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HYPERNORMALIZING GROUPS

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Abstract

All subnormal subgroups of hypernormalizing groups have by definition subnormal normalizers. It is shown that finite soluble HN-groups belong to the class of groups of Fitting length three. Finite HN-groups are considered including those with subnormal quotient isomorphic to $SL(2,5)$.

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A. Camina [1, 2, 3] considered the class of groups satisfying the following condition: normalizers of subnormal subgroups of G are subnormal in G. We adapt his notation and call these groups HN-groups.

We will consider finite HN-groups here, using Camina's results as a foundation. The consideration of join-irreducible subnormal subgroups of soluble HN-groups leads to a bound to the Fitting length. More precisely, if $\mathfrak{A}$, $\mathfrak{N}_2$, $\mathfrak{R}$, $\mathfrak{N}$ denote the classes of abelian, nilpotent of class 2, nilpotent of square-free exponent, nilpotent, groups respectively and we use P. Hall's product notation for group classes, we can show

MAIN THEOREM. *If G is a soluble* HN-*group, then*

$$G \in \mathfrak{N}\mathfrak{N}\mathfrak{R} \cap \mathfrak{N}\mathfrak{N}_2\mathfrak{N} \cap \mathfrak{N}\mathfrak{A}\mathfrak{N}\mathfrak{A} \cap \mathfrak{A}\mathfrak{N}\mathfrak{A}\mathfrak{N}.$$

The second part of the paper deals with the role that quotients isomorphic to $SL(2,5)$ play in HN-groups. We see in Section 4 that the quotient $SL(2,5)$

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may lead to perfect subnormal subgroups T in a HN-group such that $T/Z(T)$ is no longer the direct product of simple groups (Theorem 4.1); in this case, however, the HN-group G can be described as a subdirect product of two HN-groups in which one factor does not contain a subnormal subgroup as described in Theorem 4.1, while the commutator subgroup of the other factor is a direct product of such groups (Corollary 4.4). An example at the end shows that the Fitting length of the soluble quotient of a directly irreducible nonsoluble HN-group is not more reduced than the Main Theorem indicates.

1. Subnormal hulls of primary elements

In this section we will consider finite soluble HN-groups. If G is such a group, then every subnormal subgroup of it is again a soluble HN-group. We apply this to smallest subnormal subgroups containing a given element x of prime power order. The more complete information we obtain for involutions s is needed later; we treat this case separately.

LEMMA 1.1. *If x is an element of order 2 of a soluble* HN*-group, the smallest subnormal subgroup of G containing x is always metabelian.*

PROOF. If V is the smallest subnormal subgroup of G that contains x, then V/V' is cyclic of order 2.

We choose a chief factor R/S of V. There is a cyclic xS-invariant subgroup $T/S \neq 1$ of R/S in V/S, and since V/S is a HN-group, T/S is normal in V/S. But R/S is a chief factor and so $T = R$ and all chief factors of V are cyclic. This shows that V is supersoluble. Therefore V' is nilpotent, and if $V' \neq 1$, the construction of V yields that V' is the Fitting subgroup of V. We apply Theorem 6 of Camina [2] and obtain that $V/Z(V')$ is nilpotent. Since V is defined as the smallest subnormal subgroup containing the element x, the commutator subgroup V' is the intersection of all normal subgroups K of V with nilpotent quotient group V/K. So $Z(V') = V'$ and V is metabelian.

The general case is much more complicated.

LEMMA 1.2. *Assume that G is a finite soluble* HN*-group possessing an element x of order a prime power p^n such that x is not contained in any proper subnormal subgroup of G. If $M = G'G^p$ and H/K is a chief factor of G, then $(M')^2[M', G]K/K$ is contained in $C_{G/K}(H/K)$.*

PROOF. By Camina [1, Proposition 2], G is of p-length 1. So the order of G' is prime to p and we have $C = \langle x, G' \rangle$ and $G' \cap \langle x \rangle = 1$.

We consider a chief factor H/K such that $(G/K)/C_{G/K}(H/K)$ is non-abelian. For brevity we denote by U^* the subgroup UK/K of $G^* = G/K$. We distinguish several cases.

CASE I. The rank of H^* is not divisible by p. Now H^* is a minimal normal subgroup of the HN-group G^* possessing an element z ($= xK$) of order a prime power p^r which is not contained in any proper subnormal subgroup of G^*. If T is a z-invariant subgroup of H^*, then z is contained in the normalizer of the subnormal subgroup T of G^*, and so T is normal in G^*. So there are no proper z-invariant subgroups of H^*, and we have

(I,1) the rank of H^* is a divisor of $p - 1$.

