ON STONE LATTICES

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Introduction

M. H. Stone raised the problem ([1] Problem 70) of characterising the class of distributive pseudo-complemented lattices $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$ in which $a^* \vee a^{**} = 1$ holds identically. Several solutions to this problem have now been offerred — the first being by G. Grätzer and E. T. Schmidt [6], who gave this class of lattices the name Stone lattices. Later solutions were given by J. Varlet [11], O. Frink [4] and G. Grätzer [5]; see also G. Bruns [2].

The purpose of this paper is to show that suitably restated, several of the known characterisations of Stone lattices hold for a wider class of distributive lattices. New characterisations are also given which relate old ones in an illuminating way. We also study the direct decomposition of certain complete Stone lattices, and prove an embedding theorem for Stone lattices. The final section gives some hitherto unmentioned examples of Stone lattices.

1. Preliminaries

We shall assume a familiarity with the basic ideas of lattice theory — see G. Birkhoff [1]. The *centre* $\mathscr{Z}(\mathscr{L}) = \langle Z(\mathscr{L}); \vee, \wedge, 0, 1\rangle$ of the lattice $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$ is defined on page 27 of [1]. Since we are exclusively dealing with distributive lattices in this paper, $Z(\mathscr{L})$ consists of exactly those elements of L which possess a complement. The *principal ideal* generated by an element $a \in L$ in $\mathscr{L} = \langle L; \vee, \wedge\rangle$ is written (a), and the ideal generated by a subset $S \subseteqq L$ in $\mathscr{L}$ is written $(S)_{\mathscr{L}}$ or just (S). For two subsets A and B of L we define

$$A \vee B = \{t \in L : t = a \vee b,\ a \in A,\ b \in B\},$$

and

$$A \wedge B = \{t \in L : t = a \wedge b,\ a \in A,\ b \in B\}.$$

If A and B are ideals of the distributive lattice $\mathscr{L}$, then $A \vee B$ is the join of A and B in the lattice $\mathscr{I}(\mathscr{L}) = \langle I(\mathscr{L}); \vee, \cap\rangle$ of all ideals of $\mathscr{L}$.

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In a distributive lattice with zero, the existence of minimal prime ideals can readily be proved — see Lemma 2 of [6] — and we let $\mathscr{M}(\mathscr{L})$ denote the set of all *minimal prime ideals* of $\mathscr{L}$. Also let $\mathscr{P}(\mathscr{L})$ denote the set of all *prime ideals* of $\mathscr{L}$.

For a subset $A \subseteqq L$ in a lattice $\mathscr{L} = \langle L; \vee, \wedge, 0\rangle$ we define the annihilator $A^* = \{t \in L; \{t\} \wedge A = \{0\}\}$. If $A = \{a\}$ where $a \in L$, we write $\{a\}^* = (a)^*$ and call a *dense* if $(a)^* = \{0\}$. In a distributive lattice $\mathscr{L} = \langle L; \vee, \wedge, 0\rangle$ the set D of dense elements forms a dual ideal of $\mathscr{L}$. A lattice $\mathscr{L} = \langle L; \vee, \wedge, 0\rangle$ is said to be a *dense lattice* if $(a)^* = \{0\}$ for any $a \neq 0$, i.e. $D = L \diagdown \{0\}$.

A congruence often studied in distributive lattices with zero is the congruence R defined by

$$\langle a, b\rangle \in R \quad \text{iff} \quad (a)^* = (b)^*.$$

2. More general characterisations of Stone lattices

In this section we shall extend some known characterisations of Stone lattices to a wider class of distributive lattices than pseudo-complemented distributive lattices. Our extension is to the class of lattices known as distributive *-lattices introduced recently in [8]. Denoting the set of all such lattices by Δ^*, we have

DEFINITION. $\mathscr{L} \in \Delta^*$ if and only if $\mathscr{L}/R$ is a Boolean lattice.

