ALMOST-BOUNDED HOLOMORPHIC FUNCTIONS WITH PRESCRIBED AMBIGUOUS POINTS

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1. Introduction. Let f be a function mapping the open unit disk D into the extended complex plane. A point ζ on the unit circle C is called an *ambiguous point* of f if there exist two Jordan arcs J_1 and J_2, each having an end-point at ζ and lying, except for ζ, in D, such that

$$\lim_{\substack{z\to\zeta \\ z\in J_1}} f(z) \quad \text{and} \quad \lim_{\substack{z\to\zeta \\ z\in J_2}} f(z)$$

both exist and are unequal. Bagemihl **(1)** proved that *the set of ambiguous points of f is at most countable*, even if f is not required to be continuous in D.

Since bounded holomorphic functions in D have no ambiguous points **(6**, p. 303; **9**, p. 5**)**, several subsequent investigations have centred about the question of the existence of ambiguous points for functions which are "almost" bounded in some sense. Bagemihl and Seidel **(2)** proved that *if E is a denumerable subset of C, then there exists a function f, regular and of bounded characteristic in D, for which every element of E is an ambiguous point.*

A function f, regular in D, is of bounded characteristic if it satisfies the growth condition

$$\sup\left\{\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta : 0 \leqslant r < 1\right\} < \infty.$$

In this paper, we consider classes of functions which are subject to more stringent (Orlicz-type) growth conditions.

If h is a non-negative, non-decreasing function defined on the non-negative real axis, then let $H(h)$ denote the collection of holomorphic functions f in D for which

$$\sup\left\{\int_0^{2\pi} h[|f(re^{i\theta})|]d\theta : 0 \leqslant r < 1\right\} < \infty.$$

We observe that $H(x^p)$ is the Hardy class H^p $(p > 0)$ and that

$$H(e^x) \subset \bigcap_{0<p<\infty} H^p.$$

Theorem 2 of this paper asserts that *if E is a denumerable subset of C and if h is a non-negative, non-decreasing function defined on the non-negative real axis, then there exists a function f in $H(h)$ for which every point of E is an ambiguous point.* (For finite sets E, this result was anticipated by Gehring **(5)**.)

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In order to establish Theorem 2, we first prove (Theorem 1) that *if h is a non-negative, non-decreasing function defined on the non-negative real axis and if E is a subset of C whose (linear) measure is zero, then there exists a function Q in $H(h)$ such that*

$$\lim_{\substack{z \to \zeta \\ z \in D}} Q(z) = \infty$$

for each ζ in E. Some interest may attach to this result inasmuch as no regular function in D can have infinite angular limits on a set of positive measure (**6**, p. 378; **10**, p. 212). Theorem 1 (without proof) has been used in another connection (**4**).

Let ζ be a point on C. Then the familiar function

$$f(z) = \frac{1}{\zeta - z} \exp\left\{-\frac{\zeta + z}{\zeta - z}\right\}$$

is in $H(\log^+ x)$ since it is the quotient of two bounded functions (**6**, p. 345; **10**, p. 56), and it has ζ as an ambiguous point. Indeed, if z is in D,

$$\left|\exp\left\{-\frac{\zeta + z}{\zeta - z}\right\}\right| = \exp\left\{-\frac{1 - |z|^2}{|\zeta - z|^2}\right\},$$

and

$$|f(z)| = \frac{1}{|\zeta - z|} e^{-1}$$

if z is on the circle which has for its diameter the radius of D terminating at ζ. This function serves as a prototype in our proof of Theorem 2. A generalization of the factor $1/(\zeta - z)$ is embodied in the functions described in Theorem 1; and the remaining factor motivates our study of the tangential limits of functions of the form

$$\exp\left\{-\sum_1^\infty \alpha_n \frac{\zeta_n + z}{\zeta_n - z}\right\}.$$

2. Some lemmas. The primary purpose of this section is to establish Lemma 3.

LEMMA 1. *Let $E = \{\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots\}$ be a denumerable subset of C, and let $\{\alpha_n\}$ be a sequence of positive numbers such that*

