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Restricted Khinchine Inequality

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Abstract. We prove a Khintchine type inequality under the assumption that the sum of Rademacher random variables equals zero. We also show a new tail-bound for a hypergeometric random variable.

1 Introduction

The Khinchine inequality plays a crucial role in many deep results of probability and analysis (see [6, 9, 10, 12, 15, 19] among others). It says that L_p and L_2 norms of sums of weighted independent Rademacher random variables are comparable. More precisely, we say that ε_0 is a Rademacher random variable if $\mathbb{P}(\varepsilon_0 = 1) = \mathbb{P}(\varepsilon_0 = -1) = \frac{1}{2}$. Let ε_i, $i \le N$, be independent copies of ε_0 and $a \in \mathbb{R}^N$. The Khinchine inequality (see *e.g.*, [10, Theorem 2.b.3] or [6, Theorem 12.3.1]) states that for any $p \ge 2$ one has

$$\Big(\mathbb{E}\Big|\sum_{i=1}^{N} a_i\varepsilon_i\Big|^p\Big)^{\frac{1}{p}} \le \sqrt{p}\,\|a\|_2 = \sqrt{p}\Big(\mathbb{E}\Big|\sum_{i=1}^{N} a_i\varepsilon_i\Big|^2\Big)^{\frac{1}{2}}. \tag{1.1}$$

Note that the (Rademacher) random vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ in the Khinchine inequality has independent coordinates. However, in many problems of analysis and probability it is important to consider random vectors with dependent coordinates, *e.g.*, so-called log-concave random vectors, which in general have dependent coordinates, but whose behaviour is similar to that of the Rademacher random vector or some Gaussian random vector (see *e.g.*, [7] and references therein). In [13] the author considered random matrices, whose rows are independent random vectors satisfying certain conditions (so the vectors may have dependent coordinates). He studied limiting empirical distribution of eigenvalues of such matrices. As an example of such a vector, showing that the conditions cover large class of natural distributions, not covered by previously known results, O'Rourke considered the vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$, whose coordinates are Rademacher random variables under the additional condition

$$S = \sum_{i=1}^{N} \varepsilon_i = 0, \quad \text{where } N \text{ is even}, \tag{1.2}$$

(see [13, Examples 1.4 and 1.10]). For such vectors he proved a Khinchine type inequality with the factor $C\sqrt{N}p/\log N$ in front of $\|a\|_2$, which was enough for his

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purposes. The goal of this paper is to show that such random variables satisfy a Khinchine type inequality with the same factor $\sqrt{p}$ as in the standard Khinchine inequality. To shorten notation, we denote by $\mathbb{E}_S$ the conditional expectation given the event (1.2). Note that the corresponding probability space is

$$\Omega = \Big\{ \varepsilon \in \{-1,1\}^N \;\Big|\; \sum_{i=1}^N \varepsilon_i = 0 \Big\} = \Big\{ \varepsilon \in \{-1,1\}^N \mid \operatorname{card}\{ i : \varepsilon_i = 1\} = n \Big\}. \tag{1.3}$$

Our main result is the following theorem.

Theorem 1.1 *Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$, be a vector whose coordinates are Rademacher random variables under the condition (1.2). Let $a = (a_1, \dots, a_N) \in \mathbb{R}^N$ and $b = \frac{1}{N}\sum_{i=1}^N a_i$. Then*

$$\Big(\mathbb{E}_S \Big| \sum_{i=1}^N a_i \varepsilon_i \Big|^p \Big)^{1/p} \le \sqrt{2p}\, \big(\|a\|_2^2 - N b^2 \big)^{1/2} \le \sqrt{2p}\, \Big(\mathbb{E}_S \Big| \sum_{i=1}^N a_i \varepsilon_i \Big|^2 \Big)^{1/2}. \tag{1.4}$$

The first step in the proof is a reformulation in terms of random variables on the permutation group as follows. Let $N = 2n$. For the set Ω defined in (1.3), we put into correspondence the group Π_N of all permutations of the set $\{1, \dots, N\}$ as

$$\sigma \in \Pi_N \longleftrightarrow A_\sigma = \big\{ \varepsilon \in \Omega \mid \varepsilon_i = 1 \text{ if } \sigma(i) \le n;\, \varepsilon_i = -1 \text{ if } \sigma(i) > n \big\}.$$