THEOREM 2.1. ([8] 3.4) *For a distributive lattice* $\mathscr{L} = \langle L; \vee, \wedge, 0\rangle$ *the following are equivalent*

I. $\mathscr{L} \in \Delta^*$

II. *For any* $x \in L$, $(x)^{**} = (x')^*$ *for some* $x' \in L$.

III. *For any* $x \in L$, *there is* $x' \in L$ *such that* $x \wedge x' = 0$, $x \vee x' \in D$.

IV. *For every ideal* I *of* $\mathscr{L}$ *such that* $I \cap D = \square$, *we have* $I \subseteqq M$ *for some minimal prime ideal* M *of* $\mathscr{L}$.

REMARKS. (i) II is the condition we shall use below.
(ii) III was kindly supplied by J. Varlet.

Before going on with our extensions, we mention the following topological characterisation of Stone lattices. The proof of the result is omitted to avoid going into details of the hull-kernel topology. Here

$$\mathscr{P}_x = \{P \in \mathscr{P}(\mathscr{L}) : x \notin P\}.$$

PROPOSITION 2.2. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$ *be a distributive lattice. Then the following are equivalent:*

I. *$\mathscr{L}$ is a Stone lattice*
II. *For any $x \in L$, $(x)^* \vee (x)^{**} = L$*
III. *For any $x \in L$, $\mathscr{P}_x^-$ is open, in the hull-kernel topology.*

Our next result is a direct generalisation of the main result of G. Grätzer and E. T. Schmidt [6] to the class Δ^*. Our proof is modelled on theirs, but since this result is used in § **3** we shall give a full proof.

REMARK. Condition II above will be taken as our definition of a Stone lattice, when $\mathscr{L}$ is an arbitrary distributive lattice with zero and unit.

PROPOSITION 2.3. *Let $\mathscr{L} \in \Delta^*$. If the lattice theoretical join $P \vee Q$ of any two distinct minimal prime ideals P and Q of $\mathscr{L}$ is L, then $\mathscr{L}$ is a Stone lattice.*

PROOF. Suppose $\mathscr{L}$ is not a Stone lattice. Then there is an $x \in L$ with $(x)^* \vee (x)^{**} = (x)^* \vee (x')^* \subsetneqq L$.

In this case there is a prime dual ideal F not meeting $(x)^* \vee (x')^*$. Let Θ denote the least congruence with F as a (unit) congruence class and θ the induced homomorphism onto $\bar{\mathscr{L}} = \mathscr{L}/\Theta$. It is known that $\langle x, y\rangle \in \Theta$ iff $x \wedge y = (x \vee y) \wedge a$ for some $a \in F$. We shall see that the inverse image Q of a minimal prime ideal $\bar{Q}$ of $\bar{\mathscr{L}}$ under θ, is a minimal prime ideal of $\mathscr{L}$. Assume there is a minimal prime ideal Q_1 of $\mathscr{L}$ such that $Q_1 \subseteqq Q$. Clearly $\bar{Q}_1 = Q_1\theta = \bar{Q}$. Thus for any $q \in Q$ there is $q_1 \in Q_1$ with $\langle q, q_1\rangle \in \Theta$. We may take $q_1 \leqq q$. This implies $q_1 = q \wedge a$ where $a \in F$ and $F \cap Q_1 = \square$. Since $a \notin Q_1$ and $q_1 \in Q_1$ we deduce that $q \in Q_1$ and hence $Q = Q_1$.

From this result it follows that the join of any two distinct minimal prime ideals of $\bar{\mathscr{L}}$ is the whole lattice. Now it also follows from the fact that F is a prime dual ideal, that the unit $\bar{1}$ of $\bar{\mathscr{L}}$ is join-irreducible. Thus $\bar{\mathscr{L}}$ contains a single minimal prime ideal which must be the ideal $(\bar{0})$.

We have now seen that $(x)^* \vee (x')^* \subsetneqq L$ for some $x \in L$ implies there is a prime dual ideal F inducing a congruence Θ such that $\mathscr{L}/\Theta$ has a meet-irreducible zero. It can now be shown that this is a contradiction, for if

(i) $\langle x, 0\rangle \in \Theta$ then this would imply $x \wedge a = 0$ for some $a \in F$ but $F \cap (x)^* = \square$. Similarly if

(ii) $\langle x', 0\rangle \in \Theta$ then $(x')^* \cap F = (x)^{**} \cap F \neq \square$.