$$\sum_1^\infty \alpha_n < \infty.$$

Then

$$P(z) = \exp\left\{-\sum_1^\infty \alpha_n \frac{\zeta_n + z}{\zeta_n - z}\right\} \tag{1}$$

is a holomorphic function mapping D into D which has zero as a radial limit at each point of E. On the radii in question, the inequalities

$$|P(r\zeta_m)| < \exp\left\{-\alpha_m \frac{1+r}{1-r}\right\} \qquad (m = 1, 2, \ldots; 0 \leqslant r < 1) \tag{2}$$

hold.

Proof. The regularity of P is an immediate consequence of the inequality

$$\left|\alpha_n \frac{\zeta_n + z}{\zeta_n - z}\right| \leqslant \alpha_n \frac{1+|z|}{1-|z|}$$

in conjunction with the Weierstrass M-test. The remaining assertions of the lemma follow from the relations

$$|P(z)| = \exp\left\{-\sum_1^\infty \alpha_n \frac{1-|z|^2}{|\zeta_n - z|^2}\right\} < \exp\left\{-\alpha_m \frac{1-|z|^2}{|\zeta_m - z|^2}\right\} < 1$$

$$(z \in D; m = 1, 2, \ldots)$$

Convention. Throughout this paper, if Θ_1 and Θ_2 are angles, the shortest distance on C between $e^{i\Theta_1}$ and $e^{i\Theta_2}$ is denoted by $|\Theta_1 - \Theta_2|$. The expression $|\Theta - 0|$ is abbreviated to $|\Theta|$.

Let f be a complex-valued function defined on D, and let γ be a fixed number ($\gamma \geqslant 1$). Then f is said to have a *T_γ-limit at a point* $e^{i\Theta}$ on C provided there exists a complex number L such that, for each positive real number m, $f(z) \to L$ as $z \to e^{i\Theta}$, z being confined to the set

$$R(m, \Theta, \gamma) = \{z : 1 - |z| \geqslant m\,|\arg z - \Theta|^\gamma; 0 < |z| < 1\}.$$

We note that the T_1-limit is equivalent to the classical angular limit. T_γ-limits have been studied in connection with Blaschke products **(3)**; for purposes of analogy, it is sometimes convenient to think of $(1 - \alpha_n)\zeta_n$ as a pseudo-zero of the function P given by (1). If $e^{i\Theta}$ is an accumulation point of the set E in Lemma 1, then one can easily verify that 0 is in the cluster set of P at $e^{i\Theta}$. Nevertheless, we now prove the following lemma.

LEMMA 2. *Let $E = \{\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots\}$ be a denumerable subset of C, let γ be a fixed number satisfying $\gamma \geqslant 1$, and let $\{\alpha_n\}$ be a sequence of positive numbers such that*

$$\sum_1^\infty \alpha_n / |\Theta - \arg \zeta_n|^\gamma < \infty$$

for some real number Θ. Then

$$P(z) = \exp\left\{-\sum_1^\infty \alpha_n \frac{\zeta_n + z}{\zeta_n - z}\right\}$$

has a T_γ-limit of modulus 1 at $e^{i\Theta}$.

Proof. From the hypothesis, we see that $\sum \alpha_n < \infty$; hence P is defined. Clearly, it will suffice to consider the case when $\Theta = 0$.

Using Dini's theorem (**8**, p. 293), select a null sequence $\{w_n\}$ of positive numbers such that

$$\sum_{1}^{\infty} \alpha_n / w_n | \arg \zeta_n|^\gamma < \infty; \tag{3}$$

and then set

$$S_n = \{z : |\zeta_n - z| < w_n |\arg \zeta_n|^\gamma\} \qquad (n = 1, 2, \ldots).$$

Given $m > 0$, we want to prove that $P(z)$ approaches a limit of modulus 1 as $z \to 1$, z being confined to the set $R(m, 0, \gamma)$.

