ON OPERATOR ALGEBRAS AND INVARIANT SUBSPACES

CHANDLER DAVIS, HEYDAR RADJAVI, AND PETER ROSENTHAL

If $\mathfrak{A}$ is a collection of operators on the complex Hilbert space $\mathscr{H}$, then the lattice of all subspaces of $\mathscr{H}$ which are invariant under every operator in $\mathfrak{A}$ is denoted by Lat $\mathfrak{A}$. An algebra $\mathfrak{A}$ of operators on $\mathscr{H}$ is defined (**3**; **4**) to be *reflexive* if for every operator B on $\mathscr{H}$ the inclusion Lat $\mathfrak{A} \subseteq$ Lat B implies $B \in \mathfrak{A}$.

Arveson (**1**) has proved the following theorem. (The abbreviation "m.a.s.a." stands for "maximal abelian self-adjoint algebra".)

ARVESON'S THEOREM. *If $\mathfrak{A}$ is a weakly closed algebra which contains an m.a.s.a., and if* Lat $\mathfrak{A} = \{\{0\}, \mathscr{H}\}$, *then $\mathfrak{A}$ is the algebra of all operators on $\mathscr{H}$.*

A generalization of Arveson's Theorem was given in (**3**). Another generalization is Theorem 2 below, an equivalent form of which is Corollary 3. This theorem was motivated by the following very elementary proof of a special case of Arveson's Theorem.†

THEOREM 1. *If $\mathfrak{A}$ is a weakly closed algebra of operators on $\mathscr{H}$ containing an m.a.s.a. whose atoms span $\mathscr{H}$, and if* Lat $\mathfrak{A} = \{\{0\}, \mathscr{H}\}$, *then $\mathfrak{A}$ is the algebra of all operators on $\mathscr{H}$.*

Proof. By hypothesis, $\mathscr{H}$ has an orthonormal basis $\{e_\alpha\}$ consisting of eigenvectors of the m.a.s.a. contained in $\mathfrak{A}$. Let P_α denote the projection onto e_α. Then P_α is in the m.a.s.a., and hence in $\mathfrak{A}$. Now, for each fixed α, $\{Ae_\alpha: A \in \mathfrak{A}\}$ is an invariant linear manifold for $\mathfrak{A}$, and hence is dense in $\mathscr{H}$. Thus if B is any operator on $\mathscr{H}$, then BP_α is in the strong closure of $\mathfrak{A}P_\alpha$. However, $\mathfrak{A}P_\alpha \subseteq \mathfrak{A}$. Hence BP_α is in $\mathfrak{A}$, and therefore so is B, the (weak) sum of the BP_α.

If (the ranges of) the projections P_1 and P_2 are in Lat $\mathfrak{A}$, then P_2 is said to *cover* P_1 in Lat $\mathfrak{A}$ if $P_1 < P_2$ and if Lat $\mathfrak{A}$ does not contain any projections properly between P_1 and P_2.

THEOREM 2. *Let $\mathfrak{A}$ be a weakly closed algebra of operators on $\mathscr{H}$ containing an m.a.s.a. and let*

$$\mathscr{F}(\mathfrak{A}) = \{P_2 - P_1 : P_2 \textit{ covers } P_1 \textit{ in } \operatorname{Lat} \mathfrak{A}\}.$$

If the projections in $\mathscr{F}(\mathfrak{A})$ span $\mathscr{H}$, then $\mathfrak{A}$ is reflexive.

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†*Added in proof.* Arveson has informed us (oral communication) that he too was aware of this simple proof.

Proof. Denote the m.a.s.a. by $\mathfrak{R}$. Since Lat $\mathfrak{A} \subseteq$ Lat $\mathfrak{R}$, $\mathscr{F}(\mathfrak{A})$ is commutative. We shall show that if P and Q are distinct members of $\mathscr{F}(\mathfrak{A})$, then $PQ = 0$. First observe that if P, P_1, and P_2 are projections in Lat $\mathfrak{A}$ and if $P_2 - P_1 \in \mathscr{F}(\mathfrak{A})$, then either P contains $P_2 - P_1$ or $P(P_2 - P_1) = 0$. (This is so since $(P \vee P_1)P_2 \in$ Lat $\mathfrak{A}$ and $P_1 \leqq (P \vee P_1)P_2 \leqq P_2$. Since P_2 covers P_1, $(P \vee P_1)P_2$ is either P_1 or P_2.) Now if $Q = Q_1 - Q_2 \in \mathscr{F}(\mathfrak{A})$ with $Q_i \in$ Lat $\mathfrak{A}$, then $PQ = (P_2 - P_1)Q_2 - (P_2 - P_1)Q_1$, and applying the above observation twice we see that either $PQ = 0$ or $PQ = P_2 - P_1 = P$. Interchanging the roles of P_i with Q_i, we also see that either $PQ = 0$ or $PQ = Q$. Hence, if $PQ \neq 0$, then $PQ = P = Q$.

