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ADDITION OF AN IDENTITY TO AN ORDERED BANACH SPACE

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Abstract

Given an ordered Banach space $\mathfrak{B}$ equipped with an order-norm we construct a larger space $\tilde{\mathfrak{B}}$ with an order-norm and order-identity such that $\mathfrak{B}$ is isometrically order-isomorphic to a Banach subspace of $\tilde{\mathfrak{B}}$. We also discuss the extension of positive operators from $\mathfrak{B}$ to $\tilde{\mathfrak{B}}$.

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0. Introduction

The addition of an identity is a standard technique in the theory of C^*-algebras. In this note we examine a similar construction for ordered Banach spaces. Given a Banach space $\mathfrak{B}$ ordered by a positive cone $\mathfrak{B}_+$ we construct a larger space $\tilde{\mathfrak{B}} = (\mathfrak{B}, \mathbf{R})$ ordered by a positive cone $\tilde{\mathfrak{B}}_+$ and equipped with an order-norm $\|\cdot\|_+$ which ensures that $e = (0, 1)$ is an (order-) identity of $\tilde{\mathfrak{B}}$, that is, e is maximal in the unit ball of $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}}_+, \|\cdot\|_+)$. The embeddings $(\mathfrak{B}, 0) \subset \tilde{\mathfrak{B}}$ and $(\tilde{\mathfrak{B}}_+, 0) \subset \tilde{\mathfrak{B}}_+$ are however isometric order isomorphisms if, and only if, the norm $\|\cdot\|$ on $\mathfrak{B}$ coincides with the order-norm. This is the case for C^*-algebras.

1. The order-norm and order-identity

Let $(\mathfrak{B}, \|\cdot\|)$ be a real Banach space ordered by a *positive cone* $\mathfrak{B}_+$, that is, $\mathfrak{B}_+$ is a norm-closed convex cone in $\mathfrak{B}$ satisfying

$$\mathfrak{B}_+ \cap -\mathfrak{B}_+ = \{0\}$$

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and the relation $a \geqslant b$ is defined by $a - b \in \mathfrak{B}_+$. We define the *order half-norm* N_+, as a positive function over $\mathfrak{B}$, by

$$N_+(a) = \inf\{\lambda \geqslant 0; a \leqslant \lambda u \text{ for some } u \in \mathfrak{B}_1\},$$

and the *order-norm* $\|\cdot\|_+$ by

$$\begin{aligned}\|a\|_+ &= N_+(a) \vee N_+(-a)\\ &= \inf\{\lambda \geqslant 0; -\lambda u \leqslant a \leqslant \lambda v \text{ for some } u, v \in \mathfrak{B}_1\},\end{aligned}$$

where $\mathfrak{B}_1$ denotes the unit ball of $\mathfrak{B}$. Since $a = \|a\| u$ with $u = a/\|a\| \in \mathfrak{B}_1$ one has $N_+(a) \leqslant \|a\|$ and hence

$$\|a\|_+ \leqslant \|a\|$$

for all $a \in \mathfrak{B}$. But in general the two norms are inequivalent. Before further comparison of the norms we mention an alternative characterization of the order half-norm and hence the order-norm.

LEMMA 1.1. *Let $\mathfrak{B}$ be a Banach space with positive cone $\mathfrak{B}_+$ and order half-norm N_+. It follows that*

$$N_+(a) = \inf\{\|a + b\|; b \in \mathfrak{B}_+\}.$$

PROOF. If $a \leqslant \lambda u$ for some $u \in \mathfrak{B}_1$ then $\lambda u = a + b$ for some $b \in \mathfrak{B}_+$ and $|\lambda| \geqslant \|a + b\|$. Thus

$$N_+(a) \geqslant \inf\{\|a + b\|; b \in \mathfrak{B}_+\}.$$

But the converse inequality follows because one has $a \leqslant \|a + b\| u$, with $u = (a + b)/\|a + b\| \in \mathfrak{B}_1$, for each $b \in \mathfrak{B}_+$.

