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OSCILLATION AND COMPARISON THEOREMS FOR CERTAIN NEUTRAL DIFFERENCE EQUATIONS

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Abstract

Some comparison theorems and oscillation criteria are established for the neutral difference equation

$$\Delta(x_n + cx_{n-m}) + p_n x_{n-k} = 0, \qquad n = 0, 1, 2, \dots,$$

as well as for certain neutral difference equations with coefficients of arbitrary sign. Neutral difference equations with mixed arguments are also considered.

1. Introduction

The problem of oscillation and nonoscillation of solutions of delay difference equations has received a great amount of attention in the last few years. Erbe and Zhang [1], Gedrgiou, Grove and Ladas [2], Ladas, Philos and Sficas [4], and Gyori and Ladas [3] have done extensive work on this topic. The problem of oscillation of all solutions of the neutral difference equation

$$\Delta(y_n + cy_{n-m}) + p_n y_{n-k} = 0, \qquad n = 0, 1, 2, \dots, \tag{1.1}$$

has been investigated in [2, 8], where Δ denotes the forward difference operator:

$$\Delta y_n = y_{n+1} - y_n.$$

The main purpose of this paper is to establish some oscillation and comparison results for neutral difference equations.

In Section 2 we obtain some comparison results for the oscillation of (1.1). To the best of our knowledge this is the first time that the comparison results

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for (1.1) have been presented. An oscillation criterion for the neutral difference equations with positive and negative coefficients is given in Section 3. In Section 4 we consider the oscillation of some neutral difference equations with mixed arguments.

Let $M = \max\{m, k\}$, where m and k are non-negative integers. By a solution to (1.1) we mean a sequence $\{y_n\}$ which is defined for $n \geq N - M$ and which satisfies (1.1) for all $n \geq N$. Clearly if

$$y_n = A_n, \quad \text{for } n = N - M, \dots, N, \tag{1.2}$$

are given, then (1.1) has a unique solution satisfying the initial conditions (1.2), where N is an initial point. A nontrivial solution $\{y_n\}$ of (1.1) is said to be oscillatory if for every $N > 0$ there exists an $n \geq N$ such that $y_n y_{n+1} \leq 0$. Otherwise it is called nonoscillatory.

Difference equations are appropriate models for describing situations where population growth is not continuous but seasonal, with overlapping generations. For example, the difference equation

$$y_{n+1} = y_n \exp\left[r\left(1 - \frac{y_n}{K}\right)\right] \tag{1.3}$$

has been used to model various animal populations. It is also known [6] that for certain values of parameter r, the behavior of the solutions to (1.3) is chaotic. In fact for $r > 3\cdot 102$ there are orbits of period three and this implies chaos. Equation (1.3) is considered by some to be the discrete analogue of the logistic differential equation

$$y'(t) = ry(t)\left[1 - \frac{y(t)}{K}\right], \tag{1.4}$$

where r and K are the growth rate and the carrying capacity of population respectively. One can study (1.3) by combining the linearisation method employed by Ladas [6] and the comparison methods developed in this paper. From the point of view of applications it is important to study neutral difference equations, since such equations are discrete analogues of neutral delay differential equations which appear in problems dealing with networks containing lossless transmission lines. Such networks arise in high speed computers where transmission lines are used to interconnect circuits.

2. Main results

First we establish a necessary and sufficient condition for the oscillation of (1.1). This condition turns out to be very useful for deriving sufficient conditions for oscillation of neutral difference equations with mixed arguments and equations with nonlinear terms.

Our first result is the following:

THEOREM 2.1. *Assume that* $-1 < c \le 0$, $p_n > 0$. *Then every solution of* (1.1) *is oscillatory if and only if*

$$\Delta(y_n + cy_{n-m}) + p_n y_{n-k} \le 0 \tag{2.1}$$

has no eventually positive solution.

PROOF. The sufficiency is obvious. Suppose $\{y_n\}$ is an eventually positive solution of (2.1). We shall show that (1.1) has a positive solution also.