Given $a \in \mathbb{R}^N$, define $f_a \colon \Pi_N \to \mathbb{R}$ by

$$f_a(\sigma) := \sum_{i=1}^n a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)}. \tag{1.5}$$

By $\mathbb{E}_\Pi$ we denote the average over Π_N, *i.e.*, the expectation with respect to the normalized counting measure on Π_N. Note that $\mathbb{E}_S |\sum_{i=1}^N a_i \varepsilon_i|^p = \mathbb{E}_\Pi |f_a|^p$. Therefore, Theorem 1.1 is equivalent to the following theorem.

Theorem 1.2 *Let $N = 2n$, $a \in \mathbb{R}^N$. Let f_a be the function defined in (1.5). Let $b = \frac{1}{N}\sum_{i=1}^N a_i$. Then for $p \ge 2$,*

$$\big(\mathbb{E}_\Pi |f_a|^p \big)^{1/p} \le \sqrt{2p} \Big(\sum_{i=1}^N a_i^2 - N b^2 \Big)^{1/2} \le \sqrt{2p} \big(\mathbb{E}_\Pi |f_a|^2 \big)^{1/2}.$$

In Section 2 we prove Theorem 1.2. Then, in Section 3, we consider a special case of our problem, when the coordinates of the vector a are either ones or zeros. This particular case leads to the hypergeometric distribution. We obtain new bounds for the p-th central moments of such variables.

In the last section we discuss the behaviour of moments of the following function defined on the group of permutations endowed with normalized counting measure

$$f(\sigma) = \Big| \sum_{i=1}^N a_{\sigma(i)} b_i \Big|.$$

Note that the case $b_i = \pm 1$, together with $\sum_{i=1}^N b_i = 0$, corresponds to the settings of Theorem 1.2.

2 Proof of Theorem 1.2

We now compute

$$\mathbb{E}_\Pi |f_a|^2 = \mathbb{E}\Big|\sum_{i=1}^{n} a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)}\Big|^2.$$

Expanding the square and noticing that for every i, and every $k \neq i$, expectations over all permutations respectively are

$$\mathbb{E}(a_{\sigma(i)}^2) = \frac{\|a\|_2^2}{2n} \quad \text{and} \quad \mathbb{E}(a_{\sigma(i)}a_{\sigma(k)}) = \frac{(\sum_{i=1}^{2n} a_i)^2 - \|a\|_2^2}{2n(2n-1)},$$

we get that

$$\mathbb{E}_\Pi |f_a|^2 = \frac{N\|a\|_2^2 - (\sum_{i=1}^{N} a_i)^2}{(N-1)}.$$

Thus, without loss of generality we may assume that $\sum_{i=1}^{N} a_i = 0$.

For $k \leq n$, let

$$b_{k,\sigma} := a_{\sigma(k)} - a_{\sigma(n+k)} \quad \text{and} \quad H_{k,\sigma} := \sum_{i=k+1}^{n} a_{\sigma(i)} - \sum_{i=n+k+1}^{2n} a_{\sigma(i)}$$

(with $H_{n,\sigma} = 0$). Clearly,

$$\sum_{i=1}^{n} a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)} = b_{1,\sigma} + H_{1,\sigma} = b_{1,\sigma} + b_{2,\sigma} + H_{2,\sigma} = \cdots = \sum_{i=1}^{n} b_{i,\sigma}.$$

Note that $\mathbb{E}_\Pi |b_{1,\sigma} + H_{1,\sigma}|^p = \mathbb{E}_\Pi |-b_{1,\sigma} + H_{1,\sigma}|^p$. Hence,

$$\mathbb{E}_\Pi |f_a(\sigma)|^p = \mathbb{E}_\Pi \Big|\sum_{i=1}^{n} a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)}\Big|^p = \frac{\mathbb{E}_\Pi |b_{1,\sigma} + H_{1,\sigma}|^p + \mathbb{E}_\Pi |-b_{1,\sigma} + H_{1,\sigma}|^p}{2}.$$

