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ON SOME UNCONVENTIONAL PROBLEMS ON THE DIVISORS OF INTEGERS

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Dedicated to Professor K. Mahler for his 75th birthday

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Abstract

In this note a number of related problems about divisors are studied, and partial solutions obtained by elementary means. The problems are rather unconventional and seem to suggest interesting developments.

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1. Introduction

In this note we study a number of related problems concerning the divisors of an integer n. We denote these divisors by d and their number by $\tau(n)$; they are labelled in increasing order, thus $1 = d_1 < d_2 < \dots d_\tau = n$. As usual $\nu(n)$ denotes the number of distinct prime divisors of n.

All the problems considered here were raised by one or the other of us at various times: broadly speaking they are connected inasmuch as they are about relations between divisors, often between d_i and d_{i+1}, rather than arithmetic or analytic properties of individual divisors.

To give an example of the problem we have in mind, consider the following conjecture of P. Erdös that states the density of integers n which have two divisors $d_1 < d_2 < 2d_1$ is 1. P. Erdös (1964) stated that he can prove this—unfortunately this claim has to be withdrawn. More generally it was conjectured that the density of integers n which have two divisors

$$d_1 < d_2 < d_1(1 + (\log n)^{-\alpha}), \quad \alpha < \log 3 - 1,$$

is 1. We only know that if true this is best possible, that is it does not hold for $\alpha > \log 3 - 1$.

The following conjecture seems interesting: Denote by $\tau^+(n)$ the number of integers k for which n has a divisor d satisfying $2^k \leqslant d < 2^{k+1}$. Then $\tau^+(n)/\tau(n) \to 0$ if one disregards a sequence of density 0. This conjecture if true of course implies that for every ε the density of integers n which have two divisors $d_1 < d_2 < (1+\varepsilon) d_1$ is 1. The trouble is that at the moment we cannot attack this conjecture at all.

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Let $f(n) = \text{card}\{i\colon (d_i, d_{i+1}) = 1\}$. Each prime divisor of n is an admissible d_{i+1} so that $f(n) \geqslant \nu(n)$, with equality when $n = p_1 p_2 \dots p_\nu$ and $p_i > p_1 p_2 \dots p_{i-1}$ for $2 \leqslant i \leqslant \nu$. Thus the average order of $f(n)$ is at least $\log\log n$; we should like to determine the average and maximum orders. Concerning the maximum order, we have the following result.

THEOREM 1. *For every $\varepsilon > 0$ and $x > x_0(\varepsilon)$,*

$$\max_{m<x} f(m) > (\exp(\log\log x)^{2-\varepsilon}).$$

Next, let $\tau_k(n)$ denote the number of divisors of n of the form

$$d = t(t+1)\dots(t+k-1).$$

In the case $k = 2$, an equivalent definition is $\tau_2(n) = \text{card}\{i\colon d_{i+1} - d_i = 1\}$ so that $\tau_2(n) \leqslant f(n)$, with equality for a number like $n = 2.3.7.43$ where

$$n = p_1 p_2 \dots p_{\nu}, p_{i+1} = p_1 p_2 \dots p_i + 1.$$

It is easy to see that $\tau_2(m) = f(n)$ holds only for a finite number of n's. The average order of $\tau_k(n)$ is a positive constant, indeed for $k \geqslant 2$, we have

$$\sum_{n\leqslant x} \tau_k(n) = \frac{x}{(k-1)(k-1)!} + O(x^{1/k})$$

but the maximum order will be harder to determine. We have

THEOREM 2. *For each $k \geqslant 2$, and every fixed $A < e^{1/k}$, we have $\tau_k(n) > (\log n)^A$ infinitely often.*

It is certain that $\tau(n) > (\log n)^c$, infinitely often for every c, but this may be very difficult. Incidentally it is easy to see that the density of integers n for which $\tau_k(n) = r$ exists. Denote this density by $\alpha_k(r)$. We have $\sum_{r=0}^{\infty} \alpha_k(r) = 1$.