Consider now an abelian normal subgroup $A/C(H^*)$ of $G^*/C(H^*)$. If $A/C(H^*)$ is noncyclic, then H^* splits into proper A-invariant subgroups R_i such that $A/C(R_i)$ is cyclic for every i. We collect all elements in H^* with the same centralizer, and in this way we obtain a description of H^* as a direct product of homogeneous components RF_i. Conjugation by z permutes these components, and therefore their number must be divisible by p, while the rank of H^* is smaller than p. This is a contradiction, and we obtain

(I,2) abelian normal subgroups of $G^*/C(H^*)$ are cyclic.

Assume now $A = \langle a, C(H^*) \rangle$. Conjugation by a induces in H^* a linear mapping. If this linear mapping has a minimal polynomial which is not irreducible, then H^* is the direct product of some homogeneous components which again are permuted by conjugation with z, a contradiction.

We have derived that the minimal polynomial of the liner mapping induced by a is irreducible. Assume now that $aC(h^*)$ is different from $z^{-1}azC(H^*)$. Then $z^{1}azC(H^*) = a^k C(H^*)$ for some k, and a and a^k induce liner mappings with the same irreducible minimal polynomial on H^*. Since z is of order a power of p, this minimal polynomial is of degree a multiple of p and so the rank of H^* is a multiple of p, contradicting (I,1). We derive

(I,3) abelian normal subgroups of $G^*/C(H^*)$ belong to the centre of $G^*/C(H^*)$.

If B is the Fitting subgroup of $G^*/C(H^*)$, then the subgroup $Z(B')$ is obviously an abelian normal subgroup of $C^*/C(H^*)$. Now (I,3) yields

(I,4) $Z_2(B) \cap B' \subseteq Z(B') \subseteq Z(B) \cap B'$, $B' \subseteq Z(B)$ and $B_3 = 1$. The Fitting subgroup of $G^*/C(H^*)$ is of class two at most.

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We consider now a nonabelian q-Sylow subgroup S of the Fitting subgroup B of $G^*/C(H^*)$. Since H^* is a minimal normal subgroup of G^*, we have by Schur's Lemma that $Z(G^*/C(H^*))$ is cyclic and so its subgroup $Z(S)$ is also cyclic. If the order of $S' \subseteq Z(S)$ is equal to q^s, then S^{q^k} is abelian for all k satisfying $2k \geq s$. Using (I,3) again we obtain $s = 1$.

Choose a maximal abelian normal subgroup T of S. Except for $S \cong Q_8$ we may choose a noncyclic normal subgroup T, and H^* will split into homogeneous T-invariant components. Their centralizers in T are permuted by conjugation with elements of S which are not in T. The rank of H^* must therefore be multiple of $|S/T| = q^m$, and we have furthermore $|S/Z(S)| = q^{2m}$.

Now y permutes all $q^m + 1$ maximal abelian normal subgroups of S by conjugation, leaving none fixed. So p divides $q^m + 1$. These two numerical statements lead to $q^m \leq p-1$ and $p \leq q^m+1$, leaving equality in both cases as the only possibility. So $q = 2$ and p is a Fermat prime $2^m + 1$. In particular, $|S'| = 2$ and H^* is not a 2-group. The case of $S \cong Q_8$ is similar to $q = 3$.

The Fitting subgroup B of $G^*/C(H^*)$ is now the direct product of a cyclic group of odd order contained in $Z(G^*/C(H^*))$ and the group S just described. Now $B/B' = S/S'$ is a chief factor of $G^*/C(H^*)$; it is a factor of order a power of 2 and of rank smaller than p. Using the previous deduction, we obtain that $((G^*/C(H^*))/B')/C(B/B')$ cannot be nonabelian, as such a case occurs only for chief factors of odd order. Since B is the Fitting subgroup of $G^*/C(H^*)$, $C(B/B') = B/B'$, and $G^*/C(H^*)$ is the extension of the Fitting subgroup B by an abelian group (generated by $zC(H^*)$). Thus the statement of Lemma 1 is proved for Case I.

