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ON THE ALGEBRAIC CONVERGENCE OF FINITELY GENERATED KLEINIAN GROUPS IN ALL DIMENSIONS

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Abstract

Let $\{G_{r,i}\}$ be a sequence of r-generator Kleinian groups acting on $\overline{\mathbb{R}}^n$. In this paper, we prove that if $\{G_{r,i}\}$ satisfies the F-condition, then its algebraic limit group G_r is also a Kleinian group. The existence of a homomorphism from G_r to $G_{r,i}$ is also proved. These are generalisations of all known corresponding results.

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1. Introduction

In this paper, we will adopt the same definitions and notation as in [5, 7, 8], such as discrete groups G of $M(\overline{\mathbb{R}}^n)$, limit sets $L(G)$ of G, nonelementariness and so on. For example, G is a *Kleinian group* if G is discrete and nonelementary.

Let $\{G_{r,i}\}$ be a sequence of subgroups in $M(\overline{\mathbb{R}}^n)$ and each $G_{r,i}$ be generated by $g_{1,i}, g_{2,i}, \dots, g_{r,i}$ $(0 < r < \infty)$. If, for each $t \in \{1, 2, \dots, r\}$,

$$g_{t,i} \to g_t \in M(\overline{\mathbb{R}}^n) \quad \text{as} \quad i \to \infty,$$

then we say that $\{G_{r,i}\}$ converges algebraically to $G_r = \langle g_1, g_2, \dots, g_r\rangle$ and G_r is called the *algebraic limit group* of $\{G_{r,i}\}$. If, for each i, $G_{r,i}$ is a Kleinian group, then the question when G_r is still a Kleinian group has attracted much attention. For example, in [3], Jørgensen and Klein established the following classical algebraic convergence theorem.

THEOREM A [3]. *Let $\{G_{r,i}\}$ be a sequence of r-generator Kleinian groups of $M(\overline{\mathbb{R}}^2)$ converging algebraically to the group G_r. Then G_r is a Kleinian group.*

In higher dimensions, Martin observed that if the sequence $\{G_{r,i}\}$ contains elliptic elements $g_{t,i}$ such that $g_{t,i} \to g_t$ with $\mathrm{ord}(g_{t,i}) \to \infty$ as $i \to \infty$, then the algebraic limit

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group is not a Kleinian group, where ‘ord(g)’ denotes the order of g. This shows that to study when the algebraic limit group of a sequence of r-generator Kleinian groups is Kleinian some restriction is needed. In [5], Martin introduced the following restriction.

A set X of $M(\overline{\mathbb{R}}^n)$ is said to have *uniformly bounded torsion* if there is an integer $N > 0$ such that for each $g \in X$,

$$\mathrm{ord}(g) \le N \quad \text{or} \quad \mathrm{ord}(g) = \infty.$$

By using this restriction, Martin generalised Theorem A to the higher dimensional case.

Theorem B [5, Proposition 5.8]. *Let G_r be the algebraic limit group of a sequence $\{G_{r,i}\}$ of r-generator Kleinian groups of $M(\overline{\mathbb{R}}^n)$ with uniformly bounded torsion. Then G_r is a Kleinian group.*

Recently, Wang [7] and Yang [10] introduced the restrictions ‘*EP*-condition’ and ‘*Condition A*’, respectively, to weaken ‘uniformly bounded torsion’. Their results are as follows.

Theorem C [7, Theorem 1.1]. *Let G_r be the algebraic limit group of a sequence $\{G_{r,i}\}$ of r-generator Kleinian groups of $M(\overline{\mathbb{R}}^n)$. If $\{G_{r,i}\}$ satisfies the EP-condition, then G_r is a Kleinian group.*

Here a sequence $\{G_i\}$ is said to satisfy the *EP-condition* if the following two conditions are satisfied.

(1) For any sequence $\{f_{ik}\}$, $f_{ik} \in G_{ik}$ $(\in \{G_i\})$, if $\mathrm{card}(\mathrm{fix}(f_{ik})) = \infty$ and $f_{ik} \to f$ as $k \to \infty$, where f is the identity map I or a parabolic element, then $\{f_{ik}\}$ has uniformly bounded torsion.

(2) $\{G_i\}$ satisfies *Property A*, that is, $\{G_i\}$ contains no sequences $\{f_{ik}\}, \{g_{ik}\}$ which satisfy that both $f_{ik}, g_{ik} \in G_{ik}$ $(\in \{G_i\})$ are elliptic and

$$\mathrm{fix}(f_{ik}) \cap \mathrm{fix}(g_{ik}) = \emptyset, \quad \mathrm{card}(\mathrm{fix}(f_{ik})) = \mathrm{card}(\mathrm{fix}(g_{ik})) = 2,$$
$$f_{ik} \to I \quad \text{and} \quad g_{ik} \to I$$

as $k \to \infty$.