However $x \wedge x' = 0$ and so we have

$$\bar{x} \neq \bar{0},\ \bar{x}' \neq \bar{0} \quad \text{but} \quad \overline{x \wedge x'} = \bar{0}$$

which is a contradiction.

Our result is now proved.

Our next result is an extension of some results of J. Varlet [11] and O. Frink [4]. More precisely II is a generalisation of the condition $(a \wedge b)^* = a^* \vee b^*$ of Varlet and Frink, III is new, and IV and V are due to J. Varlet, stated for pseudo-complemented distributive lattices. Note that $L^{**} = \{(a)^{**} : a \in L\}$

PROPOSITION 2.4. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle \in \Delta^*$. *Then the following are equivalent.*

I. $\mathscr{L}$ *is a Stone lattice i.e.* $(x)^* \vee (x)^{**} = L$ *for all* $x \in L$.
II. $(x \wedge y)^* = (x)^* \vee (y)^*$.
III. $\mathscr{L}^{**} = \langle L^{**}; \vee, \cap\rangle$ *is a sublattice of* $\mathscr{I}(\mathscr{L}) = \langle I(\mathscr{L}); \vee, \cap\rangle$.
IV. *For any* $a \in L$, $(a)^* = (z)^*$ *where* $z \in Z(\mathscr{L})$.
V. *If* $a \wedge b = 0$, *then there is* $z \in Z(\mathscr{L})$ *with* $z \geqq a$ *and* $z' \geqq b$.

PROOF. I $\Rightarrow$ II. If $\mathscr{L}$ is a Stone lattice, $(x)^* = (x^*)$ and II is known to hold in the form $(x \wedge y)^* = x^* \vee y^*$.

II $\Rightarrow$ III. It has been shown elsewhere [9] that $(x \wedge y)^{**} = (x)^{**} \cap (y)^{**}$, and we have

$$(x \vee y)^{**} = \big((x)^* \cap (y)^*\big)^* = \big((x')^{**} \cap (y')^{**}\big)^* = (x' \wedge y')^{***} = (x' \wedge y')^*$$

and

$$(x)^{**} \vee (y)^{**} = (x')^* \vee (y')^* = (x' \wedge y')^*$$

and III follows.

III $\Rightarrow$ IV. Take $y = x'$ in III and we see that $(x)^*$ and $(x')^*$ must be complementary direct summands. This implies $(x)^* = (z)^*$ for some $z \in Z(\mathscr{L})$.

IV $\Rightarrow$ V. If $a \wedge b = 0$ and $(a)^* = (z)^*$ then b must satisfy $b \leqq z'$ and we have $(a)^{**} = (z)$ or $a \leqq z$.

V $\Rightarrow$ I. Assuming V, we let $a = x$, $b = x'$ and find $x \leqq z$, $x' \leqq z'$ where $z \in Z(\mathscr{L})$. This gives $(x)^* \supseteqq (z)^* = (z')$, $(x')^* \supseteqq (z')^* = (z)$ or $z \in (x')^*$ and $z' \in (x)^*$. Thus, since $z \vee z' = 1$, $(x)^* \vee (x')^* = (x)^* \vee (x)^{**} = L$.

The proposition is now proved.

We close this section with the remark that other characterisations of Stone lattices, due to J. Varlet [11], extend to the class Δ^*. In fact all known characterisations of Stone lattices which apply to pseudo-complemented lattices can be extended in a straight forward manner. This is interesting insofar as it suggests the property $\mathscr{L}/R$ being a Boolean algebra is the important thing in such characterisations, rather than the existence of pseudo-complements.

3. Relations between $\mathscr{Z}(\mathscr{L})$ and $\mathscr{M}(\mathscr{L})$

Our aim in this section is to give a new characterisation which unites the apparently unrelated results of Proposition 2.3 and 2.4. The first result is straightforward and has its proof omitted.

LEMMA 3.1. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$ *be a distributive lattice, and let* $P \in \mathscr{P}(\mathscr{L})$. *Then* $P \cap Z(\mathscr{L}) \in \mathscr{P}(\mathscr{Z}(\mathscr{L}))$.

