscr{H}_{\rho,\lambda}$ can be extended to $L_{2\rho,2}$, that it is a unitary transformation of that space into itself, that $\mathscr{H}_{\rho,\lambda}{}^{-1} = \mathscr{H}_{\rho,\lambda}$, and that if $f \in L_{2\rho,2}$,

$$(8.2) \qquad \mathscr{M}(\mathscr{H}_{\rho,\lambda}f)(s) = \frac{2^{s-\rho}\Gamma(\frac{1}{2}(\lambda - \rho + 1) + \frac{1}{2}s)}{\Gamma(\frac{1}{2}(\lambda + \rho + 1) - \frac{1}{2}s)}(\mathscr{M}f)(2\rho - s), \operatorname{Re} s = \rho.$$

The transformation that we are going to study in this section, $H_{\rho,\lambda,\gamma}$, is defined by

$$(8.3) \qquad H_{\rho,\lambda,\gamma} = \mathscr{H}_{\rho,\lambda+\gamma}\mathscr{H}_{\rho,\lambda},$$

where $\lambda > -1$, $\lambda + \gamma > -1$. Clearly $H_{\rho,\lambda,0}$ is the identity, but for $\gamma \neq 0$ it is non-trivial. A special case, namely

$$H_{\lambda+\frac{1}{2},\lambda-\frac{1}{2},1}$$

has been studied by Muckenhoupt and Stein [**8**, § 16], and they proved that if $\lambda \geqq 0$, it belongs to $[L_{2\lambda+1,p}]$ for $1 < p < \infty$, and our results can be considered an extension of theirs, though the results we shall obtain, when specialized to this case, show much more. We first prove a lemma.

https://doi.org/10.4153/CJM-1972-130-9 Published online by Cambridge University Press

LEMMA 8.1. (i) *If* $\lambda > -1$, $\gamma > 0$, *then on* $L_{2\rho,2}$,

$$H_{\lambda,\rho,\gamma} = (J_{2,\alpha,\eta})^{-1} I_{2,\alpha,\xi}, \tag{8.4}$$

where $\eta = \frac{1}{2}(\lambda - \rho + 1)$, $\xi = \frac{1}{2}(\lambda + \rho + 1)$, $\alpha = \frac{1}{2}\gamma$.

(ii) *If* $\gamma < 0$, $\lambda + \gamma > -1$, *then on* $L_{2\rho,2}$,

$$H_{\lambda,\rho,\gamma} = (I_{2,\alpha,\xi})^{-1} J_{2,\alpha,\eta}, \tag{8.5}$$

where $\eta = \frac{1}{2}(\lambda + \gamma - \rho + 1)$, $\xi = \frac{1}{2}(\lambda + \gamma + \rho + 1)$, $\alpha = -\frac{1}{2}\gamma$.

Proof. It is clear that $H_{\rho,\lambda,\gamma} \in [L_{2\rho,2}]$, indeed that it is unitary. From (8.2) and (8.3) it follows that if $f \in L_{2\rho,2}$, $\lambda > -1$, $\lambda + \gamma > -1$

$$\begin{aligned}(8.6)\quad & \mathcal{M}(H_{\rho,\lambda,\gamma} f)(s) \\ &= \frac{\Gamma(\frac{1}{2}(\lambda + \gamma - \rho + 1) + \frac{1}{2}s)\Gamma(\frac{1}{2}(\lambda + \rho + 1) - \frac{1}{2}s)}{\Gamma(\frac{1}{2}(\lambda - \rho + 1) + \frac{1}{2}s)\Gamma(\frac{1}{2}(\lambda + \gamma + \rho + 1) - \frac{1}{2}s)} (\mathcal{M}f)(s), \ \mathrm{Re}\, s = \rho.\end{aligned}$$

But from Definition 7.1 and Corollary 4.1, the right hand of (8.6) is

$$\mathcal{M}((J_{2,\alpha,\eta})^{-1} I_{2,\alpha,\xi} f)(s)$$

with the values of ξ, η and α given in (i) of the statement of this theorem, if $-(\alpha + \eta) < \rho/2 < \xi$ and $\alpha > 0$. The second inequality is obviously fulfilled since $\gamma > 0$, and the first can be written $-(\lambda + \gamma - \rho + 1) < \rho < \lambda + \rho + 1$, which is fulfilled if $\lambda > -1$, $\gamma > 0$. Hence for $f \in L_{2\rho,2}$, $\lambda > -1$, $\gamma > 0$

$$H_{\lambda,\rho,\gamma} f = (J_{2,\alpha,\eta})^{-1} I_{2,\alpha,\xi} f, \quad \text{a.e.}$$

where ξ, η and α are as given under (i), and (8.4) follows.