Theorem D [10, Theorem 2.4]. *Let G_r be the algebraic limit group of a sequence $\{G_{r,i}\}$ of r-generator Kleinian groups of $M(\overline{\mathbb{R}}^n)$. If $\{G_{r,i}\}$ satisfies Condition A, then G_r is a Kleinian group.*

Here we say that a sequence $\{G_i\}$ satisfies *Condition A* if there is no sequence $\{f_{ik}\}$, $f_{ik} \in G_{ik}$ $(\in \{G_i\})$ with $\mathrm{card}(\mathrm{fix}(f_{ik})) = \infty$ and $f_{ik} \to I$ as $k \to \infty$ (see [2]).

Example 1.1. Suppose that $G_2 = \langle f_1, f_2 \rangle$ is a two-generator purely hyperbolic nonelementary subgroup of $\mathrm{PSL}(2, \mathbb{R})$ and that, for each natural number i,

$$f_i = \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix},$$

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where $a_i = \cos(\theta_i\pi) + e_2e_3 \sin(\theta_i\pi)$ and each θ_i is a rational number. Let

$$G_{2,i} = \langle G_2, f_i\rangle.$$

Then, for each i, $G_{2,i}$ is a Kleinian group in $\mathrm{PSL}(2, \Gamma_4)$. If the sequence $\{\theta_i\}$ converges to a rational number θ, then the algebraic limit group G_3 of $\{G_{2,i}\}$ is also a Kleinian group; but, if the sequence $\{\theta_i\}$ converges to an irrational number θ, then G_3 is nondiscrete. Moreover, in the former case, if $\theta_i = 1/3^i$, then we know that the sequence $\{G_{2,i}\}$ does not satisfy the EP-condition nor Condition A, but G_3 is still a Kleinian group.

Motivated by Example 1.1, we introduce the following restriction.

Definition 1.2. We say that a sequence $\{G_i\}$ satisfies the *F-condition* if there is no sequence $\{f_{ik}\}$, $f_{ik} \in WY(G_{ik})$ $(\in \{G_i\})$ such that $f_{ik} \to f$ as $k \to \infty$, where f is an elliptic element with $\mathrm{ord}(f) = \infty$.

Let us recall the important notation $WY(G)$ for a Kleinian group G, which was first put forward by Wang and Yang in [8]:

$$WY(G) = \{f : f|_{M(G)} = I, f \in G\},$$

where $M(G)$ is the smallest G-invariant hyperbolic space whose boundary contains the limit set $L(G)$ of G (see [6]). It is obvious that $WY(G)$ is $\{I\}$ or a purely elliptic subgroup of G.

Remark 1.3. Obviously, if a sequence of Kleinian groups satisfies the EP-condition or Condition A, then it must satisfy the F-condition. From Example 1.1, we see that there are sequences of Kleinian groups which satisfy the F-condition but do not satisfy the EP-condition nor Condition A. Also, if a sequence $\{G_{r,i}\}$ $(\{WY(G_{r,i})\})$ of Kleinian groups has uniformly bounded torsion, then $\{G_{r,i}\}$ satisfies the F-condition.

By using the F-condition, we get the following generalisation of Theorems B, C and D.

Theorem 1.4. *Let G_r be the algebraic limit group of a sequence $\{G_{r,i}\}$ of r-generator Kleinian groups of $M(\overline{\mathbb{R}}^n)$. If $\{G_{r,i}\}$ satisfies the F-condition, then G_r is a Kleinian group.*

We have the following corollary, which is easily derived from Theorem 1.4 and Remark 1.3.

Corollary 1.5. *Let G_r be the algebraic limit group of a sequence $\{G_{r,i}\}$ of r-generator Kleinian groups of $M(\overline{\mathbb{R}}^n)$. If $\{WY(G_{r,i})\}$ has uniformly bounded torsion, then G_r is a Kleinian group.*

Moreover, we prove the following result, which is a generalisation of [5, Theorem 6.1].

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THEOREM 1.6. *Let $\{G_{r,i}\}$ be a sequence of r-generator Kleinian groups of $M(\overline{\mathbb{R}}^n)$ converging algebraically to the group G_r. Suppose that the corresponding sequence $\{WY(G_{r,i})\}$ of $\{G_{r,i}\}$ has uniformly bounded torsion and that G_r is finitely presented. Then G_r is also a Kleinian group and the correspondence from the generators of G_r to their approximants in $G_{r,i}$ extends for all sufficiently large i to a homomorphism of G_r onto $G_{r,i}$.*

2. Proofs of Theorems 1.4 and 1.6

2.1. Several lemmas. The following result due to Waterman is from [9].

LEMMA E [9, Theorem 11]. *If $\langle f, g\rangle$ is a Kleinian group of $M(\overline{\mathbb{R}}^n)$, then*

$$\|f - I\| \cdot \|g - I\| > \tfrac{1}{32}.$$

The following two lemmas are crucial for the proofs of Theorems 1.4 and 1.6.