REMARK. The order half-norm is implicit in the work of Grosberg and Krein [6] and occurs explicitly in the work of Kadison [7]. It is basic to the introduction of the order-norm on an 'order-unit' space [1], [3]. More recently Arendt, Chernoff and Kato [2] defined the order half-norm by the criterion of Lemma 1 and called it the canonical half-norm. Note that N_+ is determined by $\mathfrak{B}_+$ and conversely

$$\mathfrak{B}_+ = \{a; N_+(-a) = 0\}.$$

Equivalence of the norm and order-norm is basically a property of the positive cone $\mathfrak{B}_+$. Krein [8] was the first to introduce the appropriate notion of a normal cone.

The cone $\mathfrak{B}_+$ is defined to be *α-normal* if there is an $\alpha \geqslant 1$ such that $a \leqslant b \leqslant c$ always implies

$$\|b\| \leqslant \alpha(\|a\| \vee \|c\|).$$

There are various alternative definitions of normality (see, for example, [9] Chapter 2). In particular it can be characterized in terms of positive functions.

An element of the dual $\mathfrak{B}^*$ of $\mathfrak{B}$ is defined to be positive, $f \geqslant 0$, if

$$f(a) \geqslant 0$$

for all $a \in \mathfrak{B}_+$. The set of positive functionals $f \in \mathfrak{B}^*$ forms a norm-closed convex cone $\mathfrak{B}^*_+$ which is called the *dual cone*. The cone is said to be *α-generated* if each $f \in \mathfrak{B}^*$ has a decomposition $f = f_+ - f_-$ with $f_\pm \in \mathfrak{B}^*_+$ and

$$\alpha \| f \| \geqslant \| f_+ \| + \| f_- \|.$$

Note that if $\mathfrak{B}^*_+$ is 1-generated then it follows from the triangle inequality that each $f \in \mathfrak{B}^*$ has a *Jordan decomposition*, that is, $f = f_+ - f_-$ with $f_\pm \in \mathfrak{B}^*_+$ and

$$\| f \| = \| f_+ \| + \| f_- \|.$$

The following proposition gives criteria for equivalence, and equality, of the norm $\| \cdot \|$ and order-norm $\| \cdot \|_+$ on $\mathfrak{B}$. In particular it restates Grosberg and Krein's result on the equivalence of α-normality of $\mathfrak{B}_+$ and α-generation of $\mathfrak{B}^*_+$.

PROPOSITION 1.2. *Let $(\mathfrak{B}, \mathfrak{B}_+, \| \cdot \|)$ be an ordered Banach space with corresponding order-norm $\| \cdot \|$. The following conditions are equivalent*:

1. *$\| \cdot \|$ and $\| \cdot \|_+$ are equivalent norms,*
2. *$\mathfrak{B}_+$ is α-normal, for $\alpha \geqslant 1$,*
3. *$\mathfrak{B}^*_+$ is α-generated, for $\alpha \geqslant 1$.*

Moreover the following conditions are equivalent:

1′. $\| \cdot \| = \| \cdot \|_+$,

2′. *$\mathfrak{B}_+$ is 1-normal,*

3′. *each $f \in \mathfrak{B}^*$ has a Jordan decomposition.*

PROOF. The equivalence of Conditions 2 and 3, and hence of Conditions 2′ and 3′, was established by Grosberg and Krein [6] (see also [3] Chapter 2). We will prove $1 \Leftrightarrow 2$ and simultaneously $1' \Leftrightarrow 2'$.

$1 \Rightarrow 2$. Assume $\| b \| \leqslant \alpha \| b \|_+$ for all $b \in \mathfrak{B}$. But $a \leqslant b \leqslant c$ implies directly that $\| b \|_+ \leqslant \| a \| \vee \| c \|$ and hence

$$\| b \| \leqslant \alpha(\| a \| \vee \| c \|).$$

Setting $\alpha = 1$ one deduces that $1' \Rightarrow 2'$.