We next prove a type of converse to this result when $\mathscr{L}$ is a Stone lattice.

PROPOSITION 3.2. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$ *be a Stone lattice and* $Q \in \mathscr{P}(\mathscr{Z}(\mathscr{L}))$. *Then* $(Q)_{\mathscr{L}} = \{t \in L; t \leqq q \text{ for some } q \in Q\}$ *is a minimal prime ideal of* $\mathscr{L}$.

PROOF. It is definition that $(Q)_{\mathscr{L}}$ is an ideal of $\mathscr{L}$. Let $a \wedge b \in (Q)_{\mathscr{L}}$.

Then $a \wedge b \leqq q$ for some $q \in Q \subseteqq Z(\mathscr{L})$. Now Q is prime in $Z(\mathscr{L})$. Also $a^{**} \wedge b^{**} \leqq q$ and, since a^{**} and b^{**} belong to $Z(\mathscr{L})$ when $\mathscr{L}$ is a Stone lattice, we deduce that $a^{**} \in Q$ or $b^{**} \in Q$.

From $x \leqq x^{**}$, we see that $(Q)_{\mathscr{L}}$ is a prime ideal of $\mathscr{L}$.

It remains to show $(Q)_{\mathscr{L}}$ is a minimal prime ideal of $\mathscr{L}$. This would follow if we showed that for $x \in (Q)_{\mathscr{L}}$ there was $y \in (x)^{*} \setminus (Q)_{\mathscr{L}}$.

But $x \in (Q)_{\mathscr{L}}$ implies $x \leqq q$ for $q \in Q$, and so $x \wedge q' = 0$. Clearly $q' \notin (Q)$ since $q' \notin Q$ and so $q' \in (x)^{*} \setminus (Q)$ and our proposition is proved.

THEOREM 3.3. *Let* $\mathscr{L} = \langle L: \vee, \wedge, 0, 1\rangle \in \Delta^{*}$. *Then* $\mathscr{L}$ *is a Stone lattice iff* $M = (M \cap Z(\mathscr{L}))_{\mathscr{L}}$ *for all* $M \in \mathscr{M}(\mathscr{L})$.

PROOF. Suppose $\mathscr{L}$ is a Stone lattice. Then by 3.1, if $M \in \mathscr{M}(\mathscr{L})$, $M \cap Z(\mathscr{L}) \in \mathscr{P}(\mathscr{Z}(\mathscr{L}))$. Thus

$$M \supseteqq (M \cap Z(\mathscr{L}))_{\mathscr{L}}.$$

Now for the reverse inclusion we take $x \in M$ and find that $x^{**} \in M$ and since $x^{**} \in Z(\mathscr{L})$, we see that

$$x \in (M \cap Z(\mathscr{L}))_{\mathscr{L}}.$$

The result $M = (M \cap Z(\mathscr{L}))_{\mathscr{L}}$ thus holds when $\mathscr{L}$ is a Stone lattice.

Assume now that $M = (M \cap Z(\mathscr{L}))_{\mathscr{L}}$ for all $M \in \mathscr{M}(\mathscr{L})$. Then for M, N in $\mathscr{M}(\mathscr{L})$

$$M \cap Z(\mathscr{L}) \neq N \cap Z(\mathscr{L})$$

for equality here would imply $M = N$ on generating the ideals in $\mathscr{L}$. But in a Boolean lattice all prime ideals are maximal, and so

$$(M \cap Z(\mathscr{L})) \vee (N \cap Z(\mathscr{L})) = Z(\mathscr{L}).$$

In other words $a \vee b = 1$ for some $a \in M \cap Z(\mathscr{L})$ and some $b \in N \cap Z(\mathscr{L})$.

This implies $M \vee N = L$ and so, by Proposition 2.3. $\mathscr{L}$ is a Stone lattice. This concludes our proof.

We close this section with a result relating the factor lattice $\mathscr{L}/R$ with $Z(\mathscr{L})$. It is implicit in Frink [4] that $\mathscr{L}/R \cong Z(\mathscr{L})$ when $\mathscr{L}$ is a Stone

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lattice, and under an additional hypothesis we may prove a converse to this result.