Let

$$Z_n = y_n + cy_{n-m}. \tag{2.2}$$

Then

$$\Delta Z_n < 0, \qquad Z_n > 0 \text{ eventually.}$$

Define

$$w_n = -\frac{\Delta Z_n}{Z_n} > 0, \qquad n \ge N. \tag{2.3}$$

It is obvious that $w_n < 1$ for $n \ge N$. Inequality (2.1) can be rewritten in the form

$$\Delta Z_n + p_n Z_{n-k} - c\frac{p_n}{p_{n-m}} p_{n-m} y_{n-m-k} \le 0. \tag{2.4}$$

Dividing (2.4) by Z_n and using (2.3) we have

$$w_n \ge p_n \prod_{i=n-k}^{n-1} (1-w_i)^{-1} - c\frac{p_n}{p_{n-m}} w_{n-m} \prod_{i=n-m}^{n-1} (1-w_i)^{-1}. \tag{2.5}$$

Define

$$\{\lambda_n^{(0)}\} = \{0\}, \qquad n = N, N+1, \dots \quad \text{for } n = N, \dots, N+M-1,$$

$$\{\lambda_n^{(r)}\} = \left\{ p_n \prod_{j=n-k}^{n-1} (1-\lambda_j^{(r-1)})^{-1} - c\frac{p_n}{p_{n-m}} \lambda_{n-m}^{(r-1)} \prod_{j=n-m}^{n-1} (1-\lambda_j^{(r-1)})^{-1} \right\} \quad \text{for } n \ge N+M. \tag{2.6}$$

It is not difficult to prove that

$$\lambda_n^{(0)} \le \lambda_n^{(1)} \le \cdots \le \lambda_n^{(r)} \le \cdots \le w_n, \qquad r \ge 0, \ n \ge N.$$

Then for each fixed $n \ge N$, we have

$$\lim_{r\to\infty} \lambda_n^{(r)} = \lambda_n. \tag{2.7}$$

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Taking the limit in (2.6), we get

$$\lambda_n = p_n \prod_{j=n-k}^{n-1} (1-\lambda_j)^{-1} - c\frac{p_n}{p_{n-m}}\lambda_{n-m} \prod_{j=n-m}^{n-1} (1-\lambda_j)^{-1}, \qquad n \geq N+M,$$
$$\lambda_n = 0, \qquad n = N, \dots, N+M-1. \tag{2.8}$$

Define

$$Z_N = 1$$
$$Z_{n+1} = Z_n(1-\lambda_n) \qquad \text{for } n \geq N.$$

That is,

$$Z_{N+1} = (1-\lambda_N)$$
$$Z_{N+2} = (1-\lambda_N)(1-\lambda_{N+1})$$
$$\cdots = \cdots$$
$$Z_n = \prod_{i=N}^{n-1}(1-\lambda_i) > 0 \qquad \text{for } n \geq N.$$

Thus

$$\Delta Z_n = -Z_n\lambda_n < 0. \tag{2.9}$$

So

$$\lambda_n = -\frac{\Delta Z_n}{Z_n} > 0. \tag{2.10}$$

Substituting (2.10) into (2.8), we have

$$-\frac{\Delta Z_n}{Z_n} = p_n\frac{Z_{n-k}}{Z_n} - c\frac{p_n}{p_{n-m}}\left(l - \frac{Z_{n-m+1}}{Z_{n-m}}\right)\frac{Z_{n-m}}{Z_n}, \quad \text{for } n \geq N+M.$$

So

$$-\Delta Z_n = p_n Z_{n-k} - c\frac{p_n}{p_{n-m}}(-\Delta Z_{n-m}), \quad \text{for } n \geq N+M$$

or

$$-\frac{\Delta Z_n}{p_n} = Z_{n-k} + \frac{c}{p_{n-m}}\Delta Z_{n-m}, \quad \text{for } n \geq N+M. \tag{2.11}$$

Define

$$x_n = -\frac{\Delta Z_{n+k}}{p_{n+k}} > 0, \quad \text{for } n \geq N+M-k. \tag{2.12}$$

By combining (2.11) and (2.12), we obtain

$$x_{n-k} = Z_{n-k} - cx_{n-m-k}, \quad \text{for } n \geq N+M,$$

that is

$$Z_{n-k} = x_{n-k} + cx_{n-k-m}. \tag{2.13}$$

Now substituting (2.13) into (2.12), we get

$$\Delta(x_n + cx_{n-m}) + p_n x_{n-k} = 0, \quad \text{for } n \geq N + M.$$

Thus, $\{x_n\}$, $n \geq N + M - k$, is a positive solution of (1.1), which is a contradiction. This completes the proof of the theorem.