The right hand side of (8.6) is, from Definition 7.1 and Corollary 4.1,

$$\mathcal{M}((I_{2,\alpha,\xi})^{-1} J_{2,\beta,\eta} f)(s)$$

with the values of ξ, η, and α given in (ii) of the statement of this Theorem, if $-\eta < \rho/2 < \xi + \alpha$, and $\alpha > 0$. The second inequality is obviously fulfilled if $\gamma < 0$, and the first can be written $-(\lambda + \gamma - \rho + 1) < \rho < \lambda + \rho + 1$, which is fulfilled if $\lambda > -1$, $\lambda + \gamma > -1$, and (8.5) follows as in the case of (8.4).

This result can be interpreted in two ways. The first of these is that on $L_{2\rho,2}$, (8.4) and (8.5) give representations of $(J_{2,\alpha,\eta})^{-1} I_{2,\alpha,\xi}$ and $(I_{2,\alpha,\xi})^{-1} J_{2,\alpha,\eta}$ respectively as products of Hankel transformations. The second interpretation is that (8.4) and (8.5) give us a method of extending $H_{\rho,\lambda,\gamma}$ to other $L_{\mu,p}$ spaces, namely to define it to be the right hand side of these equations for all values of ξ, η and α for which the right hand side belongs to $[L_{\mu,p}]$. Such an extension is clearly unique since $L_{2\rho,2} \cap L_{\mu,p}$ is clearly dense in $L_{\mu,p}$. Of course $H_{\lambda,\rho,0}$ can be extended to the same spaces by defining it to be the identity.

It is this second interpretation which we adopt here, and reading off the properties of the operators on the right hand side of (8.4) and (8.5) from Theorem 7.1, we obtain the following theorem.

https://doi.org/10.4153/CJM-1972-130-9 Published online by Cambridge University Press

THEOREM 8.1. *If* $\lambda > -1$, $\lambda + \gamma > -1$, $H_{\rho,\lambda,\gamma}$ *can be uniquely extended, using* (8.4) *and* (8.5), *to* $L_{\mu,p}$ *for all* μ *and* p *satisfying*

$$-(\lambda + \gamma - \rho + 1) < \mu/p < \lambda + \rho + 1, \quad 1 < p < \infty,$$

and the extended operator, which we continue to denote by $H_{\rho,\lambda,\gamma}$, *satisfies* $H_{\rho,\lambda,\gamma} \in [L_{\mu,p}]$. *The operator is one-to-one if* (a) $1 < p \leqq 2$ *or* (b) $\gamma > 0$, *and* $-(\gamma - \rho + 1) < \mu/p < \gamma + \rho + 1$, *or* (c) $\gamma < 0$ *and* $-(\gamma + \lambda - \rho + 1) < \mu/p < (\lambda + \gamma + \rho + 1)$. *In cases* (b) *and* (c) *the mapping is onto. It is unitary on* $L_{\mu,2}$ *if* $\mu = 2\rho$.

Specializing to the case considered by Muckenhoupt and Stein [**8**] we obtain the following results.