THEOREM 3.4. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$ *be a distributive lattice. Then* $\mathscr{L}/R \cong Z(\mathscr{L})$ *implies* $\mathscr{L}$ *is a Stone lattice iff* $\mathscr{L}/R$ *is finite.*

PROOF. Assume $\mathscr{L}/R$ is finite. It has been proved elsewhere [9] that $\mathscr{L}/R \cong \mathscr{L}^{**}$ (see Proposition 2.4). Now the map $\gamma : Z(\mathscr{L}) \to \mathscr{L}^{**}$ is easily seen to be injective and, since $|Z(\mathscr{L})| < \infty$ it must be surjective. Hence $\mathscr{L}^{**}$ is a sublattice of $\mathscr{I}(\mathscr{L})$ and so by 2.4, $\mathscr{L}$ is a Stone lattice.

Now the remaining part of our theorem follows from the following counter example supplied by the referee. Let $\mathscr{B}$ be the Boolean lattice of all subsets of an infinite set. Then $\mathscr{B}\times\mathbf{2}^2 \cong \mathscr{B}$. Let $\mathscr{L} = \mathbf{2}\times N+\mathbf{1}$ where N is the chain of natural numbers. Then $\mathscr{L}$ is in Δ^* but is not a Stone lattice. Finally it can be shown that

$$\mathscr{L}\times\mathscr{B}/R \cong \mathscr{B}\times\mathbf{2}^2 \cong \mathscr{B} \cong Z(\mathscr{L}\times\mathscr{B})$$

Thus $\mathscr{L}\times\mathscr{B}$ satisfies the conditions of the Theorem but is not a Stone lattice.

4. Decompositions of distributive lattices

Before proving our final results on Stone lattices we collect some results concerning direct sum decompositions of complete distributive lattices $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$ satisfying the infinite distributive law (called I.D.-lattices)

(I.D.) $$x \wedge \bigvee_{\alpha\in A} x_\alpha = \bigvee_{\alpha\in A} x \wedge x_\alpha \quad \text{for any} \quad x, \{x_\alpha : \alpha \in A\} \subseteqq L.$$

The approach in this section is, with appropriate modifications, taken from J. von Neumann [7].

DEFINITION 4.1. A system $[a_\alpha : \alpha \in A]$ where $a_\alpha \in L$ for $\alpha \in A$, is said to be a direct sum decomposition of the I.D.-lattice $\mathscr{L} = \langle L; \vee, \wedge\rangle$, if for any $x \in L$ and $\alpha \in A$ there is a unique $x_\alpha \in (a_\alpha)$ such that

$$x = \bigvee_{\alpha\in A} x_\alpha$$

Under these circumstances we call $\{x_\alpha\}_{\alpha\in A}$ the decomposition of $x \in L$.

LEMMA 4.2. *Let* $[a_\alpha : \alpha \in A]$ *be a direct sum decomposition of the I.D.-lattice* $\mathscr{L} = \langle L; \vee, \wedge\rangle$ *and let* x *and* y *have decompositions* $\{x_\alpha\}$, $\{y_\alpha\}$ *respectively. Then*

(i) $x \leqq y$ *if and only if* $x_\alpha \leqq y_\alpha$ *for all* $\alpha \in A$.
(ii) $x_\alpha = x \wedge a_\alpha$ *for all* $\alpha \in A$.

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PROOF. (i) Since $x = \bigvee_{\alpha \in A} x_\alpha$ and $y = \bigvee_{\alpha \in A} y_\alpha$ we have

$$x \vee y = \bigvee_{\alpha \in A} (x_\alpha \vee y_\alpha)$$

and thus $\{x_\alpha \vee y_\alpha\}$ is the (unique) decomposition of $x \vee y$. Now $x \leqq y$ is equivalent to $x \vee y = y$ which gives $x_\alpha \vee y_\alpha = y_\alpha$, and the result is proved.