We can ask many questions about the function

$$t_k(n) = \min\{t \geqslant 1 : n \mid t(t+1)\dots(t+k-1)\},$$

and its restriction to the sequence of factorials. Plainly $t_{m-1}(m!) = 2$, and we can show that $t_{m-2}(m!) = O(m)$ (this is best possible, for example, if $m = 2^k$). What can be said about $t_{m-3}(m!)$? It is true that for infinitely many values of n and every $1 \leqslant i \leqslant n-1$

$$(1) \qquad t_i(n!) < t_{i-1}(n) - 1\,?$$

In particular we showed with Selfridge that (1) holds for $n = 10$.

More generally let F_n be the smallest integer with $F_n! \equiv 0 \pmod{n}$. Can one characterize the integers n for which all $1 \leqslant i \leqslant F_n$

$$(2) \qquad t_i(n) < t_{i-1}(n) - 1.$$

If F_n is very large (2) clearly cannot hold. What is the largest value of F_n for which (2) holds? For how many $n < x$ can (2) hold? The maximum order of $t_k(n)$ is easily settled since for primes $p \geqslant k$ we have $t_k(p) = p+1-k$. Here it is the average and normal orders which are of interest. We have the following result.

THEOREM 3.

$$\frac{1}{x} \sum_{n \leqslant x} t_2(n) \ll x \frac{\log\log\log x}{\log\log x}.$$

We conjecture that for some fixed $\alpha > 0$, we can replace the right-hand side by $x(\log x)^{-\alpha}$, indeed it is likely that any fixed $\alpha < \log 2$ will do. In view of the fact that $t_2(p) = p-1$, $\alpha > 1$ is impossible.

Is it true that

$$(3) \qquad \sum_{n=1}^{x} t_{i+1}(n) = o\left(\sum_{n=1}^{x} t_i(n)\right)?$$

We have not even proved (3) for $i = 2$.

Our final problem is rather different since it involves the divisors of two integers. We say that m and n interlock, and we write $m \vartriangle n$, if every pair of divisors of n are separated by a divisor of m, and conversely (with the exception that 1 and the smallest prime factor of mn obviously cannot be separated). Thus $45 \Delta 28$, in view of the pattern

$$\begin{array}{ccccccccccc} 45 & & 15 & & 9 & & 5 & & 3 & & 1 \\ & 28 & & 14 & & 7 & & 4 & & 2 & \end{array}$$

We say that n is separable if there exists an m such that $m \vartriangle n$, and we define $A(x)$ to be the number of separable $n \leqslant x$. We should like to prove the innocent-looking

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relation $A(x) = o(x)$, but have been unable to do so. In the opposite direction, we have

THEOREM 4. *For a fixed $c' > 0$, and sufficiently large x, we have*

$$A(x) > c'\, x/\log\log x.$$

We would like to mention two further conjectures concerning separable numbers. Is it true that 2^k is separable for almost all k? Notice that if $k+1$ is prime, this is not possible for $k \geqslant 4$. Secondly, let $N(k)$ be the product of the first $2k$ primes. When can we have $N(k) = mn$, $m \Delta n$? $k = 1, 2, 3, 4$ are all possible, for $k = 4$, $m = 2.5.13.19$, $n = 3.7.11.17$. It seems likely that for large k this cannot happen.

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In this section we prove our theorems and also state a few more problems. To prove Theorem 1 put

$$n_x = \prod p, \quad \log x < p < (2-\eta)\log x.$$

$n_x < x$ follows immediately from the prime number theorem. Put

$$y = [(\log\log x)^{1-2\eta}]$$

and denote by $D_1 < D_2 < \ldots < D_r$ the divisors of n_x having exactly y prime factors. Clearly by the prime number theorem and a simple computation

$$(\log x)^y < D_1 < \ldots < D_r < 2^y(\log x)^y < (\log x)^{y+1}, \tag{4}$$

and

$$r = \binom{\nu(n_x)}{y} > \left(\frac{(1-2\eta)\log x}{\log\log x}\right)^y \Big/ y! > \exp((\log\log x)^{2-3\eta}). \tag{5}$$

Now by (4) D_i and D_{i+1} are clearly consecutive divisors of n_x and if $(D_i, D_{i+1}) > 1$ then $(D_i, D_{i+1}) > \log x$, so $D_{i+1} - D_i > \log x$. Thus the number of indices i with $(D_i, D_{i+1}) > 1$ is less than

$$2^y(\log x)^{y-1} < \tfrac{1}{2}\mathrm{r}.$$

Hence finally from (5) $(\eta < \frac{1}{4}\varepsilon)$

$$f(n_x) > \tfrac{1}{2}r > \tfrac{1}{2}\exp(\log\log x)^{2-3\eta},$$

which completes the proof of Theorem 1.