Assume for a moment that $R(m, 0, \gamma)$ and

$$\bigcup_{n=n_m}^{\infty} S_n$$

are disjoint for some positive integer n_m. Then

$$\sum_{n=n_m}^{\infty} \alpha_n \frac{\zeta_n + z}{\zeta_n - z}$$

converges uniformly on $R(m, 0, \gamma)$; for, if

$$z \in D - \bigcup_{n=n_m}^{\infty} S_n,$$

then

$$\left| \alpha_n \frac{\zeta_n + z}{\zeta_n - z} \right| \leqslant \frac{2\alpha_n}{w_n | \arg \zeta_n|^\gamma} \qquad (n \geqslant n_m),$$

and the conclusion follows from (3) and the Weierstrass M-test.

In virtue of the uniform convergence,

$$\sum_{n=n_m}^{\infty} \alpha_n \frac{\zeta_n + z}{\zeta_n - z} \to \sum_{n=n_m}^{\infty} \alpha_n \frac{\zeta_n + 1}{\zeta_n - 1}$$

as $z \to 1$, z being confined to $R(m, 0, \gamma)$. Since

$$\sum_{n=1}^{n_m - 1} \alpha_n \frac{\zeta_n + z}{\zeta_n - z}$$

is continuous at the point 1, the conclusion of the lemma follows at once.

We still have to prove that n_m exists. To this end, take n_m to be an integer such that

$$w_n < m(1 - \tfrac{1}{2}\pi w_n |\arg \zeta_n|^{\gamma - 1})^\gamma \tag{4}$$

and

$$w_n |\arg \zeta_n|^\gamma < 1$$

both hold for all $n \geqslant n_m$. Then let n be any integer such that $n \geqslant n_m$, and suppose that $z_0 \in S_n \cap D$. We want to prove that $z_0 \notin R(m, 0, \gamma)$, that is, that

$$1 - |z_0| < m |\arg z_0|^\gamma. \tag{5}$$

Clearly,

(6) $$1 - |z_0| \leqslant |\zeta_n - z_0| < w_n\,|\arg \zeta_n|^\gamma.$$

An obvious geometric argument yields

$$|\arg z_0 - \arg \zeta_n| < \arc\sin\{w_n\,|\arg \zeta_n|^\gamma\} < \tfrac{1}{2}\pi w_n\,|\arg \zeta_n|^\gamma,$$

and a simple analogue of the triangle inequality for real numbers gives

$$|\arg \zeta_n| - |\arg z_0 - \arg \zeta_n| \leqslant |\arg z_0|.$$

From these last two inequalities, we conclude that

$$m(|\arg \zeta_n| - \tfrac{1}{2}\pi w_n\,|\arg \zeta_n|^\gamma)^\gamma < m\,|\arg z_0|^\gamma.$$

This, combined with (4) and (6), yields (5), as desired.

LEMMA 3. *Let $E = \{\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots\}$ be a denumerable subset of C. Then there exists a sequence $\{\alpha_n\}$ of positive numbers such that, for each m,*

$$P(z) = \exp\left\{-\sum_1^\infty \alpha_n \frac{\zeta_n + z}{\zeta_n - z}\right\}$$

is bounded away from zero on the intersection of D and the circle whose diameter is the radius of D terminating at ζ_m. (The bound depends on m.)