(ii) For a given $\beta \in A$

$$x = x \vee (x \wedge a_\beta) = \bigvee_{\alpha \in A} x_\alpha \vee (x \wedge a_\beta) = \bigvee_{\alpha \neq \beta} x_\alpha \vee x_\beta \vee (x \wedge a_\beta)$$

Now we can define

$$x'_\alpha = \begin{cases} x_\alpha & \alpha \neq \beta \\ x_\beta \vee (x \wedge a_\beta) & \alpha = \beta \end{cases}$$

and since $x = \bigvee_{\alpha \in A} x'_\alpha$ we deduce that $x_\beta = x'_\beta \geqq x \wedge a_\beta$. But $x_\beta \leqq x \wedge a_\beta$ and so $x_\beta = x \wedge a_\beta$.

The proof of the Lemma is complete.

PROPOSITION 4.3. *The system* $[a_\alpha : \alpha \in A]$ *is a direct sum decomposition of the I.D.-lattice* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$ *if and only if*

(i) $a_\alpha \wedge a_\beta = 0$ *for* $\alpha \neq \beta$, $\alpha, \beta \in A$

(ii) $\bigvee_{\alpha \in A} a_\alpha = 1$.

When these conditions are satisfied we see that $\{a_\alpha : \alpha \in A\} \subseteqq Z(\mathscr{L})$.

PROOF. Assume $[a_\alpha : \alpha \in A]$ is a direct sum decomposition of $\mathscr{L}$. Then letting $x = \bigvee_{\alpha \neq \beta} a_\alpha$ for a given $\beta \in A$ we see that

$$x_\alpha = \begin{cases} a_\alpha & \alpha \neq \beta \\ 0 & \alpha = \beta \end{cases}$$

is a decomposition of x and thus $x_\beta = 0 = a_\beta \wedge \bigvee_{\alpha \neq \beta} a_\alpha$.

This gives $a_\beta \wedge a_\alpha = 0$ for $\alpha \neq \beta$, and (i) is proved.

Also, since $1 \in L$, $1 = \bigvee_{\alpha \in A} x_\alpha$ for $x_\alpha \in (a_\alpha)$.

Now $1 = \bigvee_{\alpha \in A} x_\alpha \leqq \bigvee_{\alpha \in A} a_\alpha \leqq 1$ and (ii) is proved.

For the converse, assume (i) and (ii) are satisfied. Then

$$x = 1 \wedge x = \bigvee_{\alpha \in A} a_\alpha \wedge x = \bigvee_{\alpha \in A} (a_\alpha \wedge x) = \bigvee_{\alpha \in A} x_\alpha$$

Let $x = \bigvee_{\alpha \in A} x'_\alpha$ where $x'_\alpha \in (a_\alpha)$ for $\alpha \in A$.

We find

$$a_\beta \wedge x = a_\beta \wedge \bigvee_{\alpha \in A} x'_\alpha = \bigvee_{\alpha \in A} a_\beta \wedge x'_\alpha = x'_\beta$$

since $a_\beta \wedge a_\alpha = 0$, $\alpha \neq \beta$. This shows $x_\beta = x'_\beta = x \wedge a_\beta$ and the decomposition is unique.

5. Decompositions of Stone lattices

G. Grätzer and E. T. Schmidt have noted the following result.

PROPOSITION 5.1. *A finite distributive lattice $\mathscr{L}$ is a Stone lattice if and only if $\mathscr{L}$ is a direct product (= sum) of dense lattices.*

We investigate this approach in a wider class of lattices and, obtaining a similar decomposition theorem, use a result of G. Grätzer [5] to prove a new embedding theorem which characterises Stone Lattices. Some preliminary results are needed.

LEMMA 5.2. *If $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$ is a pseudo-complemented lattice then the principal ideal (a) for $a \in L$ is pseudo-complemented. The pseudo-complement of $x \in (a)$ in (a) is $a \wedge x^*$.*

PROOF. Let $x \in (a)$ and $t \in (a)$ be such that $t \wedge x = 0$. Then $t \leqq x^*$ and since $t \leqq a$ we have $t \leqq a \wedge x^*$.

Conversely if $t \leqq a \wedge x^*$ then $t \in (a)$ and

$$t \wedge x \leqq a \wedge x^* \wedge x = 0.$$

We thus have, for $t \in (a)$ and $x \in (a)$

$$t \wedge x = 0 \Leftrightarrow t \leqq a \wedge x^*$$

and so the lemma is proved.

