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ORDER-BOUNDED CONVERGENCE STRUCTURES ON SPACES OF CONTINUOUS FUNCTIONS

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Dedicated to Professor Kiyoshi Iseki on his 60th birthday

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Abstract

This paper deals with solid topologies and convergence structures on the vector-lattice CX (the set of all continuous real-valued functions on a space X): the closed ideals and locally convex topologies associated with such structures are studied in particular. The work stems from E. Hewitt's paper on bounded linear functionals, touches on the classical theorems of L. Nachbin, T. Shirota and others (determining when the topology of compact convergence is barrelled or bornological), and extends some recent results on the duality between X and CX.

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1. Introduction

It is well known that if X is a compact space, a linear functional on CX is bounded exactly when it is continuous with respect to the topology t_u of uniform convergence. However, Hewitt (1950) showed that this is not necessarily true of more general spaces: t_u-continuous functionals or seminorms on CX need not be bounded. Nevertheless, boundedness can be characterized analytically.

One way of doing this is to define a convergence structure q_m on CX (or, more generally, on any vector space with modulus), such that a seminorm is bounded if and only if it is q_m-continuous. Since q_m lies between t_u and the topology t_k of compact convergence, $q_m = t_u$ if X is compact. Unlike some other generalizations of t_u such as t_k or the structure q_c of continuous convergence defined by Binz (1975), q_m is algebraic in nature and provides no more information about X than CX itself.

Because of this, in studying q_m by itself one may suppose that X is a real-compact topological space. For example, by making this assumption one can show that t_u and q_m admit the same closed ideals, and the locally convex topology associated with q_m is t_k. (Actually, this fact is in essence one of the characterizations of real-compact spaces obtained in Gulick and Gulick (1976).)

More generally, for any space X the position of q_m between t_u and t_k can be estimated by comparing it with other structures in this range, such as q_c, locally uniform convergence q_{lu}, the Marinescu structure of continuous convergence q_i and their order-bounded modifications. It turns out that q_m is finer than the finest of these, oq_i. A measure of the distance between q_m and q_i or oq_i is the strength of the condition needed for equality: for instance, if X is a c-embedded space then $q_m = q_i$ if and only if X is compact and $q_m = oq_i$ if and only if X is a Lindelöf topological space.

Finally, as the order-bounded structures oq_i, oq_{lu} and oq_c are less widely known than their parents, their properties are briefly discussed. They all have the same closed ideals, namely, the pointwise closed ideals. On the other hand, Kutzler (1974) found a locally compact Hausdorff topological space such that (i) $q_c = t_k$, because of local compactness, but (ii) the locally convex topology t associated with oq_c is properly finer than t_k. A few results are given, serving mainly to outline the problems and show that in Kutzler's example, t is not a topology of uniform convergence at all.

1. Groundwork and notation

The language of convergence spaces is used throughout: readers unfamiliar with it may consult the recent treatise of Binz (1975), the pioneering papers of Fischer (1959) and Kowalsky (1954). In this paper though, it matters little whether their definition of convergence space (Limesraum) is used or a less restrictive one; the definition given below is suitable.

A pair (Q, q) is called a *convergence space* if Q is a set and q a map (known as its structure) associating with each x in Q a collection $q(x)$ of filters on Q such that

(i) the ultrafilter $\dot{x}$ belongs to $q(x)$, and

(ii) if $\mathscr{F} \in q(x)$ and $\mathscr{G}$ is finer than $\mathscr{F} \cap \dot{x}$ then $\mathscr{G} \in q(x)$.

For typographical clarity, the adherence operator associated with a convergence structure q on Q is denoted by q as well: that is, if $A \subseteq Q$ then

$$q[A] = \{x \in Q : A \in \mathscr{F} \text{ for some } \mathscr{F} \in q(x)\}.$$

Use of this non-standard convention is clearly signalled. Further, if $\mathscr{F}$ is a filter on Q and $\mathscr{F}$ meets A (meaning that $F \cap A$ is non-void for all F in $\mathscr{F}$) then $\mathscr{F} \cap A$ and $\mathscr{F} | A$ denote the filters on Q and A respectively generated by $\{F \cap A : F \in \mathscr{F}\}$.

Though the set CX of all continuous functions from a convergence space $X = (Q, q)$ to the scalar field $\mathbf{F}$ is a lattice algebra when $\mathbf{F}$ is real, its order structure is not so simple in the complex case. What is common to both is a modulus: this suggests studying 'mod-spaces' in general.

Take $\mathbf{F}$ to be the real or complex field, and let E be a vector space over $\mathbf{F}$. A map $m: E \to E$ is called a *modulus* if for all a, b in E and r in $\mathbf{F}$,

(i) $m(a) = m(m(a))$,

(ii) $m(ra) = |r| m(a)$,

(so the relation $\leqslant$ on E defined by '$a \leqslant b$ if $b-a = m(b-a)$' is reflexive and translation-invariant)

(iii) $m(a+b) \leqslant m(\alpha) + m(b)$, and

(iv) $m(a) \leqslant a$ implies $m(a) = a$.

In this case, (E, m) is said to be a *mod-space*, and $\leqslant$ is a pre-order compatible with the vector structure. Further, $m(a-b) \geqslant m(m(a) - m(b))$.

From now on, let (E, m) be a mod-space and for each a in E, define B_a to be the set $\{e: m(e) \leqslant m(a)\}$. A subset A of E is said to be *mod-closed* if $m(A) \subseteq A$, or *mod-convex* if $B_a \subseteq A$ for all a in A. (Mod-convex subspaces are an exception: they are usually called *bands*. For example, the kernel M of m is a band making m constant on the translates of M—that is, $m(b) = m(a)$ if $m(b-a) = 0$.)

Next, a structure q on E is called *homogeneous* (or *translation-invariant*) if the translations are all homeomorphisms. Further, q is said to be a *vector structure* if the operations (addition and scalar multiplication) are continuous. Clearly vector structures are homogeneous; their well-known internal characterization can be found in Fischer (1959), Satz III.9.

Let $\mathscr{M}$ be the family of all mod-convex subsets of E. For each filter $\mathscr{F}$ on E, $\mathscr{M} \cap \mathscr{F}$ is a base for a coarser filter $\mathscr{M}(\mathscr{F})$ on E, known as the *$\mathscr{M}$-closure* of $\mathscr{F}$. Clearly $\mathscr{F}$ is $\mathscr{M}$-closed (namely, $\mathscr{F} = \mathscr{M}(\mathscr{F})$) if and only if it has a base of mod-convex sets. Similarly, for any translation-invariant structure (briefly, ti-structure) q, one gets a coarser ti-structure mq by defining

$$mq(0) = \{\mathscr{G} : \mathscr{G} \text{ is finer than } \mathscr{M}(\mathscr{F}) \text{ for some } \mathscr{F} \in q(0)\}.$$

This leads one to call q *mod-convex* if $q = mq$, or equivalently if $\mathscr{M}(\mathscr{F}) \in q(0)$ whenever $\mathscr{F} \in q(0)$. The reader will easily verify the following facts.

1.1. *Let q be a homogeneous structure on E. Then mq is the finest mod-convex ti-structure coarser than q, and the modulus is mq-continuous. Further, if q is a vector structure then so is mq.*

Similarly, with each ti-structure q is associated its order-bounded modification oq: that is, the ti-structure such that $\mathscr{F} \in oq(0)$ if and only if $\mathscr{F} \in q(0)$ and $B_a \in \mathscr{F}$, for some a in E. Naturally if q is a vector structure or a mod-convex structure, then so is oq.