Denoting by δ_i, $i \leq n$, i.i.d. Rademacher random variables independent of $\varepsilon_1, \ldots, \varepsilon_N$, and using the Khinchine inequality (1.1), we obtain

$$\begin{aligned}
\mathbb{E}_\Pi |f_a(\sigma)|^p &= \mathbb{E}_\Pi \mathbb{E}_{\delta_1} |\delta_1 b_{1,\sigma} + H_{1,\sigma}|^p \\
&= \mathbb{E}_\Pi \mathbb{E}_{\delta_1} \mathbb{E}_{\delta_2} |\delta_1 b_{1,\sigma} + \delta_2 b_{2,\sigma} + H_{2,\sigma}|^p = \cdots \\
&= \mathbb{E}_\Pi \mathbb{E}_{\delta_1} \mathbb{E}_{\delta_2} \cdots \mathbb{E}_{\delta_n} \Big|\sum_{i=1}^{n} \delta_i b_{i,\sigma}\Big|^p \\
&\leq \mathbb{E}_\Pi \Big[\sqrt{p}\Big(\sum_{i=1}^{n} b_{i,\sigma}^2\Big)^{1/2}\Big]^p = p^{p/2}\, \mathbb{E}_\Pi \Big(\sum_{i=1}^{n} |a_{\sigma(i)} - a_{\sigma(i+n)}|^2\Big)^{p/2} \\
&\leq p^{p/2}\, \mathbb{E}_\Pi \Big(2\sum_{i=1}^{n} (a_{\sigma(i)}^2 + a_{\sigma(i+n)}^2)\Big)^{p/2} = (2p)^{p/2} \|a\|_2^p,
\end{aligned}$$

which completes the proof. ∎

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3 Hypergeometric Distribution

In this section we discuss a specific case of hypergeometric distribution and show how it is related to our problem. Recall that a hypergeometric random variable with parameters (N, n, ℓ) is a random variable ξ that takes values $k = 0, \ldots, \ell$ with probability

$$p_k = \frac{\binom{\ell}{k}\binom{N-\ell}{n-k}}{\binom{N}{n}}.$$

In this section we consider only the case $N = 2n$, $\ell \le n$. It is well known that $\mathbb{E}\xi = \ell/2$. In the next proposition we estimate the central moment of ξ.

Proposition 3.1 *Let $1 \le \ell \le n$. Let ξ be $(2n, n, \ell)$ hypergeometric random variable. Then for $p \ge 2$ one has*

$$\mathbb{E}\,|\xi - \mathbb{E}\,\xi|^p \le \sqrt{2}\Big(\frac{p\,\ell}{4}\Big)^{\frac{p}{2}} = C_1^p\Big(\frac{p\,\ell}{4}\Big)^{\frac{p}{2}}.$$

Remark 3.2 It is well known that the conclusion of Proposition 3.1 is equivalent to the following, so-called ψ_2 deviation inequality: there are $C_2, C_2' > 0$, such that for all $t \ge C_2'$,

$$\mathbb{P}(|\xi - \mathbb{E}\,\xi| > t) \le \exp\Big(\frac{-t^2}{C_2^2\ell}\Big).$$

Relationships between C_1, C_2, and C_2' can be found, for example, in [5, Theorem 1.1.5]. This estimate, for hypergeometric ξ, is of independent interest; in particular, it is better than the previously observed bound $\exp(-2t^2/n)$ when $\ell \ll n$ (see [8, Section 6.5] and [17, formulas (10), (14)]).

Remark 3.3 One can use Theorem 1.2 to estimate $\mathbb{E}_S|\sum_{i=1}^{2n} a_i\varepsilon_i|^p$ in the case where the vector a has 0/1 coordinates with ℓ ones. Indeed, without loss of generality assume that $a_1 = a_2 = \cdots = a_\ell = 1$ and $a_{\ell+1} = a_{\ell+2} = \cdots = a_{2n} = 0$. Then $\sum_{i=1}^{2n} a_i\varepsilon_i = \sum_{i=1}^{\ell}\varepsilon_i$. Theorem 1.2 implies the following estimate.