Next let B be any operator such that Lat $\mathfrak{A} \subseteq$ Lat B. In order to show that $B \in \mathfrak{A}$, we shall prove that $PBQ \in \mathfrak{A}$ for every P and Q in $\mathscr{F}(\mathfrak{A})$. This will suffice, since $B = \sum PBQ$ (in the weak sense) with P and Q varying over $\mathscr{F}(\mathfrak{A})$.

(i) We first show that $PBP \in \mathfrak{A}$ for every $P \in \mathscr{F}(\mathfrak{A})$. If $P \in \mathscr{F}(\mathfrak{A})$, then, since $P \in \mathfrak{R} \subseteq \mathfrak{A}$, $P\mathfrak{A}P$ is a subalgebra of $\mathfrak{A}$; this subalgebra (when considered as an algebra of operators on the range of P) contains the m.a.s.a. $P\mathfrak{R}P$. We next show that $P\mathfrak{A}P$ has no proper invariant subspaces: let $P = P_2 - P_1$, where P_2 covers P_1 in Lat $\mathfrak{A}$, and let $Q \leqq P$ such that $(PAP)Q = Q(PAP)Q$ for all $A \in \mathfrak{A}$. Then $P_1 + Q \in$ Lat $\mathfrak{A}$. (To see this, we note that $PAQ = QAQ$ and

$$AQ = AP_2Q = P_2AP_2Q = (P_1 + P)AQ = P_1AQ + QAQ = (P_1 + Q)AQ.$$

Since $P_1 \perp Q$ and $AP_1 = P_1AP_1$, we obtain $P_1 + Q \in$ Lat $\mathfrak{A}$.) It follows from the inequalities $P_1 \leqq P_1 + Q \leqq P_2$ that either $Q = 0$ or $Q = P$.

Arveson's Theorem now implies that $P\mathfrak{A}P = P\mathfrak{B}(\mathscr{H})P$, where $\mathfrak{B}(\mathscr{H})$ is the algebra of all operators on $\mathscr{H}$, since $P\mathfrak{A}P$ is weakly closed and contains an m.a.s.a. Hence $PBP \in P\mathfrak{A}P$, so that $PBP \in \mathfrak{A}$.

(ii) Let P and Q be two distinct projections in $\mathscr{F}(\mathfrak{A})$. If $Q\mathfrak{A}P = \{0\}$, then the $\mathfrak{A}$-invariant subspace $\{\mathfrak{A}Px: x \in \mathscr{H}\}$, which is also B-invariant by hypothesis, is orthogonal to Q. Hence $QBP = 0$.

We now assume that $A_0 \neq 0$, $A_0 \in Q\mathfrak{A}P$ for some A_0. We shall prove that if x and y are two vectors in the ranges of P and Q, respectively, then there exists $A \in Q\mathfrak{A}P$ such that $Ax = y$ and $A(\{x\}^\perp) = 0$. This will then imply, since $Q\mathfrak{A}P$ is weakly closed, that $Q\mathfrak{A}P = Q\mathfrak{B}(\mathscr{H})P$ and, in particular, that $QBP \in \mathfrak{A}$.

By hypothesis, there exist x_0 and y_0 in the ranges of P and Q, respectively, such that $A_0x_0 = y_0 \neq 0$. Since, by (i) above, $P\mathfrak{A}P = P\mathfrak{B}(\mathscr{H})P$ and $Q\mathfrak{A}Q = Q\mathfrak{B}(\mathscr{H})Q$, there exist $A_1 \in P\mathfrak{A}P$ and $A_2 \in Q\mathfrak{A}Q$ such that $A_1x = x_0$, $A_1(\{x\}^\perp) = 0$, and $A_2y_0 = y$. Then $A_2A_0A_1 \in Q\mathfrak{A}P$, $A_2A_0A_1x = y$, and $A_2A_0A_1(\{x\}^\perp) = 0$. This completes the proof of the theorem.

It is perhaps worth noting here that the above proof establishes the following fact about $\mathfrak{A}$ in terms of $\mathscr{F}(\mathfrak{A})$. If

$$\mathscr{C} = \{(P, Q): P \text{ and } Q \text{ in } \mathscr{F}(\mathfrak{A}) \text{ and } P\mathfrak{A}Q \neq \{0\}\},$$

then $\mathfrak{A}$ is the weak sum of all the algebras $P\mathfrak{B}(\mathscr{H})Q$ with $(P, Q) \in \mathscr{C}$.

COROLLARY 1. *Let $\mathfrak{A}$ be a weakly closed algebra of operators containing an m.a.s.a. such that* Lat $\mathfrak{A}$ *is finite. Then $\mathfrak{A}$ is reflexive.*

Proof. Use induction to form a chain

$$\{0\} = P_0 < P_1 < \ldots < P_{n-1} < P_n = \mathscr{H}$$

in Lat $\mathfrak{A}$ with $P_j - P_{j-1} \in \mathscr{F}(\mathfrak{A})$.