Amongst all vector structures on E is a finest one, q_l, in which $\mathscr{F}$ is q_l-convergent to 0 if and only if for some finite-dimensional subspace H of E, $H \in \mathscr{F}$ and $\mathscr{F} | H$ converges to 0 in the usual Euclidean topology on H. (In the language of Binz (1975), (E, q_l) is a Marinescu space, being the convergence space inductive limit of a family of locally convex topological vector spaces—in this case, all its finite dimensional subspaces.) Every linear functional or seminorm on E is q_l-continuous: in short, q_l might as well be known as the *fine vector structure* on E.

Analogously, the mod-space (E, m) admits a finest mod-convex vector structure q_m known as the *mod-fine vector structure*, namely, $q_m = mq_l$. However, a more explicit description of q_m is needed. For any a in E, set $B_a = \{e \in E\colon m(e) \leqslant m(a)\}$ as before, and $E_a = \bigcup_1^\infty nB_a$. Then E_a is a band and B_a is a mod-convex unit ball for a norm topology t_a on E_a. Clearly if $0 \leqslant a \leqslant b$ then $E_a \subseteq E_b$, the inclusion map being $t_a = t_b$-continuous.

Now one can define a homogeneous structure q on E as follows: $\mathscr{F} \in q(0)$ if for some a in E, the set E_a belongs to $\mathscr{F}$, and $\mathscr{F} | E_a$ converges to 0 in (E_a, t_a).

1.2. *The structure q defined above coincides with q_m, so that q_m is an order-bounded Marinescu structure.*

Just as with vector lattices, one says that a linear functional or seminorm p on E is *bounded* if $p(B_a)$ is bounded in **F** for all a in E. Similarly, it is said to be *full* if $p(a) = p(m(a))$ for all a in E and $p(a) \leqslant p(b)$ if $0 \leqslant a \leqslant b$ in E. Clearly, full seminorms are bounded. Conversely, the best one can hope for is for each bounded seminorm to be majorized by a full one, a property symbolized by $(B < F)$.

Though q_m has wider applications than this, I was first led to study it when I wished to restate the geometric property of boundedness in analytic terms. The restatement is obvious.

1.3. *A seminorm or linear functional is bounded if and only if it is q_m-continuous.*

Moreover, as the kernel of a seminorm is a vector subspace, one has an obvious corollary.

1.4. *The kernel of a bounded seminorm is q_m-closed: if the seminorm is full, it is actually a q_m-closed band.*

Full seminorms are often easier to deal with than others: the purpose of $(B < F)$ is to guarantee 'enough' full seminorms. Another useful property (enjoyed by all vector lattices but not all mod-spaces) is the *decomposition property*: namely,

$$B_{a+b} = B_a + B_b \quad \text{if } a \geqslant 0 \leqslant b.$$

Let p be a bounded seminorm on E, and define $p^*(a) = \sup\{p(b): b \in B_a\}$ for all a in E. By construction, p^* is full and $p^*(ra) = |r| p^*(a)$ for all r in $\mathbf{F}$ and all a in E.

Now suppose that (E, m) has the decomposition property, and let a, b belong to E. If $m(e) \leqslant m(a+b) \leqslant m(a) + m(b)$, this property provides c in B_a and d in B_b such that $e = c + d$, Thus $p(e) \leqslant p(c) + p(d) \leqslant p^*(a) + p^*(b)$ for all $e \in B_{a+b}$, so that $p^*(a+b) \leqslant p^*(a) + p^*(b)$. In short, p^* is a full seminorm majorizing p.

This fact, that the decomposition property implies $(B < F)$, was noted by Kutzler (1974), Peressini (1967), p. 105, and doubtless others too.

Locally convex vector topologies arise now: with any convergence structure q on E is associated the locally convex vector topology lq generated by all the q-continuous seminorms. In general, it is not mod-convex, but . . .

1.5. THEOREM. *Let (E, m) have the decomposition property, and suppose that q is a mod-convex vector structure. Then lq is mod-convex as well, being generated by all the q-continuous full seminorms.*

PROOF. Let p be a q-continuous seminorm. By 1.2, p is q_m-continuous and hence bounded: thus the functional p^* constructed above is a well-defined full seminorm. To show that p^* is q-continuous, it is as usual sufficient to prove its continuity at 0.

By definition, for all a in E, $p^*(a) \in \overline{p(B_a)}$, the closure of $p(B_a)$ in $\mathbf{R}$. Thus for any mod-convex set A in E, $p^*(A) \subseteq \overline{p(A)}$. Now as $\mathbf{R}$ is a regular topological space, one can see that $p^*(\mathscr{F}) \to 0$ in $\mathbf{R}$ whenever $\mathscr{F}$ is an $\mathscr{M}$-closed filter such that $p(\mathscr{F}) \to 0$ in $\mathbf{R}$. In particular, p^* is q-continuous at 0 as desired.

Consequently lq coincides with the apparently weaker vector topology generated by all the full q-continuous seminorms, a topology which is clearly mod-convex.

Finally, suppose E to be an algebra. Then (E, m) is said to be a *mod-algebra* if there is a positive real number r such that $m(ab) \leqslant r m(a) m(b)$ for all a, b in E. In this case, the multiplication is q_m-continuous, so that q_m is an algebra structure. However, a vector convergence structure q may have the property that $a\mathscr{F} \in q(0)$ when $\mathscr{F} \in q(0)$, without the multiplication's having to be jointly continuous. Such a structure is known as a *semi-algebra structure*.

2. Function algebras

Let X be a convergence space. With the usual operations defined pointwise, the set $E = CX$ is a mod-algebra. Most of the apparatus needed in the study of q-closed ideals and bands and the topology lq is described here, for any mod-convex vector structure q on E. It is then used on q_m in particular.

To start with, define a commutative binary operation $\%$ on $\mathbf{F}$ by

$$r\% s = \operatorname{sgn}(rs)\min\{|r|,|s|\}$$

for all r, s in $\mathbf{F}$ (where $\operatorname{sgn}(t) = t/|t|$ if $t \neq 0$ and $\operatorname{sgn}(0) = 0$).

2.1. *For all r, s and t in* $\mathbf{F}$,
(i) $|r\% s| = \min\{|r|,|s|\}$, *and*
(ii) $|r\% t - s\% t| \leqslant |r-s|$.

PROOF. Trivially, (i) holds true. Now let $u = |t|$, $r' = r\% u$ and $s' = s\% u$. Since $r\% t - s\% t = \operatorname{sgn}(t)(r'-s')$, it remains only to show that $|r'-s'| \leqslant |r-s|$. An easy proof can be obtained using the diagrams given below, in which the circles are centred on the origin and have radius u (Fig. 1).

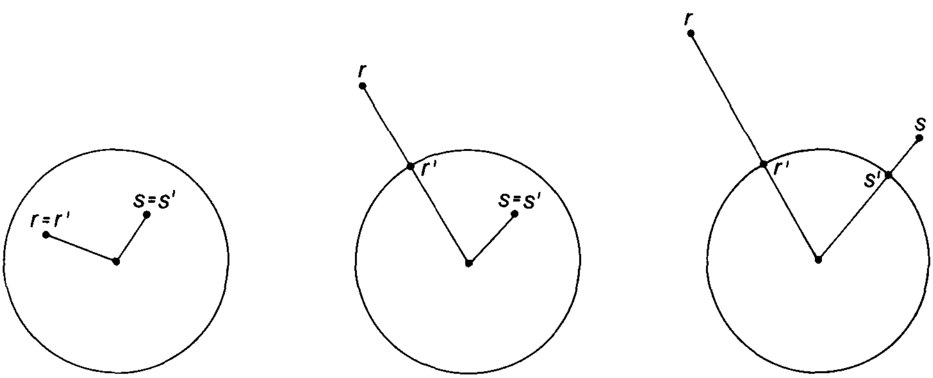

FIGURE 1.

2.2. *The operation $\%$ is continuous, regarded as a function from* $\mathbf{F}^2$ *to* $\mathbf{F}$.