REMARK 2.1. Theorem 1 in [9] is a special case of Theorem 2.1.

The following results illustrate applications of Theorem 2.1. We consider

$$\Delta(y_n + cy_{n-m}) + p_n y_{n-k} + f(n, y_{n-h_1}, \ldots, y_{n-h_l}) = 0 \tag{2.14}$$

and establish the following:

THEOREM 2.2. *Suppose that the assumptions of Theorem* 2.1 *hold and that*

$$f(n, \xi_1, \ldots, \xi_l)\xi_1 \geq 0 \text{ whenever } \xi_1\xi_j > 0, \ j = 1, \ldots, l.$$

Then the oscillation of (1.1) *implies the oscillation of* (2.14).

PROOF. If not, without loss of generality let $\{y_n\}$ be an eventually positive solution of (2.14). Then

$$\Delta(y_n + cy_{n-m}) + p_n y_{n-k} \leq 0,$$

has an eventually positive solution which contradicts the conclusion of Theorem 2.1.

REMARK 2.2. It follows that the oscillation of (1.1) implies that of

$$\Delta(y_n + cy_{n-m}) + p_n y_{n-k} + q_n y_{n+h} = 0 \tag{2.15}$$

where

$$-1 < c \leq 0, \ p_n \geq 0, \ q_n \geq 0, \ k, h > 0.$$

Next we consider the equation

$$\Delta(y_n + \bar{c}y_{n-m}) + q_n y_{n-k} = 0, \tag{2.16}$$

and derive the following result:

THEOREM 2.3. *Assume that* $\bar{c}, c \in (-1, 0]$ *and that*

$$q_n \geq p_n > 0, \tag{2.17}$$

$$\bar{c}q_n/q_{n-m} \leq cp_n/p_{n-m}. \tag{2.18}$$

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Then the oscillation of (1.1) *implies that of* (2.16).

PROOF. If not, let $\{y_n\}$ be a positive solution of (2.16), for $n \geq N$. Then as in the proof of Theorem 2.1, there exists a sequence $\{w_n\}$, $w_n \in (0, 1)$ such that

$$w_n = q_n \prod_{i=n-k}^{n-1} (1 - w_i)^{-1} - \bar{c}\frac{q_n}{q_{n-m}} w_{n-m} \prod_{i=n-m}^{n-1} (1 - w_i)^{-1}. \tag{2.19}$$

In view of (2.17) and (2.18) it follows from (2.19) that

$$w_n \geq p_n \prod_{i=n-k}^{n-1} (1 - w_i)^{-1} - c\frac{p_n}{p_{n-m}} w_{n-m} \prod_{i=n-m}^{n-1} (1 - w_i)^{-1}.$$

Thus we get the inequality (2.5), which leads to the conclusion that (1.1) has a positive solution, and hence a contradiction.

THEOREM 2.4. *If*

$$q_i \geq p_i > 0, \tag{2.20}$$

$$\sum_{i=N}^{\infty} q_i = \infty, \qquad -1 < \bar{c} \leq c < 0, \tag{2.21}$$

then the oscillation of (1.1) *implies the oscillation of* (2.16).

PROOF. Suppose the contrary, and let $\{y_n\}$ be a positive solution of (2.16). With

$$Z_n = y_n + \bar{c} y_{n-m},$$

we have

$$\Delta Z_n + q_n y_{n-k} = 0.$$

It is easy to verify (see [8]) that $Z_n \to 0$ as $n \to \infty$.