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At present we have no good upper bound for $f(n)$. It would be reasonable to expect that for every $\varepsilon > 0$

$$f(n) < \exp((\log n)^{\varepsilon}).$$

We are certain that the average of $f(n)$ is greater than any fixed power of $\log\log n$ but so far have not been able to prove it. Denote by $A(u, v; x)$ the number of integers $n \leqslant x$ for which u and v are consecutive divisors of n. Clearly,

$$\sum_{n=1}^{x} f(n) = \sum_{\substack{1 \leqslant u < v \leqslant x \\ (u,v)=1}} A(u, v; x). \tag{6}$$

The trouble is that it is very hard to estimate $A(u, v; x)$. It can happen that $A(u, v; x) = 0$ because every $n \equiv 0 \bmod ([u, v])$ has a divisor $u < d < v$. We do not at present know the number of these pairs; it is not impossible that (6) is quite useless for the estimation of $\sum_{n=1}^{x} f(n)$.

It is easy to see that for infinitely many n, $f(n) = \nu(n)$ and it is not hard to show that the density of the integers satisfying $f(n) = \nu(n)$—in fact $f(n) < (1+c)\nu(n)$—is 0 if $c > 0$ is sufficiently small. Perhaps $f(n)/\nu(n) \to \infty$ if one disregards a sequence of density 0.

Assume that n is the product of k distinct prime factors. It is easy to see that $\min f(n) = k$, but we cannot at present determine $\max f(n)$ and in fact we do not even have a good estimation for it.

Denote by $f_l(n)$, $(f_2(n) = f(n))$ the number of indices i for which $(d_{i+j_1}, d_{i+j_2}) = 1$ for every $0 \leqslant j_1 < j_2 \leqslant l-1$. Perhaps for every $l > 2$ the mean value of $f_l(n)$ is bounded.

Proof of Theorem 2. Let k be fixed, $k \geqslant 2$, and fix B, $A < B < e^{1/k}$. Put $n = \text{l.c.m.}(1, 2, \ldots, y)$. The prime number theorem implies $y = (1+o(1))\log n$. Consider the integers $m < y^B$ for which

$$m \not\equiv -i \bmod Q), \quad i = 1, 2, \ldots, k, \tag{7}$$

and Q runs through the primes and powers of primes $y/k! \leqslant Q < y^B$. The number of these m is by the well-known theorem of Mertens not less than $(y \to \infty)$

$$y^B\left(1 - k\sum\frac{1}{Q}\right) = y^B(1 - k\log\log B - o(1)) > \varepsilon y^B, \quad \varepsilon = \varepsilon(B) \tag{8}$$

By a simple argument we obtain that if m satisfies (7) then $\prod_{i=1}^{k}(m+i) \mid n$. Thus (8) implies Theorem 2.

Proof of Theorem 3. Let q be squarefree. We call the residue class $h \pmod q$ ε-good if there exist integers r and d such that $d \mid q$, $1 \leqslant r \leqslant \varepsilon d$, $(r, d) = 1$, and $h \equiv -r^{-1}(q/d)^{-1} \pmod d$.

Let $z<y<x$, $z\to\infty$ as $x\to\infty$. We write the integer $n\leqslant x$ in the form mq, where the prime factors of q all lie in $(z,y]$, and m has no prime factors in this range. We assume from now on that q is squarefree. The number of integers $n<x$ of the form mq, q not squarefree is clearly $O(x/z)$; the sum of their t_2's is $O(x^2/z)$.

Now suppose that m is in an ε-good residue class $(\operatorname{mod} q)$, and let r and d be the pair of integers specified above. Let $t=rmq/d$. Then $t+1\equiv 0 \pmod d$ and $t(t+1)\equiv 0 \pmod n$. Hence $t_2(n)\leqslant t\leqslant \varepsilon n$, and for these n, the sum of the t_2's does not exceed εx^2.