PROOF. For any r, s, t, u in $\mathbf{F}$, one has by (ii) above

$$\begin{aligned}|r\% s - t\% u| &\leqslant |r\% s - t\% s| + |t\% s - t\% u| \\ &\leqslant |r-t| + |s-u|.\end{aligned}$$

Because of this, $\%$ can be extended pointwise to a binary operation on E: for all f, g in E, $f\% g$ is defined by $(f\% g)(x) = f(x)\% g(x)$ for all x in X. Naturally the inequalities 2.1 remain true.

2.3. *The mod-algebra (E, m) has the decomposition property.*

PROOF. Suppose $a \geqslant 0 \leqslant b$ in E. Trivially, $B_{a+b} \supseteq B_a + B_b$. Conversely, let $m(f) \leqslant a+b$ and define $c = f \%\, a$. Then $m(c) \leqslant a$ by 2.1 (i), so that $c \in B_a$. Next,

$$m(f-c) = m(f\%(a+b) - f\%\,a)$$

$$\leqslant m(b), \quad \text{by 2.1(ii).}$$

Thus $f-c \in B_b$, showing that $f \in B_a + B_b$.

Next come three more technical results, of use mainly for mod-convex vector structures. Henceforth, let e stand for the unit function, constant 1.

2.4. *Let q be a structure coarser than q_m, and f belong to E. Then $f\%\,re$ is q-convergent to f as $r \to \infty$ in* $\mathbf{R}$.

PROOF. It can be assumed that $q = q_m$. Let $h = m(f) + m(f^2)$, and suppose $2 > s > 0$. Now if $r \geqslant 1/(4s) > 0$ and $z \in \mathbf{F}$ then
(i) $|z - z\%\,r| = 0 \leqslant s(|z| + |z|^2)$ if $|z| \leqslant r$, and
(ii) $|z - z\%\,r| = |z| - r \leqslant s(|z| + |z|^2)$ if $|z| > r$.
Thus $m(f - f\%\,re) \leqslant sh$ if $r \geqslant 1/(4s)$, showing that $f\%\,re \to f$ in (E_h, t_h) as desired.

2.5. LEMMA. *If q is a mod-convex group structure on E then $\%$ is q-continuous.*

PROOF. A group structure is a (necessarily translation-invariant) structure such that $+$ and $-$ are continuous. So, suppose that $\mathscr{F}, \mathscr{G} \in q(0)$ and f, g belong to E. To show the continuity of $\%$, it is enough to prove that $(\mathscr{F}+f)\%(\mathscr{G}+g)$ is q-convergent to $f\%\,g$. By mod-convexity, it can be assumed that $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{M}$-closed.

So, let F and G be mod-convex members of $\mathscr{F}$ and $\mathscr{G}$, with $a \in F$ and $b \in G$. Then by 2.1(ii),

$$m((a+f)\%(b+g) - (a+f)\%\,g) \leqslant m(b) \quad \text{and} \quad m((a+f)\%\,g - f\%\,g) \leqslant m(a).$$

Thus the former belongs to G, the latter to F and $(a+f)\%(b+g)$ to $f\%\,g + F + G$. In short, $(\mathscr{F}+f)\%(\mathscr{G}+g)$ is finer than $\mathscr{F} + \mathscr{G} + f\%\,g$, a filter q-converging to $f\%\,g$.

2.6. *If q is a mod-convex ti-structure and A is mod-convex, then $q[A]$ is also mod-convex, and furthermore, $q[A] = oq[A]$.*

PROOF. (Recall the notation for an adherence operator.) First, take $a \in q[A]$ and suppose that $m(b) \leqslant m(a)$. Then there is an $\mathscr{M}$-closed filter $\mathscr{F}$ in $q(0)$ such that

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$a+\mathscr{F}$ meets A. Choose $F\in\mathscr{F}$ and $f\in F$ such that $a+f\in A$. Then $-b\,\%\,m(f)\in F$ if F is mod-convex, while

$$m(b-b\,\%\,m(f)) = m(b\,\%\,m(a)-b\,\%\,m(f))$$
$$\leqslant m(a+f).$$

Thus $b-b\,\%\,m(f)\in(b+F)\cap A$, showing that $b+\mathscr{F}$ meets A. In short, $q[A]$ is mod-convex.

Since oq is finer than q, $oq[A]\subseteq q[A]$. Conversely, suppose $a\in q[A]$: that is, for some $\mathscr{M}$-closed filter $\mathscr{F}$ in $q(0)$, $a+\mathscr{F}$ meets A. It will now be shown that $a+\mathscr{F}\cap B_a$ also meets A, and hence that $a\in oq[A]$.

Take $F\in\mathscr{F}$. By assumption, $a+f\in A$ for some $f\in F$. By 2.1(i), $-a\,\%\,m(f)\in F\cap B_a$, if F is mod-convex, while

$$m(a-a\,\%\,m(f)) = m(a\,\%\,m(a)-a\,\%\,m(f))$$
$$\leqslant m(m(a)-m(f)), \quad \text{by 2.1(ii)}$$
$$\leqslant m(a+f).$$

(The last inequality holds in any mod-space at all, not just the 'nice' one considered here.) Thus $a-a\,\%\,m(f)\in A\cap(a+F\cap B_a)$.

2.7. THEOREM. *Let q be a mod-convex algebra structure. Then the q-adherence of a band is both a band and an ideal.*

PROOF. Let A be a band in E and $B=q[A]$. The vector operations being q-continuous, B is a vector sub-space of E, while by 2.6 it is mod-convex. Finally, let $g\in B$ and $f\in E$. By 2.4, there is a filter $\mathscr{F}$ converging to f, such that the set E_e of all bounded functions belongs to $\mathscr{F}$. Similarly there is a filter $\mathscr{G}$ converging to g, to which A belongs. As multiplication is by assumption continuous, $\mathscr{F}\mathscr{G}\to fg$. Now suppose that $F\in\mathscr{F}$ and $G\in\mathscr{G}$. Then functions $a\in F\cap E_e$ and $b\in G\cap A$ can be found, with $ab\in FG\cap A$ since A is a band. Thus $\mathscr{F}\mathscr{G}$ meets A, showing that $fg\in B$.

2.8. *If q is a mod-convex semi-algebra structure then a similar proof shows that a q-closed band is an ideal.*

Now let I be an ideal and q be a mod-convex semi-algebra structure on E. By 1.2, q_m is finer than q. In turn, the topology t_u of uniform convergence is finer than q_m: this is because t_u is a (mod-convex) group topology whose filter $\mathscr{U}$ of t_u-neighbourhoods of 0 converges to 0 in (E,q_m), since $E_e\in\mathscr{U}$ and $\mathscr{U}\,|\,E_e\to 0$ in (E_e,t_e).

Next, for any f, g in E, if g is invertible then $f\%g = fg/\max\{|f|, |g|\}$. In particular f belongs to I if and only if $f' = f\%e$ does. Take f in I, and suppose that $m(g) \leqslant m(f)$. Then $m(g') \leqslant m(f') \leqslant e$. Since $m(f')^2$ is the product of f' with its complex conjugate, $m(f')^2 \in I \cap B_e$. Let (r_n) be a sequence of polynomials on $[0, 1]$ with no constant term, converging uniformly to the cube-root function r. Then the sequence $r_n \circ m(f')^2$ converges uniformly on X to the function $f'' = r \circ m(f')^2$. By assumption, each $r_n \circ m(f')^2$ belongs to I, so that $f'' \in t_u[I]$. Thus f'' belongs to the set $q[I]$, which is an ideal since q is by assumption a semi-algebra structure. By defining

$$h(x) = \begin{cases} 0 & \text{if } g'(x) = 0, \\ g'(x)/f''(x) & \text{otherwise,} \end{cases}$$

one obtains a well-defined function h whose continuity is easily verified. But as $g' = hf''$, g' and hence g both belong to $q[I]$. This argument, due to Kutzler (1974), proves the next result.