Corollary 3.4 *Let $a \in \mathbb{R}^N$, $N = 2n$, be a vector with ℓ coordinates equal to one and $N - \ell$ zero coordinates. Then, for $p \ge 2$,*

$$\mathbb{E}_S\Big|\sum_{i=1}^{N} a_i\varepsilon_i\Big|^p \le (2\,p\,\ell)^{p/2}.$$

Proof of Proposition 3.1 Denote $X := \sum_{i=1}^{2n} a_i\varepsilon_i = \sum_{i=1}^{\ell} a_i\varepsilon_i$. Since the vector a has 0/1 coordinates with ℓ ones, $\|a\|_2 = \sqrt{\ell}$. For every $0 \le k \le \ell$ we compute the probability q_k that exactly k of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell$ equals one (in that case $X = 2k - \ell$). Since $S = \sum_{i=1}^{2n}\varepsilon_i = 0$, in order to get k ones, we have to choose k ones out of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell$ and $n - k$ ones out of $\varepsilon_{\ell+1}, \varepsilon_{\ell+2}, \ldots, \varepsilon_{2n}$. This gives us $\binom{\ell}{k}\binom{2n-\ell}{n-k}$ choices. Since

$$|\Omega| = \Big|\{\varepsilon \in \{-1,1\}^{2n} \mid \sum_{i=1}^{2n}\varepsilon_i = 0\}\Big| = \binom{2n}{n},$$

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we obtain that $q_k = p_k$, *i.e.*, $X = 2(\xi - \mathbb{E}\,\xi)$, where ξ has hypergeometric distribution with parameters $(2n, n, \ell)$. Therefore, Corollary 3.4 implies

$$\left(\mathbb{E}|\xi - \mathbb{E}\xi|^p\right)^{1/p} \le \sqrt{2\,p\,\ell}. \qquad \blacksquare$$

We would also like to note that Proposition 3.1 can be proved directly. Below we provide such a direct proof, which gives 2 in place of $\sqrt{2}$ in front of $(\frac{p\ell}{4})^{p/2}$. This proof is of interest as it can be extended to a slightly more general situation (see Remark 3.5) and can be used in another approach to the main problem (see Remark 4.3).

Direct proof of Proposition 3.1 From Stirling's formula together with the observation that $\sqrt{\pi n}\binom{2n}{n}/4^n$ increases, we observe that

$$\frac{2^{2n}}{\sqrt{2\pi n}} \le \binom{2n}{n} \le \frac{2^{2n}}{\sqrt{\pi n}}.$$

Using this, we obtain

$$\frac{\binom{2n-\ell}{n-k}}{\binom{2n}{n}} \le \frac{\binom{2n-\ell}{n-\lfloor \frac{\ell}{2}\rfloor}}{\binom{2n}{n}} \le \frac{2^{2n-\ell}}{\sqrt{\pi(n-\lfloor\frac{\ell}{2}\rfloor)}}\frac{\sqrt{2\pi n}}{2^{2n}} \le \frac{2}{2^\ell} \le 1.$$

Therefore

$$\mathbb{E}\,|\xi - \mathbb{E}\,\xi|^p = \frac{1}{2^p}\sum_{k=0}^{\ell}|2k-\ell|^p\frac{\binom{\ell}{k}\binom{2n-\ell}{n-k}}{\binom{2n}{n}} \le \frac{2}{2^{\ell+p}}\sum_{k=0}^{\ell}|2k-\ell|^p\binom{\ell}{k} = \frac{2}{2^p}\mathbb{E}|S_\ell|^p,$$

where S_ℓ is a sum of ℓ i.i.d. Rademacher random variables. By the Khinchine inequality (1.1), we have

$$\left(\mathbb{E}|S_\ell|^p\right)^{1/p} \le \sqrt{p}\,\sqrt{\ell}.$$

Thus,

$$\mathbb{E}\,|\xi - \mathbb{E}\,\xi|^p \le 2\Big(\frac{p\,\ell}{4}\Big)^{p/2}. \qquad \blacksquare$$

Remark 3.5 The above proof can be extended to a slightly larger class of hypergeometric random variables. Note that the proof works whenever $\binom{N-\ell}{n-k}/\binom{N}{n} \le 1$. Thus, if $\ell \ge N - \log_2[\sqrt{\pi}\binom{N}{n}]$, then

$$\mathbb{E}|\xi - \mathbb{E}\,\xi|^p \le 2(p\,\ell/4)^{\frac{p}{2}}$$

for a (N, n, ℓ) hypergeometric random variable ξ.