$2 \Rightarrow 1$. Since $\mathfrak{B}_+$ is α-normal the relations $-\lambda u \leqslant a \leqslant \lambda v$ with $u, v \in \mathfrak{B}_1$ imply that $\| a \| \leqslant \alpha\lambda$ and hence

$$\| a \| \leqslant \alpha \| a \|_+ .$$

But $\| a \|_+ \leqslant \| a \|$ and hence the norms are equivalent.

Again setting $\alpha = 1$ one concludes that $2' \Rightarrow 1'$.

EXAMPLE 1.3. If $\mathfrak{B}_+ = \{0\}$ then $\| \cdot \|_+ = \| \cdot \|$ and $\mathfrak{B}^*_+ = \mathfrak{B}^*$.

EXAMPLE 1.4. If $\mathfrak{B}$ is the hermitian part of a C^*-algebra $\mathfrak{A}$ ordered by the positive elements $\mathfrak{A}_+$ of the algebra then $\|\cdot\| = \|\cdot\|_+$ because each $f \in \mathfrak{A}^*$ has a Jordan decomposition [5], that is, the C^*-norm and order-norm coincide. This equality of norms can also be established by direct calculation.

EXAMPLE 1.5. Let $\mathfrak{B}$ be an order complete Banach lattice (see, for example, [10]). Then $\|\cdot\| = \|\cdot\|_+$ if, and only if, $\mathfrak{B}$ is an AM-space, that is, $\|a \vee b\| = \|a\| \vee \|b\|$ for all $a, b \in \mathfrak{B}_+$. This is established by first remarking that the dual of an AM-space is an AL-space [10] and each element of an AL-space has a Jordan decomposition, that is, Condition 3 of Proposition 1.2 is valid. Conversely each $a \in \mathfrak{B}$ has a canonical decomposition [10] $a = a_+ - a_-$ with $a_\pm \in \mathfrak{B}_+$ and $a_+ \wedge a_- = 0$. But $N_+(a) = \|a_+\|$ [2] and hence if $\|\cdot\| = \|\cdot\|_+$ then

$$\|a\| = \|a\|_+ = \|a_+\| \vee \|a_-\|.$$

Since $a_+ \wedge a_- = 0$ this is also equivalent to

$$\|a \vee b\| = \|a\| \vee \|b\|$$

for all $a, b \in \mathfrak{B}_+$ with $a \wedge b = 0$. To remove this last restriction take $a, b \in \mathfrak{B}_+$ and define $a_1, b_1 \in \mathfrak{B}_+$ by

$$a_1 = a \vee b - b, \qquad b_1 = a \vee b - a.$$

One then readily checks that $a_1 \wedge b_1 = 0$.

Next define $\mathfrak{C}$ and $\mathfrak{C}^\perp$ by

$$\mathfrak{C} = \{a;\, a \wedge b_1 = 0\}, \qquad \mathfrak{C}^\perp = \{b;\, b \wedge a = 0,\, a \in \mathfrak{C}\}.$$

It follows from the assumed order completeness that $\mathfrak{B} = \mathfrak{C} \oplus \mathfrak{C}^\perp$ ([10] Chapter 2, Theorem 2.10) and the projection P: $\mathfrak{B} \mapsto \mathfrak{C}$ is positive and continuous with $\|P\| \leqslant 1$ ([10] Chapter 2, Propositions 2.7 and 5.2). Therefore

$$P(a \vee b) - Pa = Pb_1 = 0$$

because $b_1 \in \mathfrak{C}^\perp$ and

$$(1 - P)(a \vee b) - (1 - P)b = (1 - P)a_1 = 0$$

because $a_1 \in \mathfrak{C}$. Therefore if

$$a' = P(a \vee b), \qquad b' = (1 - P)(a \vee b)$$

one has $a' \wedge b' = 0$ and $a' + b' = a' \vee b' = a \vee b$. Moreover $0 \leqslant a' = Pa \leqslant a$, $0 \leqslant b' = (1 - P)b \leqslant b$. Consequently

$$\|a \vee b\| = \|a' \vee b'\| = \|a'\| \vee \|b'\| \leqslant \|a\| \vee \|b\|$$

where the second step uses $a' \wedge b' = 0$. But $a \vee b \geqslant a$, $a \vee b \geqslant b$. Hence

$$\|a \vee b\| \geqslant \|a\| \vee \|b\|$$

and this establishes that $\mathfrak{B}$ is an AM-space.