2.9. THEOREM. *If q is a mod-convex semi-algebra structure, then any q-closed ideal is a band.*

One can do a little more than this in some cases, for example, if the operation $f \mapsto m(f)^{\frac{2}{3}}$ is q-continuous on E, then $q[I]$ itself is always a band. Without some such assumption?

3. The mod-fine vector structure

The topology t_k of compact convergence generalizes t_u in a natural way, avoiding some drawbacks t_u has when X is not compact. Various other structures (such as continuous convergence q_c, locally uniform convergence q_{lu} and the Marinescu structure of continuous convergence q_I, all described in Binz (1975)) generalize t_u as well: these all carry detailed information about X even when X is not compact. Yet another is q_m: it is the finest mod-convex vector structure coarser than t_u. Unlike the others, both t_u and q_m are entirely algebraic in nature, depending solely on the mod-space structure of E, and hence reflect a more blurred image of X.

This section is devoted almost entirely to q_m itself, its closed ideals and to the locally convex topology lq_m. Its relations with other structures and topologies on E are dealt with later.

3.1. *The structure q_m lies between t_u and t_k. Moreover, it is complete.*

PROOF. As noted earlier in the proof of 2.9, t_u is finer than q_m. But as t_k is a mod-convex vector structure, $q_m \geqslant t_k$ (1.2). Finally, because any inductive limit of

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complete convergence vector spaces is complete, by Wloka (1963), Satz III.11, q_m is complete if (E_f, t_f) is, for each f in E.

So, suppose that (g_n) is a Cauchy sequence in (E_f, t_f), that is, that for each $r>0$ one can find n in $\mathbf{N}$ such that if $n''>n'>n$ then $m(g_{n''}-g_{n'})\leqslant rm(f)$. This inequality holding pointwise, completeness of $\mathbf{F}$ yields a function g, the pointwise limit of (g_n). The reader will easily verify that g is continuous, actually belonging to E_f, and that (g_n) is t_f- convergent to g.

As mentioned before, q_m is an algebraic construct depending only on E. Thus it can be and is assumed for the rest of the section that X is a real-compact topological space, whose Stone–Čech compactification is X^*. With each f in E is associated its Stone extension f^*, a continuous function from X^* to the one point compactification of $\mathbf{F}$. As in Nanzetta and Plank (1972), a subset P of X^* is called a zero-set if $P=Z(f)=\{y\in X^*: f^*(y)=0\}$ for some f in E. Similarly, it is said to be *far* (from X) if there is a zero-set S such that $P\subseteq S\subseteq X^*\setminus X$ or, equivalently, if there is some f in E with $P\subseteq U(f)=\{y\in X^*: f^*(y)=\infty\}$.

Now let I be an ideal in E, and let N stand for its null-set $\{y\in X^*: f^*(y)=0$ for all f in $I\}$. Nanzetta and Plank (1972), Theorem 2.3, showed that

$$t_u[I]=\{f\in E\colon Z(f)\supseteq N\}.$$

A similar formula holds for $q_m[I]$ and, as will be seen, its proof owes a lot to their work.

3.2. THEOREM. *For any ideal I in E, $q_m[I]=\{f\in E\colon N\setminus Z(f)$ is far from $X\}$.*

PROOF. Let J be the set defined above and suppose first that $f\in q_m[I]$. This means that for some filter $\mathscr{F}$, $I\in\mathscr{F}\in q_m(f)$. More precisely, there is a function $g\geqslant 0$ and a sequence (f_n) in $I\cap E_g$ such that $(f_n)\to f$ in (E_g, t_g). Thus for each $s>0$ an integer l exists, such that if $n>l$ then the inequality $m(f_n-f)\leqslant sg$ holds on X. But as continuity and density guarantee similar inequalities on X^*, if $x\in N\setminus U(g)$ then $|f^*(x)|\leqslant sg^*(x)<\infty$ for all $s>0$. In short, f^* vanishes on $N\setminus U(g)$, showing that $f\in J$.

Conversely, suppose that $f\in J$, meaning that $N\subseteq Z(f)\cup U(g)$ for some g in E. Clearly g can be so chosen that $g=m(g)\geqslant e+m(f^2)$, having thus a multiplicative inverse h. Moreover $0\leqslant h\leqslant e$ and $U(f)\subseteq U(g)=Z(h)$. Let $a=fh$. By the choice of g, a^* vanishes on $Z(f)\cup Z(h)$, and in fact $Z(a)=Z(f)\cup Z(h)\supseteq N$. Now by the theorem quoted above, $a\in t_u[I]$ and so $a\in q_m[I]$. But as $q_m[I]$ is an ideal, $f=ag\in q_m[I]$, as desired.

3.3. COROLLARY. *The q_m-adherence of any ideal is a band.*

To continue, a closed subset P of X^* is said to be an *ideal set* if $\{f\in E\colon Z(f)\supseteq P\}$ is an ideal, while Nanzetta and Plank (1972), Theorem 3.1 and Corollary 2.4, showed that (a) P is an ideal set if and only if $\overline{P\setminus F}=P$ for all far sets F, and (b) that conditions (iii) and (iv) below characterize t_u-closed ideals in terms of ideal sets.

3.4. THEOREM. *The following statements are equivalent*:

(i) *I is a q_m-closed band,*

(ii) *I is a q_m-closed ideal,*

(iii) *I is a t_u-closed ideal, and*

(iv) *$I=\{f\in E: Z(f)\supseteq N\}$, for some ideal set N.*

PROOF. By 2.7, (i) implies (ii). Trivially, (ii) implies (iii), while (iii) and (iv) are equivalent. Finally, suppose N is an ideal set and $I=\{f: Z(f)\supseteq N\}$. By definition, I is an ideal whose null-set is clearly N. Furthermore, it is a band. Now by 3.3, if $g\in q_m[I]$ then the set $F=N\setminus Z(g)$ is far from X, and so $Z(g)\supseteq N\setminus F$. Consequently $Z(g)\supseteq N$ and hence $g\in I$. In short, I is a q_m-closed band.

Since q_m is finer than t_k, the same can also be said of the locally convex vector topology lq_m. In fact, $lq_m=t_k$ as will gradually be shown, the close connection between this and one other characterization of real-compact spaces being explored later.

First, the kernel of a full seminorm is a q_m-closed band (1.4) and, hence, an ideal. It is reasonable to ask if all q_m-closed ideals can arise in this way, the following lemma leading to the answer: no.

3.5. *Let p be a full seminorm. Then for each f in E, $p(f-f\%re)=0$ for some $r\geqslant 0$.*

PROOF. Take f in E. Because p is full and $m(f-f\%re)=m(f)-m(f)\%re$ if $r\geqslant 0$, it can be assumed that $0\leqslant f$. To save writing, let $g(r)=f-f\%re=\max(f-re,0)$ for all r, so that $g(r)\geqslant g(s)$ if $r\leqslant s$. Then $g(r)\to 0$ in (E,q_m) as $r\to\infty$ (2.4), and consequently $p(g(r))\to 0$ in $\mathbf{R}$.

Suppose the lemma false: then there is a sequence $0<r_1<r_2\ldots<r_n<\ldots\to\infty$ such that $p(g(r_1))>p(g(r_2))>\ldots\to 0$. Now let $h_n=g(r_n)-g(r_{n+1})\neq 0$ and note that

$$(\dagger)\ldots h_n(x)=\begin{cases}0 & \text{if } f(x)\leqslant r_n,\\ f(x)-r_n & \text{if } r_n<f(x)<r_{n+1},\\ r_{n+1}-r_n & \text{if } r_{n+1}\leqslant f(x).\end{cases}$$

Further, $p(h_n)\geqslant p(g(r_n))-p(g(r_{n+1}))>0$. As one can see from (†), for any sequence (s_n) of positive scalars whatsoever, the function $h=\sum s_n h_n$ is well-defined, finite and continuous. In particular, if $s_n=n/p(h_n)$ then $0\leqslant s_n h_n\leqslant h$ and $p(h)\geqslant n$, a clear impossibility.