LEMMA 2.1. *Let G_r be the algebraic limit group of a sequence $\{G_{r,i}\}$ of r-generator Kleinian groups of $M(\overline{\mathbb{R}}^n)$. Then:*

(1) *G_r is nonelementary; and*
(2) *G_r is nondiscrete if and only if there exists an elliptic element $f \in WY(G_r)$ with* $\mathrm{ord}(f) = \infty$.

PROOF. The first part of this lemma follows from [4, Theorem 1.4]. Now we come to prove the second part. It suffices to show that if G_r is nondiscrete, then there is an element $f \in WY(G_r)$ with $\mathrm{ord}(f) = \infty$, since the converse is obvious. Now we assume that G_r is nondiscrete. Recall that G_r is a finitely generated subgroup of $M(\overline{\mathbb{R}}^n)$. By applying the Selberg lemma, we know that G_r contains a torsion free subgroup G'_r of finite index which is nondiscrete as well. Then there exists a sequence $\{f_j\}$ in G'_r such that $f_j \to I$ as $j \to \infty$. As G'_r is nonelementary, there are finitely many loxodromic elements $g_1, g_2, \ldots, g_s$ in G'_r such that the set $\{\mathrm{fix}(g_1), \mathrm{fix}(g_2), \ldots, \mathrm{fix}(g_s)\}$ spans the boundary of $M(G'_r)$. Then, for all sufficiently large j, we have

$$\|f_j - I\| \cdot \|g_k - I\| < \tfrac{1}{32},$$

where $k \in \{1, 2, \ldots, s\}$. Let $f_{i,j}$ and $g_{i,k}$ be the corresponding elements of f_j and g_k in $G_{r,i}$, respectively. Then, for large enough i,

$$\|f_{i,j} - I\| \cdot \|g_{i,k} - I\| < \tfrac{1}{32}.$$

Lemma E implies that the subgroups $\langle f_{i,j}, g_{i,k}\rangle$ are elementary. It follows that $\mathrm{fix}(g_{i,k}) \subset \mathrm{fix}(f_{i,j})$, which shows that for $k \in \{1, 2, \ldots, s\}$ and all sufficiently large j, $\mathrm{fix}(g_k) \subset \mathrm{fix}(f_j)$. Hence, $f_j \in WY(G'_r)$, from which the conclusion follows. □

LEMMA 2.2. *Let $\{G_i\}$ be a sequence of finitely generated Kleinian groups of $M(\overline{\mathbb{R}}^n)$ converging algebraically to a group G. If there exists a sequence $\{f_{ik}\}$, $f_{ik} \in G_{ik}$ ($\in \{G_i\}$), such that $f_{ik} \to I$ as $k \to \infty$, then, for sufficiently large k, $f_{ik} \in WY(G_{ik})$.*

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PROOF. By [4, Lemma 4.2], we know that for large enough k, $f_{ik} = I$ or there is a G_{ik}-invariant hyperbolic space Π_{ik} which is fixed pointwise by f_{ik}. So, the closed set $\overline{\Pi}_{ik} \cap \overline{\mathbb{R}}^n$ is also G_{ik}-invariant. Since the limit set $L(G_{ik})$ of G_{ik} is the smallest G_{ik}-invariant subset in $\overline{\mathbb{R}}^n$, similar reasoning as in [1, Theorem 5.3.7] shows that $L(G_{ik}) \subset \overline{\Pi}_{ik} \cap \overline{\mathbb{R}}^n$, which implies that $M(G_{ik}) \subset \Pi_{ik}$. It follows that $f_{ik} \in WY(G_{ik})$. □

2.2. Proof of Theorem 1.4. By Lemma 2.1, we only need to prove that there is no elliptic element $f \in WY(G_r)$ with $\mathrm{ord}(f) = \infty$. Suppose on the contrary that there is some elliptic element $f \in WY(G_r)$ such that $\mathrm{ord}(f) = \infty$. Then there exists an integer sequence $\{n_j\}$ such that $f^{n_j} \to I$ as $n_j \to \infty$. For each n_j, let $f_i^{n_j}$ be the corresponding element in $G_{r,i}$. By Lemma 2.2 and the hypothesis that $\{G_{r,i}\}$ satisfies the F-condition, we know that $f_i^{n_j} = I$ for large enough i. It follows that $f^{n_j} = I$, which contradicts the assumption that $f \in WY(G_r)$ with $\mathrm{ord}(f) = \infty$.

2.3. Proof of Theorem 1.6. The proof easily follows from Lemma 2.2 and a similar argument as in the proof of [5, Theorem 6.1].

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