Proof. Let $\alpha_1 = 2^{-1}$, $\alpha_2 = 2^{-2}\,|\arg \zeta_1 - \arg \zeta_2|^2$, and, for $n > 2$, let

$$\alpha_n = 2^{-n}\min\{|\arg \zeta_k - \arg \zeta_n|^2 : 1 \leqslant k < n\}.$$

We note that $\sum \alpha_n < \infty$. For a fixed point ζ_m, it is clear that

$$\alpha_{m+k}/|\arg \zeta_m - \arg \zeta_{m+k}|^2 \leqslant 2^{-(m+k)} \qquad (k = 1, 2, \ldots).$$

Then, according to Lemma 2,

$$P_m(z) = \exp\left\{-\sum_{\substack{n=1\\ n\neq m}}^\infty \alpha_n \frac{\zeta_n + z}{\zeta_n - z}\right\}$$

has a T_2-limit of modulus 1 at ζ_m. Let z ($z \neq 0$, $z \neq \zeta_m$) be a point on the circle whose diameter is the radius of D terminating at ζ_m. Since, obviously, $|z| = \cos(|\arg z - \arg \zeta_m|)$, it follows that

$$1 - |z| = 2\sin^2(\tfrac{1}{2}|\arg z - \arg \zeta_m|) > 2\pi^{-2}|\arg z - \arg \zeta_m|^2.$$

Accordingly, z is in $R(2\pi^{-2}, \arg \zeta_m, 2)$, and $|P_m(z)| \to 1$ as $z \to \zeta_m$ along the circle in question. Thus, $|P(z)| \to \exp\{-\alpha_m\}$ as $z \to \zeta_m$ along the circle; and, since $P(z)$ never vanishes in D, the conclusion of the lemma follows at once.

3. Infinite limits. We now turn our attention to the problem of constructing almost-bounded functions having infinite limits on prescribed subsets of C. To effect the construction, we use a familiar technique **(10)**.

Throughout this paper, we let

$$P(r, \Theta) = \frac{1 - r^2}{1 - 2r\cos\Theta + r^2}.$$

Given a measurable subset M of the interval $[0, 2\pi]$, we denote its characteristic function by χ_M; and we denote the Poisson integral of χ_M by u_M, that is,

$$u_M(re^{i\Theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(r, t - \Theta)\chi_M(t)dt.$$

Clearly, $0 \leqslant u_M(re^{i\Theta}) \leqslant 1$ if $0 \leqslant r < 1$ and $0 \leqslant \Theta \leqslant 2\pi$. If A is a measurable subset of the real line, we denote its measure by mA or $m(A)$.

LEMMA 4. *Let N be a subset of $(0, 2\pi)$ whose measure is zero. Then, given $\epsilon > 0$, there exist an open set G and a measurable set B such that $N \subset G \subset (0,2\pi)$, $B \subset [0, 2\pi]$, $mG < \epsilon$, $mB < \epsilon$, and $u_G(re^{i\Theta}) < \epsilon$ if $0 \leqslant r < 1$ and $\Theta \in [0,2\pi] - B$.*

Proof. Select an open set S such that $mS < \epsilon/2$ and $N \subset S \subset (0, 2\pi)$. For almost all Θ in $[0, 2\pi]$,

$$\lim_{r\to 1-} u_S(re^{i\Theta})$$

exists and is equal to $\chi_S(\Theta)$, according to Fatou's theorem (**6**, p. 337). Using an extension of Egoroff's theorem (**7**, p. 124), we conclude that the convergence is uniform off some set T where $mT < \epsilon/2$. Hence, there exists a number $r_\epsilon (0 < r_\epsilon < 1)$ such that $u_S(re^{i\Theta}) < \epsilon$ if $r_\epsilon < r < 1$ and

$$\Theta \in [0, 2\pi] - B,$$

where $B = S \cup T$. Finally select an open set G such that $N \subset G \subset S$ and

$$\frac{1}{2\pi}\frac{1 + r_\epsilon}{1 - r_\epsilon}\cdot mG < \epsilon.$$

Then, clearly, $u_G(re^{i\Theta}) < \epsilon$ if $0 \leqslant r \leqslant r_\epsilon$ and $0 \leqslant \Theta \leqslant 2\pi$; and, since $u_G(re^{i\Theta}) \leqslant u_S(re^{i\Theta}) < \epsilon$ if $r_\epsilon < r < 1$ and $\Theta \in [0, 2\pi] - B$, the lemma is proved.