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3.6. *The kernel of a full seminorm is a t_u-closed ideal, whose null-set is a compact subset of X.*

PROOF. Let p be a full seminorm on E. By 3.4, the null-set N of its kernel is an ideal set. Also, for each f in E there is $r \geqslant 0$ such that $p(f - f\%re) = 0$, that is, $(f - f\%re)^*$ vanishes on N. In other words, $|f^*(x)| \leqslant r$ if $x \in N$. This means that each member of E is bounded on N, forcing N to lie inside X.

It is now easy to see for which r the conclusion of 3.5 holds: namely, $r \geqslant p_N(f) = \max\{|f(x)| : x \in N\}$. To prove this, note that if $r \geqslant p_N(f)$ then $f - f\%re$ vanishes on N, so that $p(f - f\%re) = 0$.

The next result can obviously be applied to more than just q_m. For example, Corollary 3.8 extends Feldman (1974), Proposition 1, which he proved using support sets under slightly more restrictive conditions (but much more quickly).

3.7. THEOREM. *Let q be a mod-convex structure on E. Then lq is coarser than a topology of compact convergence.*

PROOF. Since q is mod-convex and (E, m) has the decomposition property (2.3), lq is the topology generated by all the q-continuous full seminorms (1.5).

So let p be such a seminorm and N the null-set of its kernel. As noted above, if $f \in E$ and $r = p_N(f)$ then

$$p(f) = p(f\%re) \leqslant p(re) = rp(e).$$

In short, p is majorized by a multiple of p_N. Taking $\mathscr{C}$ to be the set of all such N, one can see that the topology $t(\mathscr{C})$ of $\mathscr{C}$-convergence (which is a topology of compact convergence) is finer than lq.

3.8. *The topology of compact convergence is the finest mod-convex vector topology on CX, for real-compact X.*

3.9. *The locally convex vector topology associated with q_m is the topology of compact convergence.*

PROOF. By 3.7, $t_k \geqslant lq_m$. Conversely, as $q_m \geqslant t_k$ and t_k is a locally convex vector topology, $lq_m \geqslant t_k$.

Recently, Gulick and Gulick (1976), p. 262, summarized various characterizations of real-compact spaces starting with Hewitt (1950), Theorem 22. In particular, they observed that X is real-compact if and only if (E, t_k) is the inductive limit (in the category of locally convex topological vector spaces) of the spaces $(E_f, t_f)_{f \geqslant 0}$.

In order to relate this with 3.9, note that the 'locally convex' inductive limit carries the locally convex topology associated with the 'convergence' inductive limit. Thus 3.9 says that (E, t_k) is the locally convex limit of the family (E_f, t_f).

The other half is easier: if X is a completely regular space which is not realcompact, then any point in its real-outgrowth yields a bounded semi-norm on E which is not t_k-continuous, but is lq_m-continuous.

4. Comparisons

The structure q_m lies somewhere between t_u and t_k (3.1): one can narrow its position down a little by comparing it with other structures between t_u and t_k, such as q_c, q_{lu} and q_I, the structure q_D of Dini convergence due to Kutzler (1974), and their order-bounded modifications (§ 1). Moreover by doing so, one characterizes Lindelöf spaces (and others too).

There are two ways of describing q_I, the original one given in Binz (1975), and a more geometrical one in Schroder (1976) which is easier to work with here. Denoted by q_i, the latter is defined briefly below, along with q_{lu} and q_c.

As before, let X be a convergence space and $E = CX$. With the weak topology induced by E, X becomes a completely regular space X' (possibly not Hausdorff), such that $E = CX'$.

Let $\mathscr{A}$ be a collection of subsets of X and $\underline{\mathscr{A}}$ its closure under finite unions. The topology $t(\mathscr{A})$ of uniform $\mathscr{A}$-convergence is translation-invariant, being a mod-convex group topology whose filter of neighbourhoods of 0 is denoted by $\psi(\mathscr{A})$. Also, let $B(\mathscr{A})$ be the set of all $\mathscr{A}$-bounded members of E, where f in E is said to be *$\mathscr{A}$-bounded* if $f(A)$ is bounded in $\mathbf{F}$ for all A in $\mathscr{A}$.

Turning back to X, one says that $\mathscr{A}$ is *w-closed* if all its members are closed in X', and that $\mathscr{A}$ *covers* X if $\mathscr{A}$ meets every X-convergent filter ($\mathscr{A}$ meets a filter $\mathscr{F}$ if $\mathscr{A} \cap \mathscr{F}$ is non-void). For brevity, w-closed covers are called w-covers.

Now one obtains the homogeneous structures q_{lu} and q_i as follows: $\theta \in q_{lu}(0)$ if and only if $\theta \supseteq \psi(\mathscr{A})$ and $\varphi \in q_i(0)$ if and only if $\varphi \supseteq \psi(\mathscr{B}) \cap B(\mathscr{B})$, for some w-covers $\mathscr{A}$ and $\mathscr{B}$. Next, $\rho \in q_c(f)$ if and only if for each x in X, if $\mathscr{F} \to x$ in X then the filter $\rho(\mathscr{F})$ based on sets of the form

$$R(F) = \{g(y) \colon g \in R \text{ and } y \in F\},$$

where $R \in \rho$ and $F \in \mathscr{F}$, converges to $f(x)$ in $\mathbf{F}$. So defined, these are all complete vector structures, q_i being a Marinescu structure besides.

Also, let q_i', q_{lu}' and q_c' be the corresponding structures obtained from X' instead of X. Then $q_i' \geqslant q_i$, and so on. Finally, q_D can be defined using the characterization $q_D = oq_c'$ given by Kutzler (1974).

4.1. *For any space X, $t_u \geqslant q_m \geqslant q_i \geqslant q_{lu} \geqslant q_c \geqslant t_k$.*

The only new link in this chain is the second, which follows from 1.2 and the mod-convexity of q_i. Equality can occur at each link: the last three cases were fully dealt with by Binz (1975), Kutzler (1974) and Schroder (1976), and the first two are discussed below.

4.2. *The space X is pseudo-compact if and only if $t_u = q_m$. In particular, if X is real-compact then it is compact if and only if $t_u = q_m$.*

4.3. *The following statements are equivalent, for each c-embedded space X:*
(i) *X is compact,*
(ii) *$X \in \underline{\mathscr{A}}$, for each w-cover $\mathscr{A}$ of X,*
(iii) *$t_u = t_k$, and*
(iv) *$q_m = q_i$.*

PROOF. A space Y is known to be compact if its underlying set belongs to $\underline{\mathscr{B}}$, for each cover $\mathscr{B}$ of Y. Thus (ii) implies (i), for in a c-embedded space every cover is refined by a w-cover (this may easily be deduced from Schroder (1973), Proposition 3.4, for example). Trivially, (i) implies (iii), and (iii) implies (iv) by 4.1.

Finally, suppose (iv) holds but not (ii). Then there is a w-cover $\mathscr{A}$ of X such that $X \backslash A$ is non-void for all A in $\underline{\mathscr{A}}$. The filter $\rho = \psi(\mathscr{A}) \cap E_e$ is q_i-convergent to 0 by definition, and q_m-convergent to 0 by assumption. Consequently $B_f \in \rho$ for some $f \geqslant e$. Now if $R \in \rho$ then R contains some set P of the form $E_e \cap \{h \in E\colon m(h) \leqslant s \text{ on } A\}$, for some $s > 0$ and A in $\underline{\mathscr{A}}$. Since $A \neq X$, a non-zero bounded function h can be found, vanishing on A. Clearly, all multiples of h belong to P, but not to B_f. This contradiction shows that (iv) implies (ii).