CASE II. The rank of H^* is not 2 and is divisible by p. We can see at once that every element of H^* is contained in a proper y^p-invariant subgroup of H^* and obtain

$$y^p C(H^*) \in Z(G^*/C(H^*)). \tag{II,1}$$

Assume that p^m is the order of $yC(H^*)$ in $G^*/C(H^*)$, and $(H^*)^t = 1$ for some prime t. If K is the splitting field of $s^{p^{m-1}} - 1$ over the field F of order t, we have that H^* can be considered as a vector space of dimension p over K. Assume for the moment that the Fitting subgroup B of $G^*/C(H^*)$ is nonabelian. Since B is a normal subgroup of $G^*/C(H^*)$ which contains $yC(H^*)$, the minimal normal subgroup H^* will split into the direct product of K-subspaces of equal dimension. These spaces cannot be of K-dimension 1 since B is nonabelian. So H^* does not split nontrivially as a B-module. Now every nonabelian Sylow subgroup S of B is or order prime to p. Choose some maximal abelian subgroup T of S and some element a contained in $N(T) \cap S$ not in T. Now $\langle a, T\rangle$ is subnormal in S, and by induction on the

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defect we deduce that H^* is an irreducible $\langle a, T\rangle$-module, since the number of constituents is at the same time p or 1 and has a common divisor with $|S|$. Since T is abelian and a operates nontrivially on t, H^* can no longer be an irreducible T-module, leading to a contradiction which proves

(II,2) the Fitting subgroup of $G^*/C(H^*)$ is abelian.

If U/V is any chief factor of $G^*/C(H^*)$, $y^pC(H^*)$ induces the identity on U/V by conjugation. So the rank of U/V divides $p-1$. Assume now that $p \neq 2$ and that there is a 2-group in $(G^*/C(H^*))/V$ which is not contained in $C(U/V)$. The 2-Sylow subgroup A of $G/C(H^*)$ operates irreducibly only on subgroups of 2-power rank of H^*. The normalizer of such a proper subgroup of H^* will lead to a contradiction to G being a HN-group. This shows from Case I for U/V, that $C(U/V) \supseteq ((G^*/C(H^*))/V)'$ and, in particular, that

$$(G^*/C(H^*))/C(B) \text{ is abelian.}$$

We deduce

(II,3) $G^*/C(H^*)$ is metabelian.

This shows slightly more than wanted in the statement of the lemma.

Case III. The rank of H^* is 2, and $p = 2$. If there are no proper z^2-invariant subgroups of H^*, then $G^*/C(H^*) = \langle zC(H^*)\rangle$ and nothing is to be shown. If there are proper z^2-invariant subgroups of H^*, the argument follows along the lines of Case II. Lemma 1 is shown.

Remark. The two cases mentioned in the proof of Lemma 1.2 lead to examples of HN-groups. For Case II is Camina's Example 2; see [3, page 63]. For Case I, matters are slightly more involved. We begin with a Fermat prime $p = 2^m + 1$ and another prime r, where r is not a square modulo p and is of the form $4t + 1$. We choose a 2-group T such that $Z(T) = T'$ is of order 2, T itself is of order 2^{2m+1} and admits an automorphism of order p. Such groups T exist; see Hall and Higman [5, page 33]. There is, up to operator isomorphism, only one faithful irreducible $\mathbf{Z}_r$ module of T (cf. [5, page 17]), and the group ring $\mathbf{Z}_r[T]$ is the direct sum $A \oplus B$, where A is the direct sum of w^{2m} fields of order r, while B is the full matrix ring of $2^m \times 2^m$ matrices over $\mathbf{Z}_r$. We extend B by T in the obvious manner and extend BT by a group P of automorphisms of T which is of order p. Assume that D is a minimal T-invariant subgroup of B and that D is also invariant under P. Then DTP is an illustration of our Case I. We have to show that such a minimal T-invariant subgroup D of B exists.

Assume, to the contrary, that all minimal T-invariant subgroups of B are moved by the nonidentity elements of P. Choose such a minimal T-invariant

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subgroup U of B, an element $x \neq 1$ of P and an element y of order 4 of T. We have

$$uu^x \cdots u^{x^{p-1}} = 1 \quad \text{for all } u \text{ in } U$$

and

$$B = U^x \times U^{x^2} \times \cdots \times U^{x^{p-1}}.$$

Now from

$$u^y u^{yx} \cdots u^{yx^{p-1}} = 1 \quad \text{and} \quad (uu^x \cdots u^{x^{p-1}})^y - 1$$

we obtain

$$\prod_{i=1}^{p-1} u^{x^i y - yx^i} = 1$$

where by construction $u^{x^i y - yx^i}$ belongs to U^{x^i}, and so $u^{x^i y} = u^{yx^i}$ for all i. This is impossible since y does not belong to $Z(T)$. This contradiction shows the existence of our group DTP outlined before. (This construction for Case I is probably well known, the details are included for the convenience of the reader.)