REMARK. An element $k \in B$ of a Boolean lattice $\mathscr{B} = \langle B; \vee, \wedge, 0, 1\rangle$ is called an atom if $t < k$ for $t \in B$ implies $t = 0$. The Boolean lattice $\mathscr{B}$ is called atomic if the join of all atoms in B is 1.

LEMMA 5.3. *Let $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$ be a Stone lattice. Then for $a \in Z(\mathscr{L})$ (a) is dense as a lattice if and only if a is an atom of $\mathscr{Z}(\mathscr{L})$.*

PROOF. Suppose (a) is dense. Then take $z \in Z(\mathscr{L})$

$$0 \neq z \leqq a.$$

We have

$$a \wedge z^* = a \wedge z' \in (a)$$

and since (a) is dense $a \wedge z' = 0$ which implies $a \leqq z'' = z$. But $z \leqq a$ and so $a = z$. Thus a is an atom of $\mathscr{Z}(\mathscr{L})$.

For the converse suppose a is an atom of $\mathscr{Z}(\mathscr{L})$ and take $x \in (a)$. We have $x^{**} \leqq a^{**} = a$ and so, since a is an atom, $x^{**} = a$ or $x^{**} = 0$. This proves that $a \wedge x^*$ is equal to either $a \wedge a^* = 0$ or $a \wedge 1 = a$, for $x = 0$. We have shown (a) is dense and the Lemma is proved.

THEOREM 5.4. *Let $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$ be a complete Stone lattice satisfying I.D. Then $\mathscr{L}$ has a direct sum decomposition into a family of dense lattices if and only if $\mathscr{Z}(\mathscr{L})$ is atomic.*

PROOF. Let $\mathscr{L}$ have a direct sum decomposition as indicated, say $[a_\alpha : \alpha \in A]$ where each (a_α) is dense. Then by the comment in the statement of Proposition 4.3, $a_\alpha \in Z(\mathscr{L})$ for $\alpha \in A$. Hence each a_α is an atom of $\mathscr{Z}(\mathscr{L})$ by Lemma 5.3, and since by Proposition 4.3 (ii) the join of all the a_α is 1, we deduce that $\mathscr{Z}(\mathscr{L})$ is an atomic Boolean lattice.

Conversely suppose $\mathscr{Z}(\mathscr{L})$ is atomic. Then for any atom $a \in \mathscr{Z}(\mathscr{L})$, (a) is dense. The set Z_{at} of all atoms of $\mathscr{Z}(\mathscr{L})$ is easily seen to satisfy the conditions (i), (ii) of Proposition 4.3 and so $[a : a \in Z_{at}]$ is a direct sum decomposition of $\mathscr{L}$ into dense lattices.

G. Grätzer [5] proved the following result, after a conjecture of O. Frink [4]. See also G. Bruns [2]. Note that $*$-sublattice means that pseudo-complements are preserved by the injection map.

THEOREM 5.5. (Grätzer) *To every Stone lattice* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$ *there corresponds a set* H *such that* $\mathscr{L}$ *is isomorphic to a* $*$*-sublattice of the lattice* $\mathscr{I}(\mathscr{B}) = \langle I(\mathscr{B}); \vee, \cap\rangle$ *of all ideals of* $\mathscr{B} = \langle 2^H; \cup / \cap / \square, H\rangle$. *Conversely every* $*$*-sublattice of the lattice of all ideals of a complete atomic Boolean lattice is a Stone lattice.*

We combine this result with Theorem 5.4 and obtain

THEOREM 5.6. *Let* $\mathscr{L} = \langle L; \vee, \wedge, 0, 1\rangle$, *be a Stone lattice. Then* $\mathscr{L}$ *is isomorphic to a* $*$*-sublattice of a direct sum of dense lattices. Conversely every* $*$*-sublattice of a direct sum of dense lattices is a Stone lattice.*

PROOF. Assume $\mathscr{L}$ is a Stone lattice. Then by Theorem 5.5 above $\mathscr{L}$ is isomorphic to a $*$-sublattice of $\mathscr{I}(\mathscr{B})$ where $\mathscr{B}$ is a Boolean lattice of all subsets of some set H. Now it is easily seen that $\mathscr{Z}(\mathscr{I}(\mathscr{B})) \cong \mathscr{B}$ and thus the centre of $\mathscr{I}(\mathscr{B})$ is atomic. Also it is known that the lattice of all ideals of any Boolean lattice (indeed any distributive lattice) satisfies the infinite distributive law I.D. Thus the conditions of Theorem 5.4 are satisfied and its conclusion proves the first assertion of our present theorem.