Even though q_m lies next to q_i in 4.1, they are still quite a long way apart: one sign of this is the previous result, that to bring them together one must demand compactness, no less, and another sign is the 'ideal theory' (q_m-closed ideals are not always q_i-closed, see Binz (1975), Theorem 35).

Yet another indication lies in vector duality: for any c-embedded space X, a linear functional on E is q_m-continuous if and only if its support set is compact in the real-compactification X'' of X' (3.9), while Binz (1975), Theorem 37, shows that it is q_i-continuous if and only if its support set is actually compact in X. That theorem states more, namely, that $lq_i = t_k$. This allows one to determine when $lq_m = lq_i$.

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4.4. *For any c-embedded space X, $lq_m = t_k$ if and only if each compact subset of X'' is compact in X.*

Two examples illustrate the difficulties: both are based on the full Tychonov plank. In the notation of Gilman and Jerison (1960), §8.20, they are modifications of T^*, a compact topological space. In the first, X_1, nothing is changed except that a filter may converge to t only if it is based on the right hand (short) edge, while in the second X_2, one allows convergence to t only along the top (long) edge. Both X_1 and X_2 are locally compact c-embedded spaces such that $X_1' = X_2' = T^*$, Schroder (1974).

In particular, $E = CT^* = CX_1 = CX_2$ and, further, each bounded linear functional is continuous on $C_i X_i$, while some (those whose support is the right edge of T^*, for example) are not continuous on $C_i X_2$.

In particular, X_1 and X_2 show that the conditions (i) X and X'' have the same underlying set, (ii) $C_i X$ and $C_m X$ have the same dual, and (iii) $lq_i = lq_m$ are not equivalent.

Having seen the size of the gap between q_m and q_i, to shrink it one is led to the order-bounded structures oq_i, oq_{lu} and oq_c. Again as these are mod-convex vector structures, $q_m \geqslant oq_i \geqslant oq_{lu} \geqslant oq_c$.

4.5. *For any space X, $oq_i = oq_{lu}$.*

Proof. Kutzler (1974), Satz 2.3, proved this for completely regular spaces. As noted above, $oq_i \geqslant oq_{lu}$. Conversely, let $\theta \in oq_{lu}(0)$. Then by definition, $\theta \supseteq \psi(\mathscr{A}) \cap B_g$ for some w-cover $\mathscr{A}$ and some $g \geqslant 0$. Take $[g]$ to be the countable w-cover $\{(g \leqslant n): n \in \mathrm{N}\}$, where $(g \leqslant n) = \{x \in X: |g(x)| \leqslant n\}$. Now let $\mathscr{B}$ be any w-cover refining both $\mathscr{A}$ and $[g]$, noting that $\psi(\mathscr{A}) \supseteq \psi(\mathscr{B})$ and $B_g \subseteq B(\mathscr{B})$. Thus $\theta \supseteq \psi(\mathscr{B}) \cap B(\mathscr{B})$, as desired.

The next two lemmas allow one to attack the equality of oq_i and q_m geometrically. Given a collection $\mathscr{A}$ of subsets of X and function g in E, one says that *g fits* $\mathscr{A}$ if the open cover (g) consisting of the sets $(g < n) = \{x \in X: |g(x)| < n\}$ refines $\underline{\mathscr{A}}$.

4.6. *If $\psi(\mathscr{A}) \cap B_e \in q_m(0)$ and $\mathscr{A}$ is w-closed, then g fits $\mathscr{A}$ for some $g \geqslant e$.*

Proof. The restriction to E_g of the filter $\psi(\mathscr{A}) \cap B_e$ is t_g-convergent to 0 for some g in E: it is assumed without loss of generality that $\inf\{g(x): x \in X\} = 1$. Thus if $n \in \mathrm{N}$ then $(1/n) B_g$ belongs to $\psi(\mathscr{A}) \cap B_e$, meaning that for some $s > 0$ and A in $\mathscr{A}$,

$$(*) \qquad \ldots (1/n) B_g \supseteq B_e \cap \{h: m(h) \leqslant s \text{ on } A\}.$$

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The assumption guarantees that $s \leqslant 1/n$. Also $(g<n) \subseteq A$: for otherwise $g(x)<n$ for some x outside A, yielding h in E vanishing on A, such that $0 \leqslant h \leqslant e$ and $h(x)=1$: thus $h \nleqslant g/n$, contradicting (*).

4.7. *Let g fit $\mathscr{A}$. Then for all f in E, $\psi(\mathscr{A}) \cap B_f \in q_m(0)$.*

PROOF. Take n in $\mathbf{N}$, and choose A in $\underline{\mathscr{A}}$ so that $(g<n) \subseteq A$. Let

$$k=(e+m(f))(e+m(g)) \quad \text{and} \quad P=\{h\colon m(h) \leqslant 1/n \text{ on } A\}.$$

Now take h in $B_f \cap P$. If x is in A then

$$|h(x)| \leqslant 1/n \leqslant k(x)/n,$$

while otherwise $x \notin A$, so that $|g(x)| \geqslant n$ and

$$|h(x)| \leqslant |f(x)| < 1+|f(x)|$$

$$< (1+|f(x)|)(1+|g(x)|)/n = k(x)/n.$$

In short, $m(h) \leqslant k/n$, showing that $P \cap B_f \subseteq (1/n) B_k$.

4.8. THEOREM. *For any c-embedded space X, $q_m = oq_i$ if and only if X is a Lindelöf topological space.*

PROOF. Suppose first that X is a Lindelöf topological space and that $\mathscr{B}$ covers X. By complete regularity, there is a refinement $\mathscr{U}$ of $\mathscr{B}$ consisting of sets of the form $U_x = (h_x < 1)$, where $h_x(x)=0$ and $h_x \in E$ for all x in X. Now put $V_x = (h_x \leqslant \frac{1}{2})$ for all x. The cover $\mathscr{V}$ so obtained refines $\mathscr{B}$ as well: moreover, there is a countable subset of $\mathscr{V}$ which covers X (Lindelöf), yielding an increasing (possibly finite) sequence of zero-sets (V_n) which covers X. Let (U_n) be the corresponding sequence obtained from $\mathscr{U}$. The disjoint zero-sets V_n and $X \backslash U_n$ can be separated by a function $g_n = m(g_n) \leqslant e$, vanishing on V_n and constant 1 on $X \backslash U_n$. The function $g = \sum g_n$ is well-defined, finite and continuous. Moreover, as $(g<n) \subseteq U_n$, g fits $\mathscr{B}$. This being true for every w-cover $\mathscr{A}$ in particular, $\psi(\mathscr{A}) \cap B_f$ belongs to $q_m(0)$ for all f in E (4.7). That is, $oq_{lu} \geqslant q_m$.

Conversely, suppose that $oq_i = q_m$. Then for each w-cover $\mathscr{A}$ of X, $\psi(\mathscr{A}) \cap B_e \in q_m(0)$. So by 4.6, g fits $\mathscr{A}$ for some $g \geqslant e$. In other words, $\underline{\mathscr{A}}$ has a countable refinement consisting of sets open in X'. This means (i) that X is a Lindelöf convergence space, see Binz (1975), § 5.3, and (ii) that every neighbourhood filter in X' converges in X, see Schroder (1973), Theorem 3.6. In short, $X = X'$ is a Lindelöf topological space.

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Lindelöf spaces have been characterized through E in several other ways: Kutzler showed that a completely regular space X is Lindelöf if and only if q_i coincides with another Marinescu structure $q_{I'}$ and Feldman showed that a c-embedded space X is Lindelöf if and only if $C_c X$ is first countable. For details, see Binz (1975), Theorems 84 and 82, and also 5.6.

