Canad. Math. Bull. Vol. **37** (3), 1994 pp. 365–373

A UNIMODAL PROPERTY OF PURELY IMAGINARY ZEROS OF BESSEL AND RELATED FUNCTIONS

C. G. KOKOLOGIANNAKI, M. E. MULDOON AND P. D. SIAFARIKAS

ABSTRACT. We show, among other things, that, for $n = 0, 1$, the negative of the square of a purely imaginary zero of $J_\nu^{(n)}(x)$ is unimodal on $(n-2, n-1)$. One of the important tools in the proof is the Mittag-Leffler partial fractions expansion of $J_{\nu+1}(z)/J_\nu(z)$.

1. **Introduction.** For $n = 0, 1, 2$ the smallest positive zero of $J_\nu^{(n)}(x)$ decreases to 0 as ν decreases to $n-1$. For $n = 0, 1$, this is a classical result [14, Chapter 15], while for $n = 2$ it is very recent; see [10], [12], [13], [15]. As ν decreases below $n-1$ the zero becomes purely imaginary first moving away from the origin and then returning to the origin (along the imaginary axis) as ν approaches $n-2$. This behaviour was observed numerically by Kerimov and Skorokhodov ([8] and [9]). It can be described by saying that the negative of the square of such a purely imaginary zero is *unimodal* on $(n-2, n-1)$. Curiously, this was proved analytically [4] first for the case $n = 2$, *i.e.*, for a purely imaginary zero of $J_\nu''(x)$ on $(0, 1)$. Our main purpose here is to deal with the cases $n = 0, 1$, *i.e.*, to prove the corresponding properties of the purely imaginary zeros of $J_\nu(x)$ and $J_\nu'(x)$ on $(-2, -1)$ and $(-1, 0)$ respectively. We also give a slightly simpler version of the proof in the case $n = 2$. The case $n = 0$ is illustrated in Figure 1.

We will have need of the Bessel differential equation

$$z^2 J_\nu''(z) + zJ_\nu'(z) + (z^2 - \nu^2)J_\nu(z) = 0, \tag{1.1}$$

the recurrence relations [14, p. 45]

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z), \tag{1.2}$$

and

$$zJ_\nu'(z) - \nu J_\nu(z) = -zJ_{\nu+1}(z), \tag{1.3}$$

as well as the Mittag-Leffler type expansion [14, p. 498]

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \sum_{n=1}^{\infty} \frac{2z}{j_{\nu n}^2 - z^2}, \tag{1.4}$$

Work supported by the Natural Sciences and Engineering Research Council (Canada).
Received by the editors December 8, 1992; revised March 8, 1993.
AMS subject classification: 33A40.

where $\{\pm j_{\nu n}\}$ is the sequence of zeros of the entire function $z^{-\nu}J_\nu(z)$. We note that these zeros are all real if $\nu > -1$, and according to the conventional notation [14, p. 497], $0 < j_{\nu 1} < j_{\nu 2} < \cdots$.

We use the method of the last part of [4], *i.e.*, we prove the unimodality of a function by showing that it is concave down at every turning point, hence there can be only one turning point.

2. **A general function.** All of the functions which we deal with can be subsumed in the general formula

$$f_\nu(z) = \left(cz^2 + f(\nu)\right)J_\nu(z) - azJ_{\nu+1}(z) + bzJ_{\nu-1}(z).$$

The most important special cases are:

(i) $f_\nu(z) = zJ_{\nu-1}(z)$, got by taking $b = 1$, $a = c = f(\nu) = 0$;

(ii) $f_\nu(z) = zJ'_\nu(z)$, got by taking $a = 1$, $b = c = 0, f(\nu) = \nu$ and using (1.3);

(iii) $f_\nu(z) = \alpha J_\nu(z) + zJ'_\nu(z)$, got by taking $a = 1$, $b = c = 0, f(\nu) = \alpha + \nu$;

(iv) $f_\nu(z) = -z^2J''_\nu(z)$, got by taking $a = c = 1$, $b = 0, f(\nu) = \nu - \nu^2$ and using (1.1) and (1.3).