COROLLARY 8.1.1. *If* $\lambda > -\frac{1}{2}$, $H_{\lambda+\frac{1}{2},\lambda-\frac{1}{2},1}$ *can be extended to* $L_{\mu,p}$ *for all* μ *and* p *satisfying* $-p < \mu < (2\lambda + 1)p$. *It is one-to-one if* $1 < p \leqq 2$ *or* $0 < \mu < (2\lambda + 1)p$, *and if* $0 < \mu < (2\lambda + 1)p$ *the mapping is onto.*

Two other cases can be noted. Let H_+ denote the Hilbert transformation of even functions and H_- the Hilbert transformation of odd functions. It is well-known that H_+ is the Fourier sine transformation of the Fourier cosine transformation, while H_- is the Fourier cosine transformation of the Fourier sine transformation; that is, since

$$J_{-\frac{1}{2}}(x) = (2/\pi x)^{1/2} \cos x, \quad \text{and} \quad J_{\frac{1}{2}}(x) = (2/\pi x)^{1/2} \sin x,$$
$$H_+ = \mathscr{H}_{\frac{1}{2},\frac{1}{2}} \mathscr{H}_{\frac{1}{2},-\frac{1}{2}} = H_{\frac{1}{2},-\frac{1}{2},1}, \quad \text{and} \quad H_- = \mathscr{H}_{\frac{1}{2},-\frac{1}{2}} \mathscr{H}_{\frac{1}{2},\frac{1}{2}} = H_{\frac{1}{2},\frac{1}{2},-1},$$

and specializing Theorem 8.1 to these cases, the following corollary is obtained.

COROLLARY 8.1.2. (i) *If* $-p < \mu < p$, $H_+ \in [L_{\mu,p}]$. H_+ *is one-to-one if* $1 < p < 2$ *or* $0 < \mu < p$. *In the latter case* H_+ *is onto.*

(ii) *If* $0 < \mu < 2p$, $H_- \in [L_{\mu,p}]$. H_- *is one-to-one if* $1 < p < 2$ *or* $0 < \mu < p$. *In the latter case* H_- *is onto.*

That $H_+ \in [L_{\mu,p}]$ for $-p < \mu < p$ is a known result due to Hardy and Littlewood [**5**]. However that $H_- \in [L_{\mu,p}]$ for $0 < \mu < 2p$ seems new, the best previous result being that $H_- \in [L_{\mu,p}]$ for $0 < \mu < p$, due to Babenko [**1**].

9. The range of the Hankel transformation. In this section we shall apply the results of the previous section to the study of the range of the Hankel transformation.

It is easy to show that if $\frac{1}{2} \leqq \rho \leqq \lambda + 1$, then $\mathscr{H}_{\rho,\lambda}$ is a bounded operator from $L_{2\rho,p}$ to $L_{2\rho,p'}$ for $1 \leqq p \leqq 2$. Indeed from [**10**, Theorem 7.31.2] and [**13**, § 3.31 (1)]

$$|J_\lambda(x)| \leqq K_\lambda \min(x^{-\frac{1}{2}}, x^\lambda) \leqq K_\lambda x^{\rho-1},$$

if $\frac{1}{2} \leqq \rho \leqq \lambda + 1$, and thus if $f \in L_{2\rho,1}$,

$$|(\mathscr{H}_{\rho,\lambda} f)(x)| \leqq K_\lambda \int_0^\infty t^{2\rho-1} |f(t)| dt = K_\lambda \|f\|_{2\rho,1},$$

and $\mathscr{H}_{\rho,\lambda}$ is of strong type $(1, \infty)$. As already remarked, it is also of strong type $(2, 2)$, and thus by the Riesz-Thorin convexity theorem [**14**, Chapter 12, Theorem 1.11], it is of strong type (p, p') for $1 \leqq p \leqq 2$. However, it is also well-known that the range of $\mathscr{H}_{\rho,\lambda}$ on $L_{2\rho,p}$, $\mathscr{H}_{\rho,\lambda}(L_{2\rho,p})$ is not all of $L_{2\rho,p'}$ if $1 < p < 2$. In fact an easy application of the Marcinkiewicz interpolation theorem [**14**, Chapter 12, Theorem 4.6] shows that if $f \in L_{2\rho,p}$, where $1 < p \leqq 2$, $g = \mathscr{H}_{\rho,\lambda}f$, $\frac{1}{2} \leqq \rho \leqq \lambda + 1$, then $x^{1-2/p}g(x) \in L_{2\rho,p}$; but not every member of $L_{2\rho,p'}$ has this property. In this section we shall show that if $\frac{1}{2} \leqq \rho \leqq \lambda + 1$, the range of $\mathscr{H}_{\rho,\lambda}$ on $L_{2\rho,p}$ is independent of λ. First we need a lemma.