For the converse we note firstly that the direct sum of a family of dense lattices is a Stone lattice. Also the fact that a $*$-sublattice of a Stone lattice is again a Stone lattice is evident. From these two remarks the second assertion follows and the theorem is proved.

6. Examples of Stone lattices

We close with listing some simple results which provide examples of Stone lattices. Suppose $\langle X, \mathscr{T}\rangle$ is a topological space. Then $\langle \mathscr{T}; \cup / \cap / \square\rangle$ is known to be a pseudo-complemented lattice.

PROPOSITION 6.1. *For a topological space* $\langle X, \mathscr{T}\rangle$, $\langle \mathscr{T}; \cup / \cap / \square\rangle$ *is a Stone lattice if and only if* $\langle X, \mathscr{T}\rangle$ *is extremally disconnected* (*i.e. the closure of every open set is open*).

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PROOF. Suppose $\langle X, \mathscr{T} \rangle$ is extremally disconnected — denoting the closure operator by '$^-$' and complement by '$'$' — this means

$$T^- \in \mathscr{T} \quad \text{for all} \quad T \in \mathscr{T}.$$

Now $T^* = T^{-\prime}$ is always open, and so in this case T^* is open-closed and

$$T^* \cup T^{**} = T^{-\prime} \cup T^{-\prime -\prime} = T^{-\prime} \cup T^- = X.$$

Thus $\langle \mathscr{T}; \cup / \cap / \square \rangle$ is a Stone lattice.

Conversely, if $\langle \mathscr{T}; \cup / \cap / \square \rangle$ is a Stone lattice, then $T^{-\prime} \cup T^{-\prime -\prime} = X$. Also $T^{-\prime} \cap T^{-\prime -\prime} = \square$ and so $T^{-\prime}$ is open closed. So must be T^- in which case $\langle X, \mathscr{T} \rangle$ is extremally disconnected and our result is proved.

Next we mention Post Algebras which have been discussed lattice-theoretically by G. Epstein [3] and T. Traczyk [10]. In the notation of Traczyk [10] we have

PROPOSITION 6.2. *Let* $P = \langle e_0, e_1, \cdots, e_{n-1}; B \rangle$ *be a Post Algebra. Then* P *is a Stone lattice.*

PROOF. The proof follows readily from known results, G. Epstein [3], that P is pseudo-complemented and from the peculiar property of a Post Algebra — the prime ideals occur in finite disjoint chains — we deduce that the join of any two distinct minimal prime ideals of P is P. Thus P is a Stone lattice. This result also follows directly when the pseudo-complements are identified.

The elementary theory of l-groups is given in G. Birkhoff [1]. From results there we may deduce

PROPOSITION 6.3. *Let* $G = \langle G; +, \vee, \wedge, 0 \rangle$ *be a complete l-group. Then* $\mathscr{I}(G)$*, the lattice of all l-ideals of G, is a Stone lattice.*

PROOF. It is known that $\mathscr{I}(G)$ is a complete distributive lattice. For any l-ideal I of G we define

$$I^* = \{t \in G : |t| \wedge |i| = 0 \ \text{for all} \ i \in I\}.$$

I^* can be checked to satisfy the properties of a pseudo-complement.

Now the operations in $\mathscr{I}(G)$ are join $= +$ and meet $= \cap$, and Theorem 19 p. 23 of [1] states that $J \cap J^* = (0)$, $J + J^* = G$ is equivalent to $J = J^{**}$.

We have thus proved that $\mathscr{I}(G)$ is a Stone lattice.

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