For each q, we estimate the number of ε-bad residue classes. Let p be a prime factor of q. If h is ε-bad, then $h\equiv -r^{-1}(q/p)^{-1} \pmod p$ where $\varepsilon p<r<p$. By the Chinese remainder theorem, there are at most $q(1-\varepsilon)^{\nu(q)}$ bad classes. Let us choose $y=x^{1/10}$. Then the number of $n\leqslant x$ such that $n=mq$ and m is ε-bad $(\operatorname{mod} q)$ is

$$\ll \frac{x}{q}(1-\varepsilon)^{\nu(q)}\prod_{z<p\leqslant y}\left(1-\frac{1}{p}\right)$$

and, summing over q, this is

$$\ll x\prod_{z<p\leqslant y}\left(1-\frac{\varepsilon}{p}\right)\ll x\left(\frac{\log z}{\log y}\right)^{\varepsilon}.$$

Hence

$$\sum_{n\leqslant x} t_2(n)\ll x^2\left(\frac{1}{z}+\varepsilon+\left(\frac{10\log z}{\log x}\right)^{\varepsilon}\right),$$

and we set $z=\log x$, $\varepsilon=2(\log\log\log x)/(\log\log x)$. This gives the result stated.

Proof of Theorem 4. Let $n\leqslant x$ be a squarefree number, $P^-(n)>(\log x)^{\lambda}$ ($P^-(n)$ is the least prime factor of n). For any fixed λ, it is easily shown by Brun's method that the number of such n is $\sim e^{-\gamma}x/(\lambda\log\log x)$, where γ is Euler's constant. Write $n=p_1p_2\ldots p_\nu$, and for each prime p, let p' denote the next larger prime. We consider whether $m\,\Delta\, n$ where $m=p_1'p_2'\ldots p_\nu'$. For each divisor $p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_\nu^{\alpha_\nu}$ of n, m has the corresponding, larger, divisor $p_1'^{\alpha_1}p_2'^{\alpha_2}\ldots p_\nu'^{\alpha_\nu}$ and if this is always less than the next larger divisor of n, we shall have $m\,\Delta\, n$. A sufficient condition for this is $m/n<\theta(n)$, where θ denotes the smallest ratio, greater than 1, of two divisors, of n. Choose a fixed κ, $\frac{7}{12}<\kappa<1$. It is well known that for $p>p_0(\kappa)$, we have $p'<p+p^{\kappa}$. Hence provided $p_0(\kappa)<(\log x)^{\lambda}$, as we assume, we have

$$\begin{aligned} m/n &< \prod_{p\mid n}(1+p^{\kappa-1}) \\ &< (1+(\log x)^{\lambda\kappa-\lambda})^{\nu(n)} \\ &< \exp\{2(\log x)^{\lambda\kappa-\lambda+1}\} \end{aligned}$$

since $\nu(n) < 2\log x$. We can choose a fixed λ such that the right-hand side does not exceed $1+(\log x)^{-3}$. It follows that if n cannot be interlocked, certainly we must have $\theta(n) < 1+(\log x)^{-3}$. The ratio $\theta(n)$ can be achieved with relatively prime divisors of n, hence

$$\operatorname{card}\{n \leqslant x\colon \theta(n) \leqslant \theta\} \leqslant \sum_{d \leqslant x} \sum_{d < d' \leqslant \theta d} x/dd'$$
$$\ll x(\theta-1)\log x.$$

Setting $\theta = 1+(\log x)^{-3}$, we obtain

$$A(x) \geqslant (e^{-\gamma}+o(1))\, x/\lambda \log\log x.$$

This gives the result stated. We remark that it is known (Erdös (1964), but no proof has been published) that for every $\alpha > \log 3 - 1$, there exists a positive $\varepsilon = \varepsilon(\alpha)$ such that

$$\operatorname{card}\{n \leqslant x\colon \theta(n) < 1+(\log x)^{-\alpha}\} \ll x(\log x)^{-\varepsilon}.$$

We may also assume $\nu(n) < 2\log\log x$. Thus we can obtain

$$A(x) \geqslant \frac{(5e^{-\gamma}+o(1))\,x}{12(\log 3 - 1)\log\log x}, \quad \text{as } x \to \infty.$$

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