Therefore

$$Z_n = \sum_{n}^{\infty} q_i y_{i-k}.$$

So

$$y_n = -\bar{c} y_{n-m} + \sum_{n}^{\infty} q_i y_{i-k} \geq -c y_{n-m} + \sum_{n}^{\infty} p_i y_{i-k}, \tag{2.22}$$

and hence

$$y_n \geq -c y_{n-m} \geq \cdots \geq (-c)^l y_{n-lm} = (-c)^{(n-n_0)/m} y_{n_0} = \alpha(-c)^{n/m},$$

$$\alpha = (-c)^{-n_0/m} y_{n_0}.$$

Define

$$\left.\begin{array}{ll}\{\lambda_n^{(0)}\} = \{y_n\}, & n \geq n_0 \\ \lambda_n^{(r)} = -c\lambda_{n-m}^{(r-1)} + \sum_n^\infty p_i \lambda_{i-k}^{(r-1)}, & n \geq n_0 + \max\{m, k\} \\ \lambda_n^{(r)} = \lambda_n^{(0)}, \text{ for } n_0 \leq n < n_0 + \max\{m, k\}, & r = 1, 2, 3, \dots\end{array}\right\}. \tag{2.23}$$

It is obvious that

$$\lambda_n^{(1)} \leq \lambda_n^{(0)}, \quad \text{for } n \geq n_0.$$

By induction

$$\lambda_n^{(r+1)} \leq \lambda_n^{(r)}, \quad \text{for } r = 0, 1, 2, \dots, \ n \geq n_0$$

and

$$\lambda_n^{(r)} \geq \alpha(-c)^{n/m}, \quad \text{for } r = 0, 1, 2, \dots.$$

Therefore for each $n \geq n_0$ we have

$$\lambda_n^{(r)} \to \lambda_n^* \quad \text{as } r \to \infty.$$

From (2.23), one gets

$$\lambda_n^{(*)} = -c\lambda_{n-m}^* + \sum_{i=n}^{\infty} p_i \lambda_{i-k}^*$$

and hence

$$\Delta(\lambda_n^* + c\lambda_{n-m}^*) = -p_n \lambda_{n-k}^*, \qquad n \geq n_0 + M. \tag{2.24}$$

That is, $\{\lambda_n^*\}$ is a positive solution of (1.1), which is a contradiction.

COROLLARY 2.1. *If $c = \bar{c} = 0$, then for the equations*

$$\Delta y_n + p_n y_{n-k} = 0 \tag{2.25}$$

and

$$\Delta x_n + q_n x_{n-k} = 0, \tag{2.26}$$

with $q_n \geq p_n$, $l \geq K$, the oscillation of (2.25) *implies the oscillation of* (2.26).

PROOF. Follows from that of Theorem 2.3.

REMARK 2.3. This corollary was established by Yan and Chuanxi [9] recently.

REMARK 2.4. It is easy to extend the above results to

$$\Delta(y_n + cy_{n-m}) + \sum_{i=1}^{l} p_{in} y_{n-k_i} = 0. \tag{2.27}$$

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3. Neutral difference equations with positive and negative coefficients

We consider now neutral difference equations with positive and negative coefficients, of the form

$$\Delta(y_n + cy_{n-m}) + p_n y_{n-k_1} - q_n y_{n-k_2} = 0, \tag{3.1}$$

and prove the following:

THEOREM 3.1. *Assume that*

$$-1 < c \le 0,\ p_n \ge 0,\ q_n \ge 0,\ k_1 > k_2 + 1,\ \textit{and}\ p_n - q_{n-(k_1-k_2)} \ge 0\ (\not\equiv 0),$$
$$\lim_{n\to\infty} \sum_{n-(k_1-k_2)}^{n-1} q_i = 0. \tag{3.2}$$

Further assume that every solution of

$$\Delta Z_n + (p_n - q_{n-(k_1-k_2)})Z_{n-k_1} = 0 \tag{3.3}$$

is oscillatory. Then every solution of (3.1) *is oscillatory.*

PROOF. If not, let $\{y_n\}$ be a positive solution of (3.1). Define

$$\begin{aligned} Z_n &= y_n + cy_{n-m} \\ w_n &= Z_n - \sum_{n-(k_1-k_2)}^{n-1} q_i y_{i-k_2}. \end{aligned} \tag{3.4}$$

Then

$$\begin{aligned} \Delta w_n &= \Delta Z_n - q_n y_{n-k_2} + q_{n-(k_1-k_2)} y_{n-k_1} \\ &= (q_{n-(k_1-k_2)} - p_n) y_{n-k_1} \end{aligned} \tag{3.5}$$