LEMMA 1.3. *If G is a finite soluble* HN-*group possessing an element x of order a prime power p^n such that x is not contained in any proper subnormal subgroup of G, and if $M = G'G^p$, then $(M')^2[M', C] \subseteq F(G)$.*

PROOF. For every chief factor H/K of G define $C(H;K)$ such that

$$C(H;K)/K = C_{G/K}(H/K).$$

According to Lemma 1.2 we know $(M')^2[M', C] \subseteq C(H;K)$. The statement of the lemma now follows from

$$F(G) = \bigcap\{C(H;K)\colon H/K \text{ is a chief factor of } G\}.$$

Now we can deduce two statements on soluble HN-groups in general.

LEMMA 1.4. *If G is a soluble* HN-*group and $H = G/F(G)$, then $H/F(H)$ is nilpotent of squarefree exponent.*

PROOF. If x is an element of order a power of the prime p and S is the smallest subnormal subgroup of G containing x, then, by Lemma 1.2, x^p is contained in a metanilpotent normal subgroup of S. The image of x into $H/F(H)$ by the canonical epimorphism mapping G onto $H/F(H)$ is therefore an element of order dividing p generating a cyclic subnormal subgroup of $H/F(H)$. This proves Lemma 1.4.

Lemma 1.5. *If G is a soluble* HN-*group and $H = G/F(G)$, then $H/Z_2(F(H))$ is nilpotent.*

Proof. Choose again an element x of order a power of a prime p and denote by S the smallest subnormal subgroup of G containing it. So the smallest subnormal subgroup of H containing $xF(G) = x^*$ is $SF(G)/F(G) \cong S/(S \cap F(G)) = S/F(S)$. By Lemma 1.3, this is the extension of a nilpotent p'-group of nilpotency class 2 by a cyclic p-group. Denote the maximal p'-subgroup of $F(M)$ by Q. We have $\langle x^*, Q\rangle$ is subnormal in H. Now $Q = [x^*, Q](C(x^*) \cap Q)$, and both factors are normal subgroups of Q. We obtain $[x^*, Q] = (SF(G)/F(G))'$ and so it is nilpotent of class 2.

Since Q and x^* are of relatively prime orders, we find that x^* operates without fixed points on $[x^*, Q]/[x^*, Q]'$. We deduce

$$[x^*, Q] \cap (C(x^*) \cap Q) \subseteq [x^*, Q]' \subseteq Z([x^*, Q]).$$

We consider any two elements a and b of $[x^*, Q]$. The commutator $[a, b]$ is contained in $Z([x^*, Q])$. On the other hand, if t is some element of $C(x^*) \cap Q$, we have $t^{-1}at = ac$ and $t^{-1}bt = bd$, where c and d are also contained in $Z([x^*, Q])$. Now $t^{-1}[a, b]t = [ac, bd] = [a, b]$, and we obtain $[x^*, Q] \cap (C(x^*) \cap Q) \subseteq Z(Q)$, and considering all commutators of length three with one entry from $[x^*, Q]$ we obtain $[x^*, Q] \subseteq Z_2(Q)$. Now $x^*Z_2(Q)$ generated a cyclic subnormal subgroup in $H/Z_2(Q)$ and it follows that $x^*F(H)$ generates a cyclic subnormal subgroup in $H/Z_2(F(H))$. Now Lemma 1.5 follows: $H/Z_2(F(H))$ is generated by cyclic subnormal subgroups.

2. Metanilpotent HN-groups with small nonabelian Fitting quotient

We begin by considering the smallest case of all.

Lemma 2.1. *Assume that G is a* HN-*group such that $G/F(G)$ is nonabelian of exponent p and of order p^3. If P is a p-Sylow subgroup of G and Q is the maximal p'-subgroup of G, then $Q = [Q, P'] \times C(P') \cap Q$ and $[Q, P']$ is abelian.*

Proof. Since G is a HN-group, the subgroups $[Q, P']$ and $C(P') \cap Q$ of the normal subgroup Q are normal in G. We will show first that $[Q, P']$ is abelian.

By assumption we know that $P = \langle x, y, P \cap F(G)\rangle$ such that $z = [x, y]$ is not contained in $P \cap F(G)$ but $x^p, y^p, [[x, y], y]$ and $[[x, y], x]$ belong to $P \cap F(G)$. First we assume that G is "minimal" in the following sense: $L = [Q, P'] =$

$[Q, z]$ is nonabelian and $[Q, z]/M$ is abelian for all proper normal subgroups M of G which are contained in G. We deduce that L' is the only minimal normal subgroup of G which is contained in L, and L is a q-group for some prime $q \neq p$. Now $L/L'L^q$ is the direct product of quotient groups which can be considered as irreducible faithful $\langle x, y\rangle$ modules. Some elements of these modules are centralized by x, so $C(x) \cap L \not\subseteq L'$. Since $C(x) \cap L$ is subnormal in C, so is $N(C(x) \cap L)$, which contains x. Using $L = (C(x) \cap L)[L, x]$ we obtain that $[L, x]$ must be contained in $N(C(x) \cap L)$, and we find