THEOREM 2.1. *Let $\nu > -1$ and let $i\rho$ be a purely imaginary zero of $f_\nu(z)$. Then*

$$\lambda(\nu)\frac{d\rho^2}{d\nu} = \mu(\nu), \tag{2.1}$$

where

$$\lambda(\nu) = -\frac{f(\nu) + 2b\nu}{\rho^2} - 2(b+a)\sum_{n=1}^{\infty}\frac{\rho^2}{(j_{\nu,n}^2 + \rho^2)^2}, \tag{2.2}$$

or

$$\lambda(\nu) = -c + 2(b+a)\sum_{n=1}^{\infty}\frac{j_{\nu,n}^2}{(j_{\nu,n}^2 + \rho^2)^2}, \tag{2.3}$$

and

$$\mu(\nu) = 4(b+a)\rho^2\sum_{n=1}^{\infty}\frac{j_{\nu,n}dj_{\nu n}/d\nu}{(j_{\nu,n}^2 + \rho^2)^2} - f'(\nu) - 2b. \tag{2.4}$$

For those values of ν for which $d\rho^2/d\nu = 0$, we have

$$\lambda(\nu)\frac{d^2\rho^2}{d\nu^2} = \rho^2\mu_1(\nu), \tag{2.5}$$

where

(2.6)

$$\mu_1(\nu) = 4(b+a)\left[\sum_{n=1}^{\infty}\frac{j_{\nu,n}d^2j_{\nu n}/d\nu^2}{(j_{\nu,n}^2 + \rho^2)^2} + \sum_{n=1}^{\infty}\frac{(dj_{\nu n}/d\nu)^2}{(j_{\nu,n}^2 + \rho^2)^2} - 4\sum_{n=1}^{\infty}\frac{(j_{\nu n}dj_{\nu n}/d\nu)^2}{(j_{\nu,n}^2 + \rho^2)^3}\right] - f''(\nu)/\rho^2.$$

PROOF. The equation

$$f_\nu(z) = \big(cz^2 + f(\nu)\big)J_\nu(z) - azJ_{\nu+1}(z) + bzJ_{\nu-1}(z) = 0$$

can be put in the form

$$cz^2 + f(\nu) - az\frac{J_{\nu+1}(z)}{J_\nu(z)} + bz\frac{J_{\nu-1}(z)}{J_\nu(z)} = 0,$$

except at the zeros of $J_\nu(z)$.

Using (1.2) and (1.4) this becomes

$$cz^2 + f(\nu) - 2(b+a)\sum_{k=1}^{\infty}\frac{z^2}{j_{\nu k}^2 - z^2} + 2b\nu = 0.$$

For $z = i\rho$ we get

$$c - \frac{f(\nu) + 2b\nu}{\rho^2} = 2(b+a)\sum_{k=1}^{\infty}\frac{1}{j_{\nu k}^2 + \rho^2}. \tag{2.7}$$

Differentiating this equation with respect to ν and multiplying by ρ^2, we get (2.1), with $\lambda(\nu)$ given by (2.2) and $\mu(\nu)$ by (2.4). Then, using (2.7), we can express $\lambda(\nu)$ in the form (2.3). Differentiating (2.1) and using $d\rho^2/d\nu = 0$, we get (2.5), where $\mu_1(\nu)$ is given by (2.6).

In order to justify the term-by-term differentiation we have to verify that the differentiated series or, equivalently, all the infinite series in (2.3), (2.4) and (2.6), converge uniformly in ν, in any closed subinterval $[\nu_0, \nu_1]$ of $(-1, \infty)$. In the case of the series in (2.3), (2.4) and the second and third series in (2.6), this is a consequence of the inequality [11, p. 471]

$$(\nu+1)\frac{dj_{\nu k}}{d\nu} \le j_{\nu k} \tag{2.8}$$

and the convergence of

$$\sum_{k=1}^{\infty} j_{\nu k}^{-2}.$$

To deal with the first infinite series in (2.6), we use the representation [1, p. 87]