Next we consider identity elements.

An element e of the ordered Banach space $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|)$ is defined to be an *(order-) identity*, or *unit element*, if it is maximal, with respect to the order induced by $\mathfrak{B}_+$, in the unit ball $\mathfrak{B}_1$. There are alternative characterizations:

PROPOSITION 1.6. *Let e be an element of the unit ball of the ordered Banach space* $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|)$. *The following conditions are equivalent*:

1. *e is an identity of* $\mathfrak{B}$,
2. $\{a; \|a - e\| < 1\} \subset \mathfrak{B}_+$,
3. $N_+ = N_e$ *where* N_+ *is the order half-norm and*

$$N_e(a) = \inf\{\lambda \geq 0; a \leq \lambda e\}.$$

Hence if $\mathfrak{B}$ *has an identity e the order-norm is characterized by*

$$\begin{aligned}\|a\|_+ &= \inf\{\lambda \geq 0; -\lambda u \leq a \leq \lambda v; u, v \in \mathfrak{B}_1 \cap \mathfrak{B}_+\}\\ &= \inf\{\lambda \geq 0; -\lambda e \leq a \leq \lambda e\}\end{aligned}$$

and, moreover, e is an identity of $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|_+)$.

PROOF. $1 \Rightarrow 3$. Since $e \in \mathfrak{B}_1$ one has $N_e \geq N_+$. But conversely if $v \in \mathfrak{B}_1$ then $v \leq e$, by maximality, and hence

$$N_+(a) = \inf\{\lambda \geq 0; a \leq \lambda v, v \in \mathfrak{B}_1\} \geq \inf\{\lambda \geq 0; a \leq \lambda e\} = N_e(a).$$

Thus $N_+ = N_e$.

$3 \Rightarrow 2$. The order half-norm satisfies $N_+(a) \leq \|a\|$. Hence if $N_+ = N_e$ then $a \leq \|a\| e$ for all $a \in \mathfrak{B}$. Consequently the unit ball around e is contained in $\mathfrak{B}_+$.

$2 \Rightarrow 1$. If $b = e - a/\|a\|(1 + \varepsilon)$ then $\|e - b\| \leq 1/(1 + \varepsilon)$ and hence $b \geq 0$ for $\varepsilon > 0$, that is, $a \leq \|a\|(1 + \varepsilon)e$. But since $\mathfrak{B}_+$ is norm-closed one then has

$$a \leq \|a\| e$$

and hence e is maximal in $\mathfrak{B}_1$.

Now it follows from Condition 3 that

$$(*) \qquad \|a\|_+ = \inf\{\lambda \geq 0; -\lambda e \leq a \leq \lambda e\}$$

and the other characterization results from the fact that $e \in \mathfrak{B}_1 \cap \mathfrak{B}_+$. Finally $(*)$ implies that $e \geq a/\|a\|_+$ and hence, by maximality, e is an identity of $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|_+)$.

REMARK. A little care must be taken with the last statement of Proposition 1.6. If $\mathfrak{B}_+$ is not normal with respect to $\|\cdot\|$ the space $\mathfrak{B}$ is not $\|\cdot\|_+$-complete, and the cone $\mathfrak{B}_+$ is not $\|\cdot\|_+$-closed. But $\mathfrak{B}$ can be $\|\cdot\|_+$-completed, and $\mathfrak{B}_+$ can be

$\|\cdot\|_+$-closed. The identity e of $(\mathfrak{B}, \mathfrak{B}, \|\cdot\|)$ then remains an identity for the $\|\cdot\|_+$-completed space $(\overline{\mathfrak{B}}, \overline{\mathrm{B}}_+, \|\cdot\|_+)$. The order-norm is the smallest norm with this property.