(i) $$C(x) \cap \text{ is normal in } L.$$

We want to show that L' must be trivial, We assume first that L' is not centralized by z. In this case $C(x) \cap L$ and $C(y^{-1}xy) \cap L$ are normal subgroups of L intersecting each other trivially. Now $\langle uy^{-1}uy | u \in C(x) \cap L\rangle$ is normalized by z but not by L, a contradiction to G being a HN-group. So the two normal subgroups have a nontrivial intersection, and

(ii) $$[z, L'] = 1.$$

We know that L is the product of $C(x) \cap L$ and its conjugates by powers of y, since this is true for L/L'. Since $y^{-i}xy^i = xz^i$ we find

(iii) $$[x, L'] = 1,$$

and, arguing in the same way for y instead of x, we have

(iv) $$[y, L'] = 1, \quad L' \subseteq Z(G).$$

Now the minimality condition yields

(v) $$L' \text{ is cyclic.}$$

We choose a normal subgroup R of G such that $L' \subsetneqq R \subseteq L$ and R/L' is a minimal normal subgroup of G/L'. By minimality of G, $R = [R, z]$ is nonabelian. In analogy to (i) we obtain

(vi) $$C(x) \cap R \text{ is normal in } R,$$

and since the conjugates of $C(x) \cap R$ generate R, we have

(vii) $$C(x) \cap R \text{ is nonabelian.}$$

The minimality of R also yields

(viii) $$Z(C(x) \cap R) = L'.$$

Now we find furthermore

(ix) $$R = (C(x) \cap R)[R, x] = (C(x) \cap R)(C(C(x) \cap R) \cap R),$$

where in both cases the intersection of the factors is L', and all factors are x-invariant. Since no element of $[R, x]/L'$ is left invariant by x and R/L' is therefore described in two ways as direct product of factors without operator isomorphic parts, we have

(x) $$[R,x] = C(C(x) \cap R) \cap R$$

and

(xi) $$[(C(x) \cap R), (C(y^{-1}xy^i \cap R))] = 1 \quad \text{for } i = 1, \ldots, p-1.$$

It is well known, that there are integers a, b depending on q such that $1 + a^2 + b^2 \equiv 0 \bmod q$, and we deduce that

$$T = \langle uy^{-1}u^a yy^{-2}y^b y^2 | u \in C(x) \cap R \rangle L'$$

is an abelian normal subgroup of R.Also $[T, z]$ is an abelian normal subgroup of R, and $L' \cap [T, z] = 1$. So $[T, z]$ centralizes its conjugates, and L' is not contained in the smallest normal subgroup of G which contains $[T, z]$. This contradicts the minimal choice of G, and so the minimal counterexample G does not exist. We find that $L = [Q, z]$ must be abelian.

We obtained this result for the case that G was "minimal" in the sense indicated. it is however easy to see that there is a normal subgroup K of G which is contained in $[Q, z]$ such that G/K is "minimal", whenever $[Q, z]$ is nonabelian, and we obtain a contradiction. So now the commutativity of $[Q, P']$ is proved in general.

Since Q and P' have relatively prime orders and $[Q, P']$ is abelian, we have

$$[Q, P'] \cap (C(P') \cap Q) = [Q, P'] \cap C(P') = 1.$$

Lemma 2.1 is proved. Now we are able to come to the general case.

LEMMA 2.2. *If G is a* HN-*group and $G/F(G)$ is of exponent p, then $(G/Z(F(G)))'$ is nilpotent.*

PROOF. We proceed by induction on the order of $G'F(G)/F(G)$. If

$$G'F(G)/F(G) = 1,$$

nothing is to be shown. We assume that the lemma is shown for all groups H satisfying the hypotheses and satisfying

$$|H'F(H)/F(H)| < |G'F(G)/F(G)| \neq 1.$$

We fix a p-Sylow subgroup P of G and choose an element x of P with the following property: if $xF(G) \notin Z(G/F(G))$, then $xF(G) \notin Z_2(G/F(G))$. Depending on this element x there is an element y in P such that $[x, y]$ is not contained in $F(G)$. It follows that $[x, y]F(G) \in Z(G/F(G))$, and the subnormal subgroup $\langle x, y, F(G) \rangle$ of G satisfies the hypotheses of Lemma 2.1. So, if Q is the maximal p'-subgroup of G, we have $Q = [Q, [x, y]] \times (C[x, y]) \cap Q)$