$$j'' = 2\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}I(\nu, t)\,dt \tag{2.9}$$

where

$$I(\nu, t) = 2\nu j'\tanh t + j'\tanh^2 t - 2jt. \tag{2.10}$$

Here we are using the notation $j = j_{\nu k}$ and the primes denote differentiation with respect to ν. For the uniform convergence of the first infinite series in (2.6), it is sufficient to show that

$$\frac{d^2 j_{\nu k}}{d\nu^2} < F(\nu)j_{\nu k}, \quad \nu_0 \le \nu \le \nu_1, \tag{2.11}$$

https://doi.org/10.4153/CMB-1994-054-3 Published online by Cambridge University Press

where $F(\nu)$ is bounded on $\nu_0 \leq \nu \leq \nu_1$. Now, from (2.10), j'' is bounded by

$$j'(2|\nu|+1)2\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}\,dt + 4j\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}t\,dt. \tag{2.12}$$

Using [14, p. 508],

$$j' = 2j\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}\,dt, \tag{2.13}$$

and (2.8), we find that the first term here is bounded by $(2|\nu|+1)(\nu+1)^{-2}j$. Using $j_{\nu k} > \nu+1$, $\sinh t > t$ and the decreasing character of $K_0(t)$ as a function of t, $t > 0$, we find that the second term is bounded by

$$4j\int_0^\infty K_0\big(2(\nu+1)t\big)e^{-2\nu t}t\,dt.$$

Thus (2.11) holds and this completes the verification of the validity of the differentiation and the proof of Theorem 2.1.

3. **Zeros of $J_\nu(z)$.** We first deal with the case $f_\nu(z) = zJ_{\nu-1}(z)$, got by taking $b = 1$, $a = c = f(\nu) = 0$. It is well known [14, p. 483] that, for $-2 < \nu < -1$, $J_\nu(z)$ has a pair of purely imaginary zeros.

THEOREM 3.1. *Let $\pm i\rho$ be the purely imaginary zeros of $J_\nu(z)$ for $-2 < \nu < -1$. Then ρ^2 is unimodal on* $(-2,-1)$.

PROOF. We will actually work with the zeros of $J_{\nu-1}(z)$ on $(-1,0)$. From §2, we have (2.1) with

$$\lambda(\nu) = 2\sum_{n=1}^\infty \frac{j_{\nu,n}^2}{(j_{\nu,n}^2+\rho^2)^2}, \tag{3.1}$$

and

$$\mu(\nu) = 4\rho^2\sum_{n=1}^\infty \frac{j_{\nu,n}dj_{\nu n}/d\nu}{(j_{\nu,n}^2+\rho^2)^2} - 2. \tag{3.2}$$

CASE (i): $-1 < \nu \leq -0.8$. In this range we will show that $d\rho^2/d\nu > 0$, so that ρ^2 is increasing. Since $\lambda(\nu) > 0$, we need to show that $\mu(\nu) > 0$ for this range of values of ν. We will need the following results [5]:

$$j_{\nu n}\frac{dj_{\nu n}}{d\nu} > 4 - \frac{8(\nu+1)(\nu+3)}{j_{\nu n}^2} + \frac{32(\nu+1)^2(\nu+2)^2}{j_{\nu n}^4}, \quad \nu > -1; \tag{3.3}$$

$$4(\nu+1) < j_{\nu 1}^2 < 4(\nu+1)(\nu+2), \quad \nu > -1. \tag{3.4}$$

We also need inequalities for $j_{\nu 1}^2$ in the case $-2 < \nu < -1$, when it is negative. Some simple bounds in this case are [7]

$$2(\nu+1)(\nu+3) < j_{\nu 1}^2 < 4(\nu+1)(\nu+2), \quad -2 < \nu < -1. \tag{3.5}$$

or, in terms of our present notation,

$$-4\nu(\nu+1) < \rho^2 < -2\nu(\nu+2), \quad -1 < \nu < 0. \tag{3.6}$$

https://doi.org/10.4153/CMB-1994-054-3 Published online by Cambridge University Press