Note that if $\mathfrak{B}$ has an identity then it is unique because maximality of both e_1 and e_2 in $\mathfrak{B}_1$ implies $\pm(e_1 - e_2) \in \mathfrak{B}_+$ and hence $e_1 = e_2$. But not all ordered Banach spaces have an identity e. In fact Condition 2 of Proposition 1.6 demonstrates that e is an interior point of $\mathfrak{B}_+$ and in many cases $\mathfrak{B}_+$ has an empty interior.

EXAMPLE 1.7. If $\mathfrak{B}$ is the hermitian part of a C^*-algebra $\mathfrak{A}$ ordered by the positive elements $\mathfrak{A}_+$ of the algebra then $\mathfrak{B}$ has an (order-) identity if, and only if, $\mathfrak{A}$ has an (algebraic-) identity and in this case the two coincide.

EXAMPLE 1.8. If $(\mathfrak{B}, \|\cdot\|)$ is a Banach lattice then $\mathfrak{B}_+$ has interior points if, and only if, $\mathfrak{B}$ is lattice isomorphic to $C(X)$ for some compact Hausdorff space X. (See, for example [4].) Moreover if u is an interior point of $\mathfrak{B}_+$ then each $a \in \mathfrak{B}$ can be majorized by a multiple of u and hence one can introduce the norm

$$\|a\|_u = \inf\{\lambda \geqslant 0;\ -\lambda u \leqslant a \leqslant \lambda u\}.$$

It follows that $\|\cdot\|_u$ is equivalent to $\|\cdot\|$ and $(\mathfrak{B}, \|\cdot\|_u)$ is an AM-space with identity u (again see [4]).

2. Addition of an identity

Next we consider the embedding of an ordered Banach space $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|)$ in a larger space $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}}_+, \|\cdot\|_+)$ with an identity e and corresponding order-norm $\|\cdot\|_+$. This embedding is an order-isomorphism but is not necessarily isometric. But again we remark that $\mathfrak{B}$ can be completed with respect to the order-norm $\|\cdot\|_+$, and $\mathfrak{B}_+$ can be closed. The embedding theorem then gives an isometric order-isomorphism of the completed space $(\overline{\mathfrak{B}}, \overline{\mathfrak{B}}_+, \|\cdot\|_+)$ in the space $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}}_+, \|\cdot\|_+)$ with identity. For simplicity we will not distinguish between $\mathfrak{B}$ and $\overline{\mathfrak{B}}$, $\mathfrak{B}_+$ and $\overline{\mathfrak{B}}_+$, in the sequel.

THEOREM 2.1. *Let $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|_+)$ be an ordered Banach space equipped with the order norm $\|\cdot\|_+$. Consider the space $\tilde{\mathfrak{B}} = (\mathfrak{B}, \mathbf{R})$ of pairs (a, t) with $a \in \mathfrak{B}$, $t \in \mathbf{R}$, with the operations*

$$\lambda(a, t) = (\lambda a, \lambda t), \qquad (a, t) + (b, s) = (a + b, s + t),$$

with the norm

$$\|(a, t)\|_+ = (N_+(a) + t) \vee (N_+(-a) - t),$$

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and with the cone

$$\tilde{\mathfrak{B}}_+ = \{(a,t);\, t \geqslant N_+(-a)\}.$$

It follows that $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}}_+, \|\cdot\|_+)$ *is an ordered Banach space,* $\mathfrak{B} = (\mathfrak{B}, 0)$ *is a Banach subspace of* $\tilde{\mathfrak{B}}$, $\mathfrak{B}_+ = (\mathfrak{B}_+, 0)$ *is a positive subcone of* $\tilde{\mathfrak{B}}_+$, $e = (0,1)$ *is an identity of* $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}}_+, \|\cdot\|_+)$ *and* $\|\cdot\|_+$ *is the order-norm on* $\tilde{\mathfrak{B}}$ *defined by* $\tilde{\mathfrak{B}}_+$.