and $[Q,[x,y]]$ is an abelian normal subgroup of $\langle x,y,F(G)\rangle$. By construction, $\langle [x,y], P\cap F(G)\rangle$ is normal in P, and therefore we have that P is contained in the normalizer of $[Q,\langle [x,y], P\cap F(G)\rangle] = [Q,[x,y]]$ and in the normalizer of $C(\langle [x,y], P\cap F(G)\rangle)\cap Q = C([x,y])\cap Q$. The quotient group $G/[Q,[x,y]] \cong (C([x,y])\cap Q)P$ satisfies the hypotheses of Lemma 2.2 and the induction hypothesis. We obtain that $((C([x,y])\cap Q)P/Z(F(C([x,y])\cap Q)))'$ is nilpotent, and under the hypotheses of the lemma this is equivalent to the statement

$$[C([x,y])\cap Q, P'] \subseteq Z(C([x,y])\cap Q).$$

Now $Z(Q) = [Q,[x,y]]\times Z(C([x,y])\cap Q)$ and therefore $[Q,P']\subseteq Z(Q)$. Lemma 2.2 now follows from the nilpotency of $P'Q/Z(Q)$.

LEMMA 2.3. *Assume that G is a soluble* HN-*group with Fitting subgroup R. Then $(G/R)/Z(F(G/R))$ has nilpotent commutator subgroup.*

PROOF. Let $K = G/R$. By Lemma 1.4 we know that $K/F(K)$ is nilpotent of squarefree exponent. Choose a p-Sylow subgroup P of K. Then $F(K)P$ is a normal subgroup of K satisfying the hypotheses of Theorem 2.2, and we obtain that $P'F(K)/Z(F(K))$ is nilpotent. Since this is true for all primes p dividing the order of K, we obtain $K'F(K)/Z(F(K))$ is nilpotent, and Lemma 2.3 follows easily.

3. M-groups and HN-groups

Following Camina's definition we call a group G an M-group, if G is soluble and $G/Z(F(G))$ is nilpotent. We obtain the following first statement.

LEMMA 3.1. *If G is an* HN-*group and X and Y are two normal subgroups of G which are M-groups, then XY is also an M-group.*

PROOF. Consider an element t of order a power of p which is contained in X: denote by T the smallest subnormal subgroup of G which contains t, Since X is an M-group, $[t,T]$ is a abelian p'-group. Denote the maximal p'-subgroup of $F(XY)$ by Q. Then $[t,T] = [t,Q]$ and $C(t)\cap Q$ are normal subgroups of Q and we have

$$[t,Q]\cap(C()t)\cap Q) = 1 \quad \text{and} \quad [t,Q](C(t)\cap Q) = Q$$

So the nilpotent group Q is the direct product of two factors, one of which is abelian and consequently contained in $Z(Q)$. Now $\langle t\rangle Z(F(XY))$ is subnormal in XY, and the same happens for t in Y instead of X. Since XY is generated by the elements of prime power order which are contained in X or in Y, we have that $XY/Z(F(XY))$ is nilpotent. This proves Lemma 3.1.

COROLLARY 3.2. *If G is a* HN-*group generated by subnormal M-groups, then G is an M-group.*

The proof is done by an obvious induction argument on the defects.

LEMMA 3.3. *If G is a* HN-*group with nilpotent normal subgroup N such that N is a 2-group, then G is an M-group.*

PROOF. We proceed by induction on the order of $G/F(G)$ which is obviously a 2-group. The lemma is true if $G/F(G) = 1$. Assume that $|G/F(G)| = 2^k$ and the lemma is shown for all H satisfying the hypotheses and $|F/F(H)| < 2^k$.

We distinguish two cases: $G/F(G)$ is cyclic or not.

Assume first that $G/F(G)$ is noncyclic. Then G possesses two proper normal subgroups K and L containing $F(G)$ such that $KL = G$. By induction hypothesis, K and L are M-groups. Now G is an M-group by Lemma 3.1.

Assume now the second possibility and $G = \langle x, F(G)\rangle$ for some x of order 2^r. Denote by W the maximal subgroup of odd order of G. By construction, W is a normal subgroup of G contained in $F(G)$. By induction hypothesis we have

$$[x^2, W] \subseteq Z(W) \quad \text{and} \quad W = [x^2, W] \times (C(x^2) \cap W).$$

Again $\langle x, W\rangle$ is an abelian by nilpotent subnormal subgroup of G and therefore an HN-group. Now $\langle x, W\rangle/[x^2, W] \cong \langle x, C(x^2) \cap W\rangle$ is a HN-group and abelian by nilpotent. We obtain

$$C(x^2) \cap W = [x, C(x^2) \cap W] \times (C(x) \cap W) \quad \text{and} \quad W = [x, W] \times (C(x) \cap W).$$

Since $[x, W]$ is the direct product of abelian groups, it is abelian and contained in $Z(W)$. We have shown that $G/Z(W)$ is nilpotent, and Lemma 3.3 follows easily.