We have

(3.7) $$\mu(\nu) = -2 + 4\rho^2 \sum_{n=1}^{\infty} \frac{j_{\nu,n} dj_{\nu n}/d\nu}{j_{\nu,n}^4} \left[1 + \frac{\rho^2}{j_{\nu n}^2}\right]^{-2}.$$

Clearly

$$\left[1 + \frac{\rho^2}{j_{\nu n}^2}\right]^{-1} > \left[1 + \frac{\rho^2}{j_{\nu 1}^2}\right]^{-1} > 1 - \frac{\rho^2}{j_{\nu 1}^2} > 1 + \frac{2\nu(\nu+2)}{j_{\nu 1}^2} > 1 + \frac{\nu(\nu+2)}{2(\nu+1)},$$

where we have used the upper bound in (3.6) and the lower bound in (3.4). On the other hand, from (3.3),

$$\sum_{n=1}^{\infty} \frac{j_{\nu n} dj_{\nu n}/d\nu}{j_{\nu n}^4} > 4\sigma_\nu^{(2)} - 8(\nu+1)(\nu+3)\sigma_\nu^{(3)} + 32(\nu+1)^2(\nu+2)^2\sigma_\nu^{(4)}.$$

Here we have used the notation

$$\sigma_\nu^{(n)} = \sum_{k=1}^{\infty} j_{\nu k}^{-2n}$$

and the closed form expressions for these sums in [14, p. 502]. This gives

$$\sum_{n=1}^{\infty} \frac{j_{\nu n} dj_{\nu n}/d\nu}{j_{\nu n}^4} > \frac{5\nu + 11}{8(\nu+1)^2(\nu+3)(\nu+4)}.$$

Using the above bounds and the lower bound for ρ^2 in (3.6), we get from (3.7)

$$\mu(\nu) > -2 - \frac{\nu(\nu^2 + 4\nu + 2)^2(5\nu + 11)}{2(\nu+1)^3(\nu+3)(\nu+4)},$$

and it is easy to check that this is positive for $-1 < \nu \le -0.8$.

CASE (ii): $-0.8 < \nu < 0$. Here we show that $d^2\rho^2/d\nu^2$ is negative at the turning points of ρ^2. In the present case (2.6) can be written

(3.8) $$\mu_1(\nu) = 4\left[\sum_{n=1}^{\infty} \frac{j_{\nu n} d^2 j_{\nu n}/d\nu^2}{(j_{\nu n}^2 + \rho^2)^2} + \sum_{n=1}^{\infty} \frac{(dj_{\nu n}/d\nu)^2}{(j_{\nu n}^2 + \rho^2)^2} - 4\sum_{n=1}^{\infty} \frac{(j_{\nu n} dj_{\nu n}/d\nu)^2}{(j_{\nu n}^2 + \rho^2)^3}\right].$$

Now the first term on the right here is negative [1] and the sum of the two remaining terms will certainly be negative provided that

(3.9) $$\rho^2 - 3j_{\nu 1}^2 < 0.$$

But

$$\rho^2 - 3j_{\nu 1}^2 < -2(\nu^2 + 8\nu + 6)$$

from (3.4) and (3.6) and this is certainly negative in case $-0.8 < \nu < 0$. Thus the second derivative of ρ^2 with respect to ν is negative at points where the first derivative is 0; hence there can be only one such point and it is a relative maximum. This proves the unimodal property and completes the proof of Theorem 3.1.