The next task is to find out when $q_m = oq_c$. The answer to this problem is given by an 'order-bounded' extension of theorems in Binz (1975), Kutzler (1974) and Schroder (1976) dealing with the equality of q_{lu} and q_c.

4.9. THEOREM. *For any space X, the following are equivalent*:

(i) $oq_c \geqslant q_i$,

(ii) $oq_c \geqslant q_{lu}$,

(iii) $oq_c = oq_{lu} = oq_i$,

(iv) $q_c = q_{lu}$, *and*

(v) *the set of all w-covers of X is weakly countably directed.*

PROOF. Condition (v) means by definition that for any sequence $(\mathscr{A}_i)$ of w-covers, one can find a w-cover $\mathscr{A}$ refining each $\mathscr{A}_i$. The implications '(iv)⇒(i)⇒(ii)⇒(iii)' are either trivial or depend on 4.5, while (iv) and (v) are equivalent by Schroder (1976), Theorem 3.4. Finally, the proof given there or in Binz (1975), § 5.4 can be easily adapted to show that (iii) implies (iv).

This raises one obvious question, of the topological meaning of 4.9(v). Answers to this and other questions arising from it are given in the next section.

5. Topology

Condition 4.9(v) is easy to define and easy to work with when one is interested in E rather than X, but what it means for X in topological terms is not so clear. Let X be a completely regular Hausdorff topological space. Binz (1975), Theorem 8.5 (ii), showed that $q_{lu} = q_c$ if and only if the neighbourhood filter of X in the Stone–Čech-compactification X^* of X is closed under countable intersections, a property denoted here by Count($X : X^*$). Someone somewhere (I can neither find nor remember) observed that if X_{nl} is the set of points in X without a compact neighbourhood, then Count($X : X^*$) follows from Count($X_{nl} : X$). More than this is true, as indicated below.

5.1. THEOREM. *Let X be a completely regular Hausdorff topological space. Then the statements below are all equivalent:*

(i) *X satisfies* 4.9(v),

(ii) *X has Count$(X : X^*)$*,

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(iii) *X has* $Count(X_{nl} : X)$, *and*

(iv) *in the Stone-outgrowth* $X^* \setminus X$, *the union of any sequence of compact sets has compact closure.*

The reader should have no difficulty in finding a purely topological proof for himself. More generally, for c-embedded convergence spaces the problem remains. Let X be a c-embedded convergence space. A filter on X is said to be *compact* if it meets the family $\mathscr{K}$ of compact subsets of X (and a point *locally compact* if every filter converging to it is compact). Also, if Ξ is a collection of filters on X and $\mathscr{A}$ a family of subsets of X, one says that $\mathscr{A}$ covers Ξ if all its members meet $\mathscr{A}$. Finally, let Ξ be the set of all non-compact convergent ultra-filters and Λ the set of all non-compact convergent filters.

5.2. THEOREM. *For any c-embedded convergence space X, the following are equivalent:*

(i) *the set of all w-covers of X is weakly countably directed,*

(ii) *the set of all w-covers of* Λ *is weakly countably directed, and*

(iii) *the set of all w-covers of* Ξ *is weakly countably directed.*

Here too the proof is left to the reader: it involves nothing more than set-theory and Schroder (1976), Proposition 1.1. The connection between 5.1 and 5.2 is revealed by noting that if X is completely regular then X_{nl} is the set of points to which the members of Λ and Ξ converge.

Returning now to order-bounded structures on E, one recalls that $oq_i = oq_{lu}$, and that $oq_{lu} = oq_c$ if and only if the c-embedded space satisfies the conditions just discussed. Remembering that a (completely) regular topological space X is Lindelöf if and only if each compact subset of $X^* \setminus X$ is far from X, one can combine 4.8, 4.9 and 5.1 to find out when $oq_c = q_m$.

5.3. THEOREM. *For any c-embedded space X, the following are equivalent:*

(i) $q_m = oq_c$,

(ii) *X is a Lindelöf topological space with* $Count(X_{nl} : X)$, *and*

(iii) *the union of any sequence of compact subsets of* $X^* \setminus X$ *is far from X.*

5.4. THEOREM. *For any c-embedded convergence space, the following are equivalent:*

(i) *X is locally compact,*

(ii) $q_c = t_k$, *and*

(iii) $oq_c = ot_k$.

PROOF. The equivalence of (i) and (ii) was proved in Binz (1975), Theorem 32. In any case, '(i)⇒(ii)⇒(iii)' is a triviality. Finally, to prove that (iii) implies (i), it suffices to show that if X is not locally compact (that is, $\mathscr{K}$ does not cover X)

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then the filter $\sigma = \psi(\mathscr{K}) \cap B_e$ does not oq_c-converge to 0. So, let $\mathscr{F}$ be a non-compact filter converging to x in X. Take S in σ, that is $S \supseteq B_e \cap \{h\colon m(h) \leqslant s \text{ on } K\}$ for some $s > 0$ and some K in $\mathscr{K}$. As before, for each F in $\mathscr{F}$ a point y in $F \setminus K$ can be found: choose h in E, vanishing on K and 1 at x, such that $0 \leqslant h \leqslant 1$. Then h belongs to S, so that $1 \in S(F)$. As a result $\sigma(\mathscr{F}) \not\to 0$ in $\mathbf{F}$.

5.5. THEOREM. *Let X be a c-embedded space. Then*

(i) $q_m = ot_k$ *if X is a Lindelöf locally compact topological space (that is, a locally compact hemi-compact topological space), and*

(ii) $q_m = ot_s$ *(where t_s is the topology of pointwise convergence) if and only if X is a countable discrete space.*

PROOF. Claim (i) follows from 5.3 and 5.4, and claim (ii) from (i) and 4.6 (using in 4.6 the set $\mathscr{P}$ of finite subsets of X).

Now one can quickly sketch the properties of another Marinescu structure q_j in which $\sigma \in q_j(0)$ if and only if $\sigma \supseteq \psi([g]) \cap B([g])$ for some g in E (recall that $[g]$ is the w-cover $\{(g \leqslant n)\colon n \in \mathbf{N}\}$). This structure is the convergence space analogue of $q_{I'}$ described in Binz (1975), p. 119: in fact, if X is completely regular then $q_j = q_{I'}$.

Clearly $q_m \geqslant q_j \geqslant q_i$, while one can use 4.7 to show that $q_m = oq_j$: at last, a structure 'near' q_m! As might be expected from its definition, q_j can also be used to characterize Lindelöf spaces.

5.6. THEOREM. *Let X be a c-embedded space. Then the following are equivalent:*

(i) *X is a Lindelöf topological space,*

(ii) $q_j = q_i$, *and*

(iii) $(q_m =)\, oq_j = oq_i (= oq_{lu})$.

PROOF. Clearly (iii) follows from (ii), and (i) from (iii) by 4.8. So suppose X is Lindelöf and that $\mathscr{A}$ w-covers X. The argument in the proof of 4.8 produces a function g in E such that $[g]$ refines $\mathscr{A}$. Consequently the topology of $\mathscr{A}$-convergence is finer: that is $\psi(\mathscr{A}) \supseteq \psi([g])$ as desired.

This theory is here to be used, first on a rather trivial but necessary example showing that q_m does in fact differ from oq_i sometimes. Let X be a discrete copy of the real line. It is real-compact and locally compact: in fact $q_i = t_s$. But by 5.5(ii), $q_m \neq oq_i = ot_s$, nor are q_m and q_j equal.

Before considering the ideal theory of these order-bounded structures, one should find out when—if ever—they differ from their parents: the reader will easily verify the following facts.