PROOF. First consider the space $\tilde{\mathfrak{B}} = (\mathfrak{B}, \mathbf{R})$ with the norm

$$\|(a,t)\| = \|a\|_+ + |t|\,.$$

Since $\mathfrak{B}$ is $\|\cdot\|_+$-complete $\tilde{\mathfrak{B}}$ is automatically $\|\cdot\|$-complete and one can identify $\mathfrak{B} = (\mathfrak{B}, 0)$ as a Banach subspace. Now since N_+ is a half-norm $\tilde{\mathfrak{B}}_+$ is a proper norm-closed convex cone. For example if $\pm(a,t) \in \tilde{\mathfrak{B}}_+$ then $\pm t \geqslant N_+(\pm a)$ and $t = 0 = N_+(a) = N_+(-a)$, because N_+ is positive. Hence $a = 0 = t$. Note that if $a \in \mathfrak{B}_+$ then $N_+(-a) = 0$ and hence $\mathfrak{B}_+$ can be identified as the norm-closed subcone $(\mathfrak{B}_+, 0)$ of $\tilde{\mathfrak{B}}_+$.

Next we prove 3-normality of $\tilde{\mathfrak{B}}_+$. If $(a,r) \leqslant (b,s) \leqslant (c,t)$ then

$$s - r \geqslant N_+(a-b), \qquad t - s \geqslant N_+(b-c).$$

In particular $t \geqslant s \geqslant r$. There are two cases to consider.

Case 1. $s \geqslant 0$. If $s \geqslant 0$ then $|t| \geqslant |s|$. Hence

$$\begin{aligned}
\|(b,s)\| &= N_+(b) \vee N_+(-b) + |s| \\
&\leqslant (N_+(c) + t - s) \vee (N_+(-a) + s - r) + |t| \\
&\leqslant (N_+(c) + |t|) \vee (N_+(-a) + |t| - r) + |t| \\
&= N_+(c) \vee (N_+(-a) - r) + 2|t| \\
&\leqslant \|(c,t)\| \vee \|(a,r)\| + 2\|(c,t)\| \\
&\leqslant 3\|(a,r)\| \vee \|(c,t)\|.
\end{aligned}$$

Case 2. $s \leqslant 0$. If $s \leqslant 0$ then $|r| \geqslant |s|$. Hence

$$\begin{aligned}
\|(b,s)\| &\leqslant (N_+(c) + t - s) \vee (N_+(-a) + s - r) + |r| \\
&\leqslant (N_+(c) + t + |r|) \vee (N_+(-a) + |r|) + |r| \\
&= (N_+(c) + t) \vee N_+(-a) + 2|r| \\
&\leqslant \|(c,t)\| \vee \|(a,r)\| + 2\|(a,r)\| \\
&\leqslant 3\|(a,r)\| \vee \|(c,t)\|.
\end{aligned}$$

Since $\tilde{\mathfrak{B}}_+$ is 3-normal the norm $\|\cdot\|$ and order-norm $\|\cdot\|_+$ on $\tilde{\mathfrak{B}}$ are equivalent, by Proposition 1.2. Thus $\tilde{\mathfrak{B}}$ is $\|\cdot\|_+$-complete and $\tilde{\mathfrak{B}}_+$ is $\|\cdot\|_+$-closed.

Next if $e = (0,1)$ then

$$e\|(a,t)\| - (a,t) = (-a, \|a\|_+ + |t| - t) \in \tilde{\mathfrak{B}}_+$$

because $\|a\|_+ + |t| - t \geq N_+(a)$. Therefore e is maximal in the unit ball of $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}}_+, \|\cdot\|)$, that is, e is an identity of $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}}_+, \|\cdot\|)$. It now follows from Proposition 1.6 that e is an identity of $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}}_+, \|\cdot\|_+)$ where $\|\cdot\|_+$ denotes the order-norm on $\tilde{\mathfrak{B}}$. Moreover this norm is given by

$$\|(a, t)\|_+ = \inf\{\lambda \geq 0;\ -\lambda e \leq (a, t) \leq \lambda e\}.$$

Since $(a, t) \leq \lambda e$ is equivalent to $(-a, \lambda - t) \in \tilde{\mathfrak{B}}_+$ one must have

$$\lambda \geq N_+(a) + t.$$

Similarly $-\lambda e \leq (a, t)$ gives $\lambda \geq N_+(-a) - t$. Therefore the order half-norm and order-norm on $\tilde{\mathfrak{B}}$ are given by

$$N_+((a, t)) = (N_+(a) + t) \vee 0$$

and

$$\|(a, t)\|_+ = (N_+(a) + t) \vee (N_+(-a) - t).$$

This verifies the last statement of the theorem.