Since $q_{n-(k_1-k_2)} - p_n \le 0$ we have $\Delta w_n \le 0$. If $w_n \to -\infty$ as $n \to \infty$, then y_n must be unbounded. This implies that there exists an integer N such that $w_N < 0$,

$$y_N = \max_{n\le N}\{y_n > 0\} \quad \text{and} \quad 1 - c - \sum_{N-(k_1-k_2)}^{N-1} q_i \ge 0.$$

On the other hand,

$$\begin{aligned} 0 > w_N &= Z_N - \sum_{N-(k_1-k_2)}^{N-1} q_i y_{i-k_2} \\ &= y_N + c y_{N-M} + \sum_{N-(k_1-k_2)}^{N-1} q_i y_{i-k_2} \\ &\geq y_N \left(1 + c - \sum_{N-(k_1-k_2)}^{N-1} q_i\right) \geq 0, \end{aligned}$$

which is impossible.

Thus we must have $\lim_{n\to\infty} w_n = l$, where l is finite. Thus y_n is bounded and from (3.4) we have $\lim_{n\to\infty} Z_n = l$. If $l < 0$, then for n sufficiently large

$$y_n + c y_{n-m} \leq \frac{l}{2} < 0.$$

It follows that $y_n \to 0$ and hence $Z_n \to 0$ as $n \to \infty$. Consequently l cannot be negative. Therefore $l \geq 0$, and since $\Delta w_n \leq 0$, we have

$$w_n = Z_n - \sum_{n-(k_1-k_2)}^{n-1} q_i y_{i-k_2} \geq 0.$$

That is, $y_n \geq Z_n \geq w_n$. Substituting this into (3.5) we have

$$\Delta w_n + (p_n - q_{n-(k_1-k_2)}) w_{n-k_1} \leq 0.$$

Since

$$p_n - q_{n-(k_1-k_2)} \not\equiv 0, \qquad \Delta w_n \not\equiv 0,$$

it follows that $w_n > 0$ eventually, which implies that (3.3) has a positive solution. This contradicts the assumption that (3.3) is oscillatory. Thus the proof of the theorem is complete.

REMARK 3.1. It is known (see [1]) that every solution of (3.3) is oscillatory if

$$\liminf_{n\to\infty} (p_n - q_{n-(k_1-k_2)}) > \frac{k_1^{k_1}}{(1+k_1)^{1+k_1}}.$$

EXAMPLE 3.1. Consider

$$\Delta y_n + p_n y_{n-4} - q_n y_{n-1} = 0, \qquad n > 4, \tag{3.6}$$

where

$$p_n = \frac{2(n-4)}{n+1}, \qquad q_n = \frac{n-1}{n(n+1)}.$$

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Equation (3.6) satisfies all the assumptions of Theorem 3.1. Therefore every solution of (3.6) is oscillatory. In fact $\{y_n\} = \{(-1)^n \frac{1}{n}\}$ is such a solution.

4. Difference equations of mixed type

For the following difference equation of the mixed type:

$$\Delta(y_n + cy_{n-m}) + \sum_{i=1}^{l} p_i y_{n-k_i} = 0, \tag{4.1}$$

we have the following:

THEOREM 4.1. *Assume that*

(i) $-1 < c < 0$,

(ii) $p_i > 0$ *are constants,* m, k_i *are integers,* $m > 0$, k_i *with arbitrary sign,*

(iii)
$$\sum_{i=1}^{l} p_i (-c)^{\sigma_i} \left(\frac{\overline{k_i}+1}{\overline{k_i}}\right)^{\overline{k_i}} \left(\overline{k_i} + 1 + \frac{cm}{1+c}\right) > 1 + c, \tag{4.2}$$

where σ_i *is the smallest nonnegative integer such that*

$$\overline{k_i} = \sigma_i m + k_i > 0, \qquad i = 1, 2, \ldots, l.$$

Then every solution of (4.1) *is oscillatory.*

PROOF. If not, let $\{y_n\}$ be a positive solution of (4.1). Put

$$Z_n = y_n + cy_{n-m}. \tag{4.3}$$

It is easy to see that there exists a sufficiently large integer n_0 such that

$$Z_n > 0, \ \Delta Z_n < 0, \quad \text{for } n \geq n_0.$$

From (4.3), we have

$$\begin{aligned} y_n &= \sum_{j=0}^{h} (-c)^j Z_{n-jm} + (-c)^{h+1} y_{n-(h+1)m} \\ &> \sum_{j=0}^{h} (-c)^j Z_{n-jm}, \ n \geq n_0 + (h+1)m \end{aligned} \tag{4.4}$$

$$y_{n-k_i} > \sum_{j=0}^{h} (-c)^j Z_{n-k_i-jm}, \qquad n \geq n_0 + (h+1)m + \tau,$$

where $\tau = \max_{1 \leq i \leq l}\{m, |k_i|\}$.