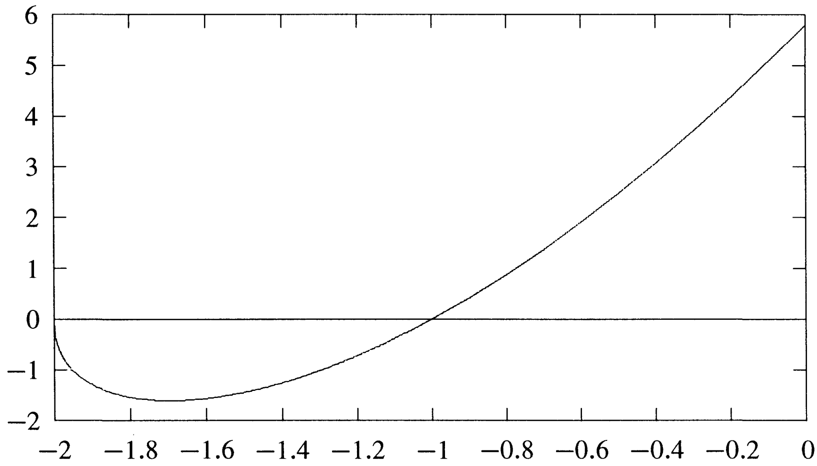

FIGURE 1: $j_{\nu 1}^2$ *vs.* ν

In Figure 1, we give the graph of $j_{\nu 1}^2$ versus ν for $-2 \le \nu \le 0$. This graph, based on a computation described in [7], suggests strongly that $j_{\nu 1}^2$ is convex on $-2 < \nu < 0$. It is shown [2] that it is convex on $(0, \infty)$ and conjectured that the convexity extends to $(-1, \infty)$.

A numerical calculation indicates that the smallest value of $j_{\nu 1}^2$ is -1.6075 to 5 digit accuracy and it occurs for ν between -1.698 and -1.697.

4. **Dini functions and derivatives of Bessel functions.** Here we deal with the case $f_\nu(z) = H_\nu(z) = \alpha J_\nu(z) + zJ_\nu'(z)$, got by taking $a = 1$, $b = c = 0, f(\nu) = \alpha + \nu$. We call these Dini functions because they arise in expansions due to Dini [14, Chapter 18]. In the special case $\alpha = 0$, we are, of course, dealing with the zeros of $J_\nu'(z)$. There do not appear to be many results in the literature on the monotonicity of purely imaginary zeros $\pm i\rho$ of $H_\nu(z)$, though it is shown in [6, pp. 78–79] that if $\alpha < 0$, then ρ^2 is decreasing on $(0, -\alpha)$. We will prove:

THEOREM 4.1. *Let* $H_\nu(z, \alpha) = \alpha J_\nu(z) + zJ_\nu'(z)$, *where* $-1/2 \le \alpha < 1$ *and* $-1 < \nu < -\alpha$. $H_\nu(z, \alpha)$ *has a pair of purely imaginary zeros* $\pm i\rho(\nu, \alpha)$. $\rho^2(\nu, \alpha)$ *is unimodal on* $(-1, -\alpha)$, *i.e., there exists a number* $\nu_0(\alpha)$ *such that* $\rho^2(\nu, \alpha)$ *increases on* $\big(-1, \nu_0(\alpha)\big)$ *and decreases on* $\big(\nu_0(\alpha), -\alpha\big)$.

COROLLARY. *If* $\pm i\rho$ *are purely imaginary zeros of* $J_\nu'(z)$ *then* ρ^2 *is unimodal on* $(-1, 0)$.

PROOF OF THEOREM 4.1. The question of existence of such zeros is equivalent to the question of whether equation (2.7) which is, in this case,

$$-(\nu+\alpha)=2\sum_{n=1}^{\infty}\frac{1}{j_{\nu n}^2/\rho^2+1}, \tag{4.1}$$

can be satisfied. The right-hand side here increases from 0 to ∞ as ρ increases from 0 to ∞, whereas the left-hand side remains constant and positive. Thus the existence of $\rho(\nu,\alpha)$ is clear. It is also clear from

$$\rho^2<-\frac{\alpha+\nu}{2+\alpha+\nu}j_{\nu 1}^2 \tag{4.2}$$

[6, (3.2)] that $\rho(\nu,\alpha)$ vanishes as $\nu\longrightarrow-\alpha^-$ and, since $j_{\nu 1}\longrightarrow 0$, it also vanishes as $\nu\longrightarrow -1^+$. In the present case, (2.5) holds with $\lambda(\nu)$ and $\mu_1(\nu)$ given by (3.1) and (3.8), *but with ρ interpreted as in the current section.* Now, as in §3, the first term on the right of (3.8) is negative [1] and the sum of the two remaining terms will certainly be negative provided that

$$\rho^2-3j_{\nu 1}^2<0. \tag{4.3}$$

But this follows from (4.2). Thus the second derivative of ρ^2 with respect to ν is negative at points where the first derivative is 0; hence there can be only one such point and it is a relative maximum. This proves the unimodal property.