EXAMPLE 2.2. If $\mathfrak{B}_+ = \{0\}$ then $N_+(a) = \|a\|$ and and $\|a\|_+ = \|a\|$. Thus $\tilde{\mathfrak{B}}_+ = \{(a, t);\ t \geq \|a\|\}$ and $\|(a, t)\| = \|a\| + |t| = (\|a\| + t) \vee (\|a\| - t) = \|(a, t)\|_+$. Therefore the norm and order-norm on $\tilde{\mathfrak{B}}$ coincide. Note that $\tilde{\mathfrak{B}}_+$ is non-trivial despite the triviality of $\mathfrak{B}_+$.

EXAMPLE 2.3. If $\mathfrak{B}$ is the hermitian part of a C^*-algebra $\mathfrak{A}$, ordered by the positive elements $\mathfrak{A}_+$ of $\mathfrak{A}$, then the construction of Theorem 2.1 coincides with the addition of an algebraic identity [5]. Since $N_+(a) = \|a_+\|$, where a_+ is the positive part of a, and since the C^*-norm and order-norm coincide (Example 1.4) one has the connection

$$\|(a, t)\| = (\|a_+\| + t) \vee (\|a_-\| - t)$$

between the C^*-norms on $\tilde{\mathfrak{B}}$ and $\mathfrak{B}$.

3. Positive operators

In this section we examine the extension of operators from the ordered Banach space $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|)$ to the space $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}}_+, \|\cdot\|_+)$ with identity e constructed in Theorem 2.1. If $\mathfrak{B}_+$ is normal $\|\cdot\|$ and $\|\cdot\|_+$ are equivalent norms by Proposition 1.2, $\mathfrak{B}$ is $\|\cdot\|_+$-complete, $\mathfrak{B}_+$ is $\|\cdot\|_+$-closed, and each bounded linear operator A on $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|)$ defines a bounded operator on the renormed space $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|_+)$. Therefore we can unambiguously consider the extension of A from $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|)$ to $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}}_+, \|\cdot\|_+)$.

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First we give a characterization of positive operators, that is, operators A with the property that $A\mathfrak{B}_+ \subseteq \mathfrak{B}_+$.

LEMMA 3.1. *Let A be a bounded linear operator on the ordered Banach space $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|)$. The following conditions are equivalent*:

1. $A\mathfrak{B}_+ \subseteq \mathfrak{B}_+$,
2. $N_+(Aa) \leqslant \|A\| N_+(a)$, $a \in \mathfrak{B}$,
3. $N_+(Aa) \leqslant \alpha N_+(a)$, $a \in \mathfrak{B}$, *for some* $\alpha \geqslant \|A\|$.

PROOF. $1 \Rightarrow 2$. One has

$$\begin{aligned} N_+(Aa) &= \inf\{\|Aa + b\|; b \in \mathfrak{B}_+\} \\ &\leqslant \inf\{\|Aa + Ab\|; b \in \mathfrak{B}_+\} \\ &\leqslant \|A\| N_+(a). \end{aligned}$$

$2 \Rightarrow 3$. This is trivial.

$3 \Rightarrow 1$. If $a \in \mathfrak{B}_+$ then $N_+(-a) = 0$ and hence $N_+(-Aa) = 0$. But this is equivalent to $Aa \in \mathfrak{B}_+$.