LEMMA 5. *Let h be a positive, increasing, continuous function defined on the non-negative real axis; and let N be a subset of $(0, 2\pi)$ whose measure is zero. Then there exist open sets G_k $(k = 1, 2, \ldots)$ such that*

$$N \subset G_k \subset (0, 2\pi), \qquad \sum_1^\infty mG_k < \infty,$$

and

$$\int_0^{2\pi} h\left[\exp\left\{\sum_1^n u_{G_k}(re^{i\Theta})\right\}\right]d\Theta < 2\pi h(e) + 1 \tag{7}$$

if $0 \leqslant r < 1$ and $n = 1, 2, \ldots$.

Proof. For $k = 1, 2, \ldots$, let $\epsilon_k = 2^{-k}\min\{1, 1/h(e^{k+1})\}$. Using Lemma 4,

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select an open set G_k and a set B_k such that $N \subset G_k \subset (0, 2\pi)$, $B_k \subset [0, 2\pi]$, $mG_k < \epsilon_k$, $mB_k < \epsilon_k$, and

$$u_{G_k}(re^{i\theta}) < \epsilon_k$$

if $0 \leqslant r < 1$ and $\theta \in [0, 2\pi] - B_k$.

Letting $B_k^* = [0, 2\pi] - B_k$, we see that, for each positive integer n, $[0, 2\pi]$ is the union of the disjoint sets

$$\begin{aligned} &B_1^* \cap B_2^* \cap \ldots \cap B_n^*, \\ &B_1 \cap B_2^* \cap \ldots \cap B_n^*, \\ &B_2 \cap B_3^* \cap \ldots \cap B_n^*, \\ &\qquad\vdots \\ &B_{n-1} \cap B_n^*, \\ &B_n. \end{aligned} \tag{8}$$

For $\theta \in B_1^* \cap \ldots \cap B_n^*$, we have

$$0 \leqslant u_{G_k}(re^{i\theta}) < \epsilon_k \qquad (k = 1, 2, \ldots, n; 0 \leqslant r < 1);$$

consequently,

$$\sum_1^n u_{G_k}(re^{i\theta}) < \sum_1^n \epsilon_k < 1,$$

$$h\left[\exp\left\{\sum_1^n u_{G_k}(re^{i\theta})\right\}\right] < h(e),$$

and

$$\int_{B_1^* \cap \ldots \cap B_n^*} h\left[\exp\left\{\sum_1^n u_{G_k}(re^{i\theta})\right\}\right] d\theta < 2\pi h(e). \tag{9}$$

Likewise, if $\theta \in B_j \cap B_{j+1}^* \cap \ldots \cap B_n^*$ $(j = 1, 2, \ldots, n-1)$, then

$$\sum_1^n u_{G_k}(re^{i\theta}) < j + \epsilon_{j+1} + \ldots + \epsilon_n < j + 1,$$

and

$$\int_{B_j \cap B_{j+1}^* \cap \ldots \cap B_n^*} h\left[\exp\left\{\sum_1^n u_{G_k}(re^{i\theta})\right\}\right] d\theta \leqslant m(B_j) h(e^{j+1}) < 2^{-j}. \tag{10}$$

Finally,

$$\int_{B_n} h\left[\exp\left\{\sum_1^n u_{G_k}(re^{i\theta})\right\}\right] d\theta \leqslant m(B_n) h(e^{n+1}) < 2^{-n}. \tag{11}$$

The integral appearing in (7) can be decomposed into a sum of integrals over the disjoint sets in (8). The desired inequality then follows from (9), (10), and (11).

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LEMMA 6. *Let g be an extended real-valued function which is defined and summable in* $[0, 2\pi]$. *If*

$$\lim_{t \to t_0} g(t) = +\infty$$

for some t_0 *in* $(0, 2\pi)$, *then*

$$\frac{1}{2\pi} \int_0^{2\pi} P(r, t - \Theta) g(t) dt \to +\infty$$

as

$$re^{i\Theta} \to e^{it_0}$$

from within D.