Unfortunately, it does not seem to be possible to handle the case where $\alpha<-1/2$ in this way. The problem is that the inequality (4.3) seems to break down in this case. This is because $j_{\nu 1}^2\sim 4(\nu+1)$ and $\rho^2\sim-4(\nu+1)(\alpha-1)/(\alpha+1)$ as $\nu\longrightarrow -1^+$ [7].

With regard to the Corollary, it is of interest to point out that if $j'_{\nu 1}$ denotes the purely imaginary zero of $J'_\nu(x)$, then the smallest value of $j'^2_{\nu 1}$ is -0.60602 to 5 digit accuracy and it occurs for ν between -0.5699 and -0.5696.

REMARK. Since the purely imaginary zeros of $J_\nu(z)$ are real zeros of $I_\nu(z)$, we may restate the results on $j'_{\nu 1}$ in the following way: *The unique positive zero of $I'_\nu(x)$ increases from* 0 *to* i_0 (= 0.7759 *to four digits*) *as ν increases from* -1 *to* $\nu_0(0)$ $(-0.5699<\nu_0(0)<-0.5697)$ *and then decreases again to* 0 *as ν increases from* $\nu_0(0)$ *to* 0.

5. **Zeros of $J''_\nu(z)$.** Here we discuss the case $f_\nu(z)=-z^2J''_\nu(z)$, got by taking $a=c=1$, $b=0$ and $f(\nu)=\nu-\nu^2$. In this situation, we have (2.1) and (2.5) where it is better to leave $\lambda(\nu)$ in the form given by (2.2):

$$\lambda(\nu)=\frac{\nu^2-\nu}{\rho^2}-2\sum_{n=1}^{\infty}\frac{\rho^2}{(j_{\nu n}^2+\rho^2)^2}, \tag{5.1}$$

and the function $\mu_1(\nu)$ in (2.5) is given by

$$\mu_1(\nu)=4\left[\sum_{n=1}^{\infty}\frac{j_{\nu n}d^2j_{\nu n}/d\nu^2}{(j_{\nu n}^2+\rho^2)^2}+\sum_{n=1}^{\infty}\frac{(dj_{\nu n}/d\nu)^2}{(j_{\nu n}^2+\rho^2)^2}-4\sum_{n=1}^{\infty}\frac{(j_{\nu n}dj_{\nu n}/d\nu)^2}{(j_{\nu n}^2+\rho^2)^3}\right]+\frac{2}{\rho^2}. \tag{5.2}$$

https://doi.org/10.4153/CMB-1994-054-3 Published online by Cambridge University Press

We note that $\lambda(\nu) < 0$, $0 < \nu < 1$, so in order to get the unimodality of $j_{\nu 1}'^{2}$, we need to show that $\mu_1(\nu) > 0$ in this interval. Apart from a change in notation (ν being replaced by $-\nu$) the approach and formulas here agree with those in [4]. However, we can simplify the proof given in [4] by noting that the first two terms in (5.2), when combined, are equal to

$$2\sum_{n=1}^{\infty} \frac{(d^2 j_{\nu n}^2/d\nu^2)}{(j_{\nu n}^2 + \rho^2)^2}$$

which is clearly positive since $d^2 j_{\nu n}^2/d\nu^2 > 0$, $\nu > 0$ [2]. Thus it remains to show that the sum of the remaining two terms is positive, *i.e.*, that

$$8\sum_{n=1}^{\infty} \frac{(j_{\nu n} dj_{\nu n}/d\nu)^2}{(j_{\nu n}^2 + \rho^2)^3} < \frac{1}{\rho^2}. \tag{5.3}$$