5.7. *The space X is pseudo-compact if and only if $q_m = q_j$.*

PROOF. If X is not pseudo-compact, take an unbounded function g in E and verify that $\psi([g]) \cap B([g])$ contains no B_f, so it cannot q_m-converge to 0. Conversely, if X is pseudo-compact and $g \in E$, then $B([g]) = E$ and $X \in [g]$. Thus $B_e \in \psi([g])$ and $\psi([g])$ is t_u-convergent to 0. (Compare with 4.2.)

5.8. THEOREM. *Let X be a c-embedded space. Then the following are equivalent:*
(i) *X is compact,*
(ii) *the structures q_i, q_{iu}, q_c and t_k are all order-bounded, and*
(iii) *any one of them is order-bounded.*

PROOF. Trivially (ii) implies (iii), while (i) implies (ii) because all four structures coincide with the order-bounded topology t_u if X is compact. Finally by 4.1, to prove that (iii) implies (i) one need only show that X is compact if q_i is order-bounded. This may be done much as in 4.3.

6. More ideals and seminorms

Again let X be a convergence space and $E = CX$. For any ideal I in E, the X-null-set of I is defined to be the set $\{x \in X : f(x) = 0 \text{ for all } f \text{ in } I\}$. It may of course be empty, but whether it is or not, it is closed in X'.

6.1. THEOREM. *Let I be an ideal in E. For any convergence structure q between oq'_i and t_s, the following are equivalent:*
(i) *I is a q-closed ideal,*
(ii) *I is a q-closed band, and*
(iii) *$f \in I$ if and only if f vanishes on the X-null-set of I.*

PROOF. Assume (ii). Then I is an oq'_i-closed band, and so q'_i-closed (2.6). Thus (iii) holds, by Binz (1975), Theorem 15. Also, (iii) implies (i), since any ideal satisfying (iii) is t_s-closed. Finally, suppose (i). Then I is an oq'_i-closed ideal and, hence, a band (2.9). By '(ii)$\Rightarrow$(iii)', I is t_s-closed and so (ii) holds.

Feldman (1974), Theorem 1, proved that if X is real-compact and t is an A-convex vector-lattice topology on E (with $\mathbf{F} = \mathbf{R}$), then every t-closed ideal is full (that is, it satisfies 6.1(iii)). In fact, more is true as one can see from 3.8.

6.2. *If X is real-compact and t is a mod-convex vector topology on E, then every t-closed ideal is full.*

While 6.1 may hold for a still wider range of convergence structures, the range does not include q_j. (Note that like q_m, q_j is an algebraic construct.)

6.3. *The q_j-closed ideals are exactly the t_u-closed ones.*

PROOF. One implication is trivial, so suppose I is t_u-closed. By 3.4 it is a q_m-closed band. Thus as $q_m = oq_j$, it is also q_j-closed (2.6).

6.4. *The locally convex topologies associated with q_m and q_j coincide.*

For any c-embedded space X, if $q_i \geqslant q \geqslant t_k$ then $lq = t_k$ as shown in Binz (1975), Theorem 37. However, Kutzler (1974) showed that this is no longer true for order-bounded structures by proving that the completely regular pseudo-compact space $X = \mathbf{R}^* \setminus (\mathbf{N}^* \setminus \mathbf{N})$ admits a q_D-continuous linear functional on E without compact support in X (in fact, its support is $X^* = \mathbf{R}^*$).

Though the situation is untidy, there are some results and many problems. Let X be a c-embedded space. A subset B of X is said to be *bounded* if $f(B)$ is bounded in $\mathbf{F}$ for all f in E. Clearly the closure in X' of any bounded set is bounded: also the set $\mathscr{B}$ of all bounded subsets of X is closed under finite unions. (One can show that a set is bounded if and only if its closure in the real-compactification X'' of X' is compact.)

6.5. THEOREM. *Let q be a mod-convex vector structure lying between oq_i' and t_s. Then*

$$lq \leqslant t(\mathscr{D}) \text{ for some } \mathscr{D} \subseteq \mathscr{B}.$$

PROOF. Let p be a q-continuous full seminorm. As in 3.7, the null-set N of its kernel is compact in X'', while by 6.1 its X-null-set M is closed in X'. Since $p(f) = 0 \Leftrightarrow f$ vanishes on $N \Leftrightarrow f$ vanishes on M, N is the closure of M in X''. That is, the X-null-set of every full q-continuous seminorm is bounded, and one can take for $\mathscr{D}$ the collection of all these X-null-sets, by 3.7.

To continue, let $\mathscr{C}$ be the X''-nullsets of the q-continuous full semi-norms. Since $q \geqslant t_s$, $X \subseteq X_q = \bigcup \mathscr{C}$. Now if $lq = t(\mathscr{C})$ then each point x of X_q yields a $t(\mathscr{C})$-continuous homomorphism $\hat{x}$ from E to $\mathbf{F}$.

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6.6. *The only points of X'' yielding oq_i'-continuous homomorphisms are those of X.*

PROOF. Let $x \in X'' \setminus X$. For each y in X choose f_y in E such that $f_y(y) = 0$, $f_y(x) = 2$ and $0 \leqslant f_y \leqslant 2e$. Then if $V_y = \{z \in X'' : f_y(z) \leqslant 1\}$ and $W_y = X \cap V_y$, the cover $\mathscr{W}$ of X' so constructed can be used (by the reader) to show that $\hat{x}$ is not oq_i'-continuous, since $\hat{x}(\psi(\mathscr{W}) \cap B_{2e}) \not\to 0$.

What this means is quite simple: if X_q contains X properly then lq is not a topology of uniform convergence. (One can easily prove that if lq is a topology of uniform convergence, it has to be $t(\mathscr{C}) = t(\mathscr{D})$.) Kutzler's example cited earlier shows that this can happen. Moreover, his example is locally compact, so that $oq_i = oq_c = q_D = ot_k$. Consequently one cannot hope to prove even the first case: $lot_k = t_k$.

The well-known characterization given by Nachbin (1954) of those completely regular spaces X for which $C_k X$ is barrelled is just as true for c-embedded X. In the language used here, it states that $C_k X$ is barrelled if and only if every w-closed bounded subset of X is compact. Kutzler (1974) proved that if this is so, then $lq_D = t_k$.

6.7. THEOREM. *Let X be a c-embedded space. Then the following are equivalent:*
(i) *$C_k X$ is barrelled,*
(ii) *$\mathscr{B} = \mathscr{K}$, the family of compacta in X, and*
(iii) *$t(\mathscr{B}) = loq_i$, in which case all of $t(\mathscr{B})$, loq_i, loq_c and lot_k coincide with t_k.*

PROOF. Nachbin's theorem shows the equivalence of (i) and (ii). Suppose (i) holds: then $t(\mathscr{B}) \geqslant loq_i$ (6.5), while $loq_i \geqslant t_k = t(\mathscr{B})$. Next, assume (iii) and take H in $\mathscr{B}$. Then the seminorm p_H is oq_i-continuous, so that if $\mathscr{A}$ w-covers X, then $p_H(\psi(\mathscr{A}) \cap B_e) \to 0$ in $\mathbf{R}$. It is now easy to verify that $H \subseteq A$, for some A in $\underline{\mathscr{A}}$. As X is c-embedded and each cover can thus be refined by a w-cover, this shows that H is compact in X.

Finally, since $t(\mathscr{B}) \geqslant loq_i = loq_{lu} \geqslant loq_c \geqslant lot_k \geqslant t_k = t(\mathscr{K})$, all these topologies coincide if $\mathscr{B} = \mathscr{K}$.

To end with some of the problems: when are (i) loq_i and loq_c, (ii) loq_c and lot_k, and (iii) lot_s and t_s equal? Under what conditions are they topologies of uniform convergence, and when they are not, what are they?

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