Proof. A proof of this classical result may be obtained by (correcting and) slightly modifying the proof of a somewhat weaker result given in (**10**, pp. 20–21).

THEOREM 1. *Let h be a non-negative, non-decreasing function defined on the non-negative real axis, and let E be a subset of C whose (linear) measure is zero. Then there exists a holomorphic function Q in D such that*

$$\lim_{\substack{z \to \zeta \\ z \in D}} Q(z) = \infty$$

for each ζ *in E and*

$$\sup\left\{ \int_0^{2\pi} h[|Q(re^{i\Theta})|] d\Theta : 0 \leqslant r < 1 \right\} < \infty.$$

Proof. Since we can always find a positive, increasing, continuous function h^* such that $h(x) \leqslant h^*(x)$ for all x in $[0, \infty)$, there is no loss in generality in assuming that h itself has these properties. Moreover, we may assume that $1 \notin E$ and work with the set $N = \{t : e^{it} \in E, 0 < t < 2\pi\}$.

Let G_k $(k = 1, 2, \ldots)$ be the sets constructed in Lemma 5. Since $\sum mG_k < \infty$, the function

$$\sum \chi_{G_k}(t)$$

is summable in $[0, 2\pi]$ by Beppo Levi's theorem. A straightforward argument shows that

$$Q(z) = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \sum_1^\infty \chi_{G_k}(t) dt \right\}$$

is holomorphic in D and that

$$|Q(re^{i\Theta})| = \exp\left\{ \sum_1^\infty u_{G_k}(re^{i\Theta}) \right\}.$$

If $t_0 \in N$, then, clearly,

$$\lim_{t \to t_0} \sum_1^\infty \chi_{G_k}(t) = +\infty.$$

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We conclude from Lemma 6 that, for $\zeta_0 = e^{it_0}$,

$$\lim_{\substack{z \to \zeta_0 \\ z \in D}} Q(z) = \infty.$$

Next, we observe that

$$\lim_{n\to\infty} \int_0^{2\pi} h\left[\exp\left\{\sum_1^n u_{G_k}(re^{i\Theta})\right\}\right]d\Theta = \int_0^{2\pi} h\left[\exp\left\{\sum_1^\infty u_{G_k}(re^{i\Theta})\right\}\right]d\Theta$$

$$= \int_0^{2\pi} h[|Q(re^{i\Theta})|]d\Theta,$$

which, in conjunction with (7), completes the proof of the theorem.

4. Functions with prescribed ambiguous points. We are now ready to prove the main theorem of the paper.

THEOREM 2. *Let $E = \{\zeta_1, \zeta_2, \ldots, \zeta_n, \ldots\}$ be a denumerable subset of C, and let h be a non-negative, non-decreasing function defined on the non-negative real axis. Then there exists a holomorphic function f in D which has each point of E as an ambiguous point, and which satisfies the condition*

$$\sup\left\{\int_0^{2\pi} h[|f(re^{i\Theta})|]d\Theta : 0 \leqslant r < 1\right\} < \infty. \tag{12}$$

Proof. We may assume that $h(x) \geqslant x$ for all x in $[0, \infty)$; for, otherwise, we could prove the theorem for the function $h^*(x) = h(x) + x$.

Let $f(z) = P(z)Q(z)$, where P is the function described in Lemma 3 and Q is the function described in Theorem 1. Then, by Lemma 1, $|f(z)| \leqslant |Q(z)|$ if z is in D, and (12) obviously holds.

We see at once that, for each m, $f(z) \to \infty$ as $z \to \zeta_m$, z being confined to the circle whose diameter is the radius of D terminating at ζ_m.