LEMMA 3.4. *If G is a soluble* HN-*group, then G is an extension of an M-group by an M-group.*

PROOF. Consider the smallest subnormal subgroup T of G containing the given element t of order a power of a prime p. We denote by V the maximal normal subgroup of T containing $F(T)$ such that $V/F(T)$ is a 2-group. By Lemma 3.3, V is an M-group, and by Lemma 1.2, T/V is a metabelian A-group and so an M-group by Camina [2, Corollary, page 364]. Let R be the normal subgroup of G containing $F(G)$ such that $R/F(G)$ is the maximal normal 2-subgroup of $G/F(G)$. We have that R is an M-group, and $T \cap R = V$. Now G/R is generated by its subnormal subgroups $TR/R \cong T(R \cap T) = R/V$ which are M-groups, and G/R is an M-group by Corollary 3.2. This completes the proof.

Now the proof of the Main Theorem follows from Lemmas 1.4, 1.5, 2.3 and 3.4.

4. The factor $SL(2,5)$ in HN-groups

We begin with a construction. Assume that p is a prime such that $p+1$ is divisible by 60. There are integers u, v, w such that, for a given power $p^k = q$ of p, we have

$$u^2 + v^2 \equiv -1 \mod q \quad \text{and} \quad w^2 \equiv 5 \mod q,$$

since these congruences have solutions modulo p. The 2×2-matrices over $\mathbf{Z}_q$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{4}\begin{pmatrix} u(1+w)-2 & vw+v+w-1 \\ vw+v-w+1 & -u(1+w)-2 \end{pmatrix}$$

generate a group isomorphic to $SL(2,5)$ since A^2 is central and $\langle A, B\rangle/\langle A^2\rangle$ yields Hamilton's representation of A_5 (see Coexeter and Moser [4, Table 5, page 138]). Using this representation of $SL(2,5)$ in $\mathrm{Aut}(C_q \times C_q)$ we find that there is an extension of $C_q \times C_q = N$ by $SL(2,5)$ with trivial centre. Since $p-1$ is not divisible by 4, 3 or 5, no noncentral element of $SL(2,5)$ leaves invariant a subgroup of order p, and by an obvious induction argument we see that only the characteristic subgroups of N are left invariant by noncentral elements of $SL(2,5)$. This shows that the extension just constructed is a HN-group. We will see that this example is in a sense typical. It shows that the condition FP on Corollaries 1–4 of Camina [1, page 67] is indispensible. The next theorem shows that condition FP can be reformulated as FP* in the form: there is no subnormal subgroup isomorphic to $SL(2,5)$ in $G/F(G)$.

THEOREM 4.1. *Assume that K is a* HN-*group with only one maximal normal subgroup, L, say, and that $K/L = PSL(2,5)$. Then one of the following is true:*

(i) $L = 1$;

(ii) *L is of order* 2 *and* $K = SL(2,5)$;

(iii) *L is nonabelian, L/L' is of order* 2 *and L' is the direct product of two cyclic groups of order m, where all prime divisors of m are of the form* $60t-1$.

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PROOF. Consider a chief factor R/S of K. If R/S is nonabelian, it must be simple by Camina [1, Corollary, page 64]. Since the group of outer automorphisms $\mathrm{Aut}(W)/\mathrm{Inn}(W)$ of a finite nonabelian simple group is (by the classification of these groups) soluble, the only nonabelian chief factor of K must be K/L, and so L is soluble. Assume now $L \supseteq R \supset S \supseteq L'$ and choose an element $z \notin L$ with $z^2 \in L$. There is a cyclic zS-invariant subgroup $\langle t, S\rangle/S$ in R/S. Since K is a HN-group, the normalizer $N(\langle t, S\rangle)$ is subnormal in K and contains the smallest subnormal subgroup of K which contains z. So, by construction, $\langle t, S\rangle$ is normal in K and R/S is cyclic. Now K is perfect, so R/S is central, that is $S \subseteq [K, R]$. We deduce $L' = [K, L]$.

Now L/L' is isomorphic to a subgroup of the Schur multiplier of $PSL(2,5)$ which is known to be of order 2. This yields that either $L = 1$ or L/L' is of order 2. By Lemma 1, L' must be abelian.