Now suppose that $\mathfrak{B}_+$ is normal and hence the bounded linear operator A on $\mathfrak{B}$ is a bounded operator on the renormed space $(\mathfrak{B}, \|\cdot\|_+)$. The simplest form of extension of A from $\mathfrak{B}$ to $\tilde{\mathfrak{B}}$ is defined by

$$A_\alpha(a, t) = (Aa, \alpha t)$$

where $\alpha \in \mathbf{R}$. Note that A_α is automatically linear and we next examine criteria for it to be positive.

THEOREM 3.2. *Let A be a bounded linear operator on the ordered Banach space $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|)$. Assume $\mathfrak{B}_+$ is normal and consider the extension A_α of A to the extended space $(\tilde{B}, \tilde{B}_+, \|\cdot\|_+)$ with identity e. If $\alpha \geqslant \|A\|$ the following conditions are equivalent*:

1. $A\mathfrak{B}_+ \subseteq \mathfrak{B}_+$,
2. $A_\alpha\tilde{\mathfrak{B}}_+ \subseteq \tilde{\mathfrak{B}}_+$,
3. $\|A_\alpha\| = \alpha$,
4. $\|A_\alpha\| \leqslant \alpha$.

PROOF. $2 \Rightarrow 1$. If $a \in \mathfrak{B}_+$ then $(a, 0) \in \tilde{\mathfrak{B}}_+$ and $A_\alpha(a, 0) = (Aa, 0) \subseteq \tilde{\mathfrak{B}}_+$ by assumption. Thus $N_+(-Aa) = 0$ and $Aa \in \mathfrak{B}_+$.

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$1 \Rightarrow 3$. Because $\alpha \geqslant \|A\|$ Condition 1 is equivalent to $N_+(Aa) \leqslant \alpha N_+(a)$ for all $a \in \mathfrak{B}$ by Lemma 3.1. Therefore

$$\begin{aligned} N_+(A_\alpha(a, t)) &= (N_+(Aa) + \alpha t) \vee 0 \\ &\leqslant \alpha(N_+(a) + t) \vee 0 \\ &= \alpha N_+((a, t)). \end{aligned}$$

Consequently

$$\|A_\alpha(a, t)\|_+ \leqslant \alpha \|(a, t)\|_+ .$$

But one also has $\|A_\alpha e\|_+ = \alpha \|e\|_+ = \alpha$ and hence $\|A_\alpha\| = \alpha$.

$3 \Rightarrow 4$. This is evident.

To conclude the proof we recall that since e is an identity of $(\tilde{\mathfrak{B}}, \tilde{\mathfrak{B}}_+, \|\cdot\|_+)$

$$\|e - (a, t)\|_+ \leqslant 1$$

implies $(a, t) \in \tilde{\mathfrak{B}}_+$. Conversely, if $(a, t) \in \tilde{\mathfrak{B}}_+$ and $\|(a, t)\|_+ \leqslant 1$ then $N_+(-a) \leqslant t$ and $N_+(a) + t \leqslant 1$. Therefore

$$\|e - (a, t)\|_+ = (N_+(-a) + 1 - t) \vee (N_+(a) - 1 + t) \leqslant 1.$$

$4 \Rightarrow 2$. Assume $(a, t) \in \tilde{\mathfrak{B}}_+$ and $\|(a, t)\|_+ \leqslant 1$. Then setting $B = A_\alpha / \alpha$ one has $Be = e$ and hence

$$\begin{aligned} \|e - B(a, t)\|_+ &= \|A_\alpha(e - (a, t))\|_+ / \alpha \\ &= (\|A_\alpha\| / \alpha) \|(e - (a, t))\|_+ \leqslant 1 \end{aligned}$$

by Condition 4 and the above. Therefore $B(a, t) \in \tilde{\mathfrak{B}}_+$ and consequently $A_\alpha \tilde{\mathfrak{B}}_+ \subseteq \tilde{\mathfrak{B}}_+$.

Remark. The equivalence of Conditions 2 and 3 is an analogue of the C^*-algebraic result that an operator which leaves the identity fixed is positive if, and only if, it has norm one (see, for example, [5] Corollary 3.2.6).

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