Finally, we note that

$$||Q|| = \sup\left\{\frac{1}{2\pi}\int_0^{2\pi} |Q(re^{i\Theta})|d\Theta : 0 \leqslant r < 1\right\}$$

is finite since $x \leqslant h(x)$. This, in turn, implies that

$$|Q(z)| \leqslant (1 - |z|)^{-1}\,||Q|| \tag{13}$$

for all z in D. Indeed, if $Q(z) = \sum \xi_n z^n$, then

$$\xi_n = \frac{1}{2\pi r^n}\int_0^{2\pi} Q(re^{i\Theta})e^{-in\Theta}d\Theta \qquad (0 < r < 1),$$

and

$$|\xi_n| \leqslant \frac{1}{2\pi r^n}\int_0^{2\pi} |Q(re^{i\Theta})|d\Theta.$$

Thus, $|\xi_n| \leqslant ||Q||$ $(n = 0, 1, 2, \ldots)$; and, since $|Q(z)| \leqslant \sum |\xi_n|\,|z|^n$, the result follows at once (see **11**, p. 103; **10**, p. 58).

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Inequality (13) and inequality (2) of Lemma 1 yield

$$|f(r\zeta_m)| < \frac{||Q||}{1-r} \exp\left\{-\alpha_m \frac{1+r}{1-r}\right\} \qquad (m = 1, 2, \ldots; 0 < r < 1),$$

so that $f(z) \to 0$ as $z \to \zeta_m$ radially. This completes the proof of the theorem.

5. Conclusion. Theorem 1 of this paper was called to the author's attention by Professor Piranian, who has devised an elegant proof of Lemma 4 which is entirely elementary, the elaborate machinery of Fatou's theorem being avoided altogether. We take the liberty of sketching his proof.

Let B be an open set for which $mB < \epsilon$ and $N \subset B \subset (0, 2\pi)$. Divide each component of B into a set of intervals whose end-points lie in the complement of N and whose ordering by position is isomorphic to the usual ordering of the integers. Order into a single sequence $\{I_k\}$ the set of all intervals thus constructed in B, and let d_k denote the distance between I_k and the complement of B. For each k, cover the set $N \cap I_k$ with an open covering that lies in I_k and has measure less than $2^{-k}d_k\epsilon$. Let G denote the union of these coverings. If $t \in I_k$ and $\Theta \in [0, 2\pi] - B$, then

$$\left|\frac{e^{it} + re^{i\Theta}}{e^{it} - re^{i\Theta}}\right| < \frac{2\pi}{d_k};$$

this, in turn, implies that

$$\int_{I_k} P(r, t - \Theta)\chi_G(t)dt < 2\pi 2^{-k}\epsilon,$$

from which the desired result follows at once.

References

1. F. Bagemihl, *Curvilinear cluster sets of arbitrary functions*, Proc. Natl. Acad. Sci. U.S., *41* (1955), 379–382.
2. F. Bagemihl and W. Seidel, *Functions of bounded characteristic with prescribed ambiguous points*, Mich. Math. J., *3* (1955–56), 77–81.
3. G. T. Cargo, *Angular and tangential limits of Blaschke products and their successive derivatives*, Can. J. Math., *14* (1962), 334–348.
4. ——— *Radial and angular limits of meromorphic functions*, Can. J. Math., *15* (1963), 471–474.
5. F. W. Gehring, *The asymptotic values for analytic functions with bounded characteristic*, Quart. J. Math. Oxford Ser. (2), *9* (1958), 282–289.
6. G. M. Golusin, *Geometrische Funktionentheorie* (Berlin, 1957).
7. H. Hahn and A. Rosenthal, *Set functions* (Albuquerque, 1948).
8. K. Knopp, *Theory and application of infinite series* (2nd English ed.; New York, 1947).
9. K. Noshiro, *Cluster sets* (Berlin–Göttingen–Heidelberg, 1960).
10. I. I. Priwalow, *Randeigenschaften analytischer Funktionen* (Berlin, 1956).
11. A. E. Taylor, *Introduction to functional analysis* (New York, 1958).

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