COROLLARY 2. *Every algebra of operators on a finite-dimensional Hilbert space which contains an m.a.s.a. is reflexive.*

Proof. The lattice of invariant subspaces of an m.a.s.a. on a finite-dimensional Hilbert space is finite.

Under certain hypotheses, the assumption that the algebra $\mathfrak{A}$ in Theorem 2 contains an m.a.s.a. is not needed.

THEOREM 3. *Let $\mathfrak{A}$ be a weakly closed algebra of operators containing the projections onto its invariant subspaces. If $\mathscr{F}(\mathfrak{A})$ contains a spanning subset of mutually orthogonal finite-dimensional projections, then $\mathfrak{A}$ is reflexive.*

Proof. We follow the lines in the proof of Theorem 2 using, instead of $\mathscr{F}(\mathfrak{A})$, the spanning subset described above. In proving that $P\mathfrak{A}P = P\mathfrak{B}(\mathscr{H})P$ in part (i) of the proof of Theorem 2 we use Burnside's Theorem (**2**, p. 276) rather than Arveson's, thus not requiring the existence of an m.a.s.a. (Burnside's Theorem states that an algebra $\mathfrak{A}$ of operators on a finite-dimensional space is $\mathfrak{B}(\mathscr{H})$ if and only if Lat $\mathfrak{A} = \{\{0\}, \mathscr{H}\}$.)

The following theorem provides an alternate description of $\mathscr{F}(\mathfrak{A})$. Letting $\mathfrak{A}^* = \{A: A^* \in \mathfrak{A}\}$, we note that $\mathfrak{A} \cap \mathfrak{A}^*$ is the largest von Neumann algebra contained in the weakly closed algebra $\mathfrak{A}$.

THEOREM 4. *If $\mathfrak{A}$ is a weakly closed algebra containing an m.a.s.a., and if $\mathfrak{S}$ is the commutant of $\mathfrak{A} \cap \mathfrak{A}^*$, then P is in $\mathscr{F}(\mathfrak{A})$ if and only if P is an atomic projection in $\mathfrak{S}$.*

Proof. Suppose first that $P \in \mathscr{F}(\mathfrak{A})$ and $P = P_2 - P_1$ with P_2 covering P_1. Then $P_i \in$ Lat $\mathfrak{A}$ implies $P_i \in \text{Lat}(\mathfrak{A} \cap \mathfrak{A}^*)$, and hence $P_i \in \mathfrak{S}$. Thus $P \in \mathfrak{S}$. To see that P is an atom, suppose that Q is a projection in $\mathfrak{S}$ and that $Q \leqq P$. As shown in the proof of Theorem 2, $P\mathfrak{A}P = P\mathfrak{B}(\mathscr{H})P$, so that $P\mathfrak{A}P \subseteq \mathfrak{A} \cap \mathfrak{A}^*$. Thus $Q \in \mathfrak{S}$ and $Q \leqq P$ imply that Q commutes with $P\mathfrak{B}(\mathscr{H})P$. Hence $Q = 0$ or $Q = P$.

Conversely, suppose that P is an atom in $\mathfrak{S}$. Let P_2 be the smallest member of Lat $\mathfrak{A}$ which contains P, and let P_1 be the largest member of Lat $\mathfrak{A}$ which is contained in P_2 and is orthogonal to P. Let $P_0 = P_2 - P_1$. We must show that $P_0 \in \mathscr{F}(\mathfrak{A})$ and that $P_0 = P$.

Suppose that $P_1 \leqq Q \leqq P_2$ and that Q is in Lat $\mathfrak{A}$. Then $PQ \in \mathfrak{S}$ and $PQ \leqq P$. Thus $PQ = P$ or $PQ = 0$. If $PQ = P$, then $Q \geqq P$ and therefore $Q \geqq P_2$; thus $Q = P_2$. If $PQ = 0$, then $Q \perp P$ and therefore $Q \leqq P_1$, and then $Q = P_1$. This shows that $P_0 \in \mathscr{F}(\mathfrak{A})$.

It is clear that $P \leqq P_0$. As we have seen in the first part of this proof, the fact that $P_0 \in \mathscr{F}(\mathfrak{A})$ implies that $P_0\mathfrak{A}P_0 = P_0\mathfrak{B}(\mathscr{H})P_0$. Thus P commutes with $P_0\mathfrak{B}(\mathscr{H})P_0$ and must be P_0.

COROLLARY 3. *If $\mathfrak{A}$ is a weakly closed algebra containing an m.a.s.a., and if the collection of atomic projections in the commutant of $\mathfrak{A} \cap \mathfrak{A}^*$ spans $\mathscr{H}$, then $\mathfrak{A}$ is reflexive.*

Proof. This follows immediately from Theorems 2 and 4.

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University of Toronto,
Toronto, Ontario