LEMMA 9.1. *Suppose* $\lambda > -1$, $\lambda + \gamma > -1$, $\frac{1}{2} \leqq \rho \leqq \lambda + 1$, $1 < p \leqq 2$. *Then* $H_{\rho,\lambda,\gamma} \in [L_{2\rho,p}]$, *and*

$$\mathscr{H}_{\rho,\lambda+\gamma}H_{\rho,\lambda,\gamma} = \mathscr{H}_{\rho,\lambda} \tag{9.1}$$

Proof. Since $1 < p \leqq 2$,

$$-(\lambda + \gamma - \rho + 1) = \rho - (\lambda + \gamma + 1) < \rho \leqq 2\rho/p < 2\rho \leqq \rho + \lambda + 1,$$

and from Theorem 8.1, $H_{\rho,\lambda,\gamma} \in [L_{2\rho,p}]$. If $f \in L_{2\rho,2}$, $\mathscr{H}_{\rho,\lambda,+\gamma}H_{\rho,\lambda,\gamma}f = \mathscr{H}_{\rho,\lambda+\gamma}\mathscr{H}_{\rho,\lambda+\gamma}\mathscr{H}_{\rho,\lambda}f = \mathscr{H}_{\rho,\lambda}f$, and thus (9.1) holds on a dense subset of $L_{2\rho,p}$, and thus throughout $L_{2\rho,p}$.

THEOREM 9.1. *Suppose* $1 < p < 2$, $\frac{1}{2} \leqq \rho \leqq \min(\lambda + 1, \lambda + \gamma + 1)$. *Then* $\mathscr{H}_{\rho,\lambda+\gamma}(L_{2\rho,p}) = \mathscr{H}_{\rho,\lambda}(L_{2\rho,p})$.

Proof. Clearly $\lambda > -1$ and $\lambda + \gamma > -1$, and hence by Lemma 9.1, $H_{\rho,\lambda+\gamma} \in [L_{2\rho,p}]$. Let $f \in L_{2\rho,p}$, $g = H_{\rho,\lambda,\gamma}f$. Then by (9.1) $\mathscr{H}_{\rho,\gamma}f = \mathscr{H}_{\rho,\lambda+\gamma}g \in \mathscr{H}_{\rho,\lambda+\gamma}(L_{2\rho,p})$, and hence

$$\mathscr{H}_{\rho,\lambda}(L_{2\rho,p}) \subseteq \mathscr{H}_{\rho,\lambda+\gamma}(L_{2\rho,p}).$$

But the hypotheses of the theorem are symmetric in λ and $\lambda + \gamma$, and thus the reverse inclusion must hold, and the result follows.

A particular case is of some interest. Denoting $\mathscr{H}_{\frac{1}{2},\lambda}$ by $\mathscr{H}_{\lambda}$, it follows, since $\mathscr{H}_{-\frac{1}{2}}$ is the Fourier cosine transformation, $\mathscr{F}_c$, and $L_{1,p} = L_p(0, \infty)$, that if $1 < p < 2, \mathscr{H}_{\lambda}(L_p(0, \infty)) = \mathscr{F}_c(L_p(0, \infty))$.

10. Further problems. There are several further problems connected with our fractional integrals that one can study. For example one can ask when $J_{\nu_1,\beta,\eta}(L_{\mu,p}) \supseteq I_{\nu_2,\alpha,\xi}(L_{\mu,p})$, where $\nu_1 \neq \nu_2$, and whether $(J_{\nu_1,\beta,\eta})^{-1}I_{\nu_2,\alpha,\xi}$ is bounded. The methods used here answer these questions efficiently; for example, the above inclusion is true if

$$-\nu_1 \operatorname{Re}\beta < \mu/p < \nu_2 \operatorname{Re}\xi \quad \text{and} \quad \operatorname{Re}\beta \leqq \operatorname{Re}\alpha.$$

Alternatively, some authors have modified the kernels of the fractional integrals to obtain new fractional integrals with many of the same formal properties, but bounded for different ranges of values of the parameters in question; see [**3**], for example. We shall study some of these in a later paper, and prove similar results for them. Applications to Hankel transformations whose kernels are "cut" Bessel functions will be made.

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