4 Concluding Remarks

In this section we would like to prove Theorem 1.2 in a more general context; namely, we study behaviour of moments of $f(\sigma) = |\sum_{i=1}^N a_{\sigma(i)}b_i|$, where σ is permutation function. A possible approach to this problem is to use the concentration on the group Π_N (endowed with the distance $d_N(\sigma,\pi) = |\{i : \sigma(i) \ne \pi(i)\}|$). The following theorem is proved by Maurey ([11], see also [16]).

Theorem 4.1 *Let $f\colon \Pi_N \to \mathbb{R}$ be a 1-Lipschitz function. Then for all $t > 0$*

$$\mu\big(\{\sigma : |f(\sigma) - \mathbb{E}f| \ge t\}\big) \le 2e^{-t^2/(32N)}.$$

Let us mention here the following open question posed by G. Schechtman in [16]:

Is there an equivalent (with constants independent of N) metric on Π_N for which the isoperimetric problem can be solved?

Theorem 4.1 implies the following estimate.

Corollary 4.2 *Let $a, b \in \mathbb{R}^N$. Let $f\colon \Pi_N \to \mathbb{R}$ be defined by*

$$f(\sigma) := \Big|\sum_{i=1}^{N} a_{\sigma(i)} b_i\Big|.$$

Then

$$\big(\mathbb{E}|f|^p\big)^{1/p} \le \mathbb{E}|f| + 4\sqrt{p}\sqrt{N}\|a\|_\infty \|b\|_\infty.$$

Proof It is easy to see that f is a Lipschitz function with Lipschitz constant $2\|a\|_\infty \|b\|_\infty$, indeed,

$$\begin{aligned} |f(\sigma) - f(\pi)| &\le \Big|\sum_{i=1}^{N} a_{\sigma(i)} b_i - \sum_{i=1}^{N} a_{\pi(i)} b_i\Big| \\ &\le \sum_{i=1}^{N} |b_i||a_{\sigma(i)} - a_{\pi(i)}| \le 2\|a\|_\infty \|b\|_\infty d_N(\sigma, \pi). \end{aligned}$$

Using Theorem 4.1 and the bound $\Gamma(x) \le x^{x-1}$ for all $x \ge 1$ (see, for example, [4]), we obtain

$$\begin{aligned} \mathbb{E}|f - \mathbb{E}f|^p = \int_0^\infty \mu_N(|f - \mathbb{E}f|^p \ge t^p) dt^p &\le 2p \int_0^\infty e^{-t^2/(32N\|a\|_\infty^2 \|b\|_\infty^2)} t^{p-1} dt \\ &\le 4^p\, \Gamma\Big(\frac{p}{2}\Big) N^{p/2} \|a\|_\infty^p \|b\|_\infty^p \\ &\le 4^p\, N^{p/2} p^{p/2} \|a\|_\infty^p \|b\|_\infty^p. \end{aligned}$$

Thus,

$$\big(\mathbb{E}|f|^p\big)^{1/p} \le \mathbb{E}|f| + 4\sqrt{p}\sqrt{N}\|a\|_\infty \|b\|_\infty \le \sqrt{\mathbb{E}|f|^2} + 4\sqrt{p}\sqrt{N}\|a\|_\infty \|b\|_\infty. \quad \blacksquare$$

Remark 4.3 In the case where $b_i = \pm 1$ with condition $\sum_{i=1}^N b_i = 0$, Corollary 4.2 gives an additional factor $\sqrt{N}$ in the upper estimate in (1.4). Using the chaining argument similar to the one used in [1–3] and Proposition 3.1, the factor $\sqrt{N}$ can be reduced to $\sqrt{\ln N}$ (the details are provided in [18]).

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