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Substituting (4.4) into (4.1) we get

$$\Delta z_n + \sum_{i=1}^{l} p_i(-c)^{\sigma_i} \sum_{j=0}^{h-\sigma_i} (-c)^j Z_{n-\overline{k_i}-jm} \leq 0. \tag{4.5}$$

By Theorem 2.1, (4.5) implies that

$$\Delta Z_n + \sum_{i=1}^{l} p_i(-c)^{\sigma_i} \sum_{j=0}^{h-\sigma_i} (-c)^j Z_{n-\overline{k_i}-jm} = 0, \tag{4.6}$$

has a positive solution. On the other hand, from (4.2),

$$\sum_{i=1}^{l} p_i(-c)^{\sigma_i} \left(\frac{\overline{k_i}+1}{\overline{k_i}}\right)^{\overline{k_i}} \left(\overline{k_i} + 1 - \frac{cm}{1+c}\right) > 1 + c,$$

or

$$\sum_{i=1}^{l} p_i(-c)^{\sigma_i} \left(\frac{\overline{k_1}+1}{\overline{k_i}}\right)^{\overline{k_i}} \left[(\overline{k_i}+1)\sum_{j=0}^{\infty}(-c)^j - cm\sum_{j=0}^{\infty}(j+1)(-c)^j\right] > 1,$$

which we have obtained by dividing both sides by $1+c$, and by using the fact that

$$\left(\sum_{j=0}^{\infty} r^j\right)\left(\sum_{j=0}^{\infty} r^j\right) = \sum_{j=0}^{\infty}(j+1)r^j, \quad \text{for } 0 < r < 1.$$

Now we can choose $h > \sigma_i$ so that

$$\sum_{i=1}^{l} p_i(-c)^{\sigma_i} \left(\frac{\overline{k_i}+1}{\overline{k_i}}\right)^{\overline{k_i}} \left[(\overline{k_i}+1)\sum_{j=0}^{h-\sigma_i}(-c)^j - cm\sum_{j=0}^{h-\sigma_i-1}(j+1)(-c)^j\right] > 1$$

$$\text{i.e. } \sum_{i=1}^{l} p_i(-c)^j \left(\frac{\overline{k_i}+1}{\overline{k_i}}\right)^{\overline{k_i}} \left[\sum_{j=0}^{h-\sigma_i}(-c)^j(\overline{k_i}+1+jm)\right] > 1. \tag{4.7}$$

Since $(\frac{x+1}{x})^x$ is an increasing function for $x > 0$, we have

$$\left(\frac{\overline{k_i}+1}{\overline{k_i}}\right)^{\overline{k_i}} < \left(\frac{\overline{k_i}+1+jm}{\overline{k_i}+jm}\right)^{\overline{k_i}+jm}, \qquad j > 0. \tag{4.8}$$

We substitute (4.8) into (4.7) to obtain

$$\sum_{i=1}^{l} p_i(-c)^{\sigma_i} \sum_{j=0}^{h-\sigma_i}(-c)^j \frac{(\overline{k_i}+1+jm)^{\overline{k_i}+1+jm}}{(\overline{k_i}+jm)^{\overline{k_i}+jm}} > 1. \tag{4.9}$$

By a known result [1], (4.9) implies that every solution of (4.6) is oscillatory. This contradiction proves the theorem.

REMARK 4.1. If $k_i > 0$, then $\sigma_i = 0$, $\overline{k_i} = k_i$, and then (4.2) reduces to

$$\sum_{i=1}^{l} p_i \left(\frac{k_i+1}{k_i}\right)^{k_i} \left(k_i + 1 + \frac{cm}{1+c}\right) > 1 + c. \tag{4.10}$$

If we further assume that $c = 0$, then (4.10) becomes

$$\sum_{i=1}^{l} p_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1,$$

which coincides with the result in [1].

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