But from [3, (1.5)], we have

$$\frac{1}{\rho^2} > \frac{2\nu + 1}{2\nu(1 - \nu^2)}. \tag{5.4}$$

while the inequality (2.8) and the Rayleigh sum [14, p. 502]

$$\sum_{n=1}^{\infty} j_{\nu n}^{-2} = 1/[4(\nu + 1)]$$

show that the left-hand side of (5.3) is $< 2/(\nu + 1)^3$. It is a simple matter to show that

$$\frac{2}{(\nu + 1)^3} < \frac{2\nu + 1}{2\nu(1 - \nu^2)}$$

for $0 < \nu < 1$, so we find that $\mu_1(\nu) > 0$, $0 < \nu < 1$ and this completes the proof.

ACKNOWLEDGMENTS. We are grateful to a referee whose careful reading led to the elimination of a number of errors. We thank Lee Lorch for some helpful comments.

REFERENCES

1. Á. Elbert, *Concavity of the zeros of Bessel functions*, Studia Sci. Math. Hungar. **12**(1977), 81–88.
2. Á. Elbert and A. Laforgia, *On the square of the zeros of Bessel functions*, SIAM J. Math. Anal. **15**(1984), 206–212.
3. E. K. Ifantis, P. D. Siafarikas and C. B. Kouris, *The imaginary zeros of a mixed Bessel function*, Z. Angew. Math. Phys. **39**(1988), 157–165.
4. E. K. Ifantis and P. D. Siafarikas, *A result on the imaginary zeros of $J_\nu''(z)$*, J. Approx. Theory **62**(1990), 192–196.
5. M. E. H. Ismail and M. E. Muldoon, *On the variation with respect to a parameter of zeros of Bessel and q-Bessel functions*, J. Math. Anal. Appl. **135**(1988), 187–207.
6. ———, *Zeros of combinations of Bessel functions and their derivatives*, Appl. Anal. **31**(1988), 72–90.
7. ———, *Bounds for the small real and purely imaginary zeros of Bessel and related functions*, Meth. Appl. Anal., to appear.
8. M. K. Kerimov and S. L. Skorokhodov, *Evaluation of complex zeros of Bessel functions $J_\nu(z)$ and $I_\nu(z)$ and their derivatives*, Comput. Math. Math. Phys. **24**(1984), 131–141; Russian original, Zh. Vychisl. Mat. i Mat. Fiz. **24**(1984), 1497–1513.
9. ———, *Calculation of the multiple zeros of the derivatives of cylindrical Bessel functions $J_\nu(z)$ and $Y_\nu(z)$*, Comput. Math. Math. Phys. **25**(1985), 101–107; Russian original, Zh. Vychisl. Mat. i Mat. Fiz. **25**(1985), 1749–1760.

10. C. G. Kokologiannaki and P. D. Siafarikas, *An alternative proof of the monotonicity of* $j_{\nu 1}''$, Boll. Un. Mat. Ital. (7-A) **7**(1993), 373–376.
11. A. Laforgia and M. E. Muldoon, *Monotonicity and concavity properties of zeros of Bessel functions*, J. Math. Anal. Appl. **98**(1984), 470–477.
12. L. Lorch and P. Szego, *On the points of inflection of Bessel functions of positive order*, Canad. J. Math. **42**(1990), 933–948; *ibid.*, 1132.
13. A. McD. Mercer, *The zeros of* $az^2J_\nu''(z)+bzJ_\nu'(z)+cJ_\nu'(z)$ *as a function of order*, Internat. J. Math. Math. Sci. **15**(1992), 319–322.
14. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, 1944.
15. R. Wong and T. Lang, *On the points of inflection of Bessel functions of positive order, II*, Canad. J. Math. **43**(1991), 628–651.

Department of Mathematics
University of Patras
261 10 Patras
Greece

Department of Mathematics and Statistics
York University
North York, Ontario
M3J 1P3

Department of Mathematics
University of Patras
261 10 Patras
Greece