Consider now a p-chief factor R/S with $R \subseteq L'$. Then R/S cannot be cyclic since it is not a central chief factor and K is perfect. The chief factor R/S must be irreducible with respect to every subgroup outside L, in particular with respect to subgroups of order 4, 3 and 5. Considering the subgroups of order 4 and 3, we find that R/S must have rank 2 and that the prime q involved must be congruent to -1 modulo 3 and modulo 4. Considering the subgroups of order 5 we obtain in addition that q must be congruent to -1 also modulo 5, so we have $p \equiv 1 \bmod 60$.

Assume now that there are two different minimal normal subgroups A, B of K which are of order p^2, and choose elements a, b different from 1 out of A and B. If x is an element of order 4, in K, the subnormal subgroup $\langle ab, x^{-1}abx\rangle$ is normalized by x and so normal in K. So AB is the union of $p^2 + 1$ normal subgroups of K, a contradiction. This shows that L' has rank 2, and (iii) holds.

THEOREM 4.2. *If G is a* HN-*group and K and K^+ are two different subnormal subgroups of G satisfying the hypotheses of Theorem* 4.1, *if L and L^+ are their only maximal normal subgroups, then the orders of L' and $(L^+)'$ are relatively prime.*

PROOF. Assume to the contrary that K and K^+ possess isomorphic minimal normal subgroups T and T^+. Since K and K^+ are subnormal and perfect and possess only one maximal normal subgroup, K and K^+ are normal in $\langle K, K^*\rangle$ by a famous theorem of Wieldandt [6, (20)*, page 225]. Since all normal subgroups of K and K^+ are characteristic in K and K^+ respectively, T and T^+ are normal in KK^+. If $T = T^+$, we have that $K^+/C(T) \cap KK^+$ is isomorphic to the central product of two copies of $SL(2,5)$, which is impossible since T must have rank 2.

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In particular, we find that K and K^+ intersect each other trivially. We choose an element $u \neq 1$ from T and another element $v \neq 1$ from T^+, also an element y of order 4 from K and another such element z from K^+. The subgroup $N = \langle uv, y^{-1}uyz^{-1}vz\rangle$ is subnormal in KK^+ and is normalized by yz, which is not contained in any maximal normal subgroup of KK^+. Now N must be normal in KK^+ since KK^+ is a HN-group, and N has trivial intersection with K and with K^+. So N is contained in the centre of KK^+ which is trivial. This contradiction shows that the pair T, T^+ does not exist, and that L' and $(L^+)'$ are of coprime orders.

THEOREM 4.3. *Assume that G is a* HN-*group and that K is a subnormal subgroup of G satisfying the hypotheses of Theorem* 4.1 *with L nonabelian. Then K is normal in G, and G is the subdirect product of two* HN-*groups M and G/K, where M' is isomorphic to K.*

PROOF. K is a normal subgroup of G by Theorem 4.2. From $1 = Z(K) = K \cap C_G(K)$ we see that G is a subdirect product of G/K and $G/C_G(K)$. The HN-group $G/C_G(K) = M$ is a subgroup of $\mathrm{Aut}(K)$, which in turn is an extension of $\mathrm{Inn}(K) \cong K$ by an abelian group (represented by power automorphisms of L'). The proof is complete.

By iteration of Theorem 4.3, we obtain

COROLLARY 4.4. *Every finite* HN-*group is a subdirect product of groups A_i whose commutator subgroups A_i' are groups as described in Theorem* 4.1 *together with one FP^**-HN-*group B.*

5. An example of a nonsoluble HN-group

It is easily seen that $U = GL(7, 5^6)$ can be described as a direct product, namely $U = T \times (Z(U))^7$ where $Z(T) \cong T/T^2$ is cyclic and $T^7/Z(T) \cong PSL(7, 5^6)$ is simple. The group T possesses outer automorphisms α, β induced by the field automorphisms of $GF(5^6)$ which are of orders 2 and 3 respectively. We choose two isomorphic copies T_1, T_2 of T and form an extension of their direct product. Let an isomorphism τ mapping T_1 onto T_2 be given. We define K to be generated by $x, y, z, T_1 \times T_2$ subject to the relations

$$x^3 = y^3 = z^4 = [x, y] = [x, z^2] = [y, z^2] = 1,$$
$$x^{-1}ux = u^\beta \quad \text{for } u \in T_1, \quad y^{-1}vy = v^\beta \quad \text{for } v \in T_2,$$
$$[x, v] = [y, u] = 1 \quad \text{for } u \in T_1 \text{ and } v \in T_2,$$
$$z^{-1}uz = u^\tau \quad \text{for } u \in T_1, \quad z^{-1}vz = v^{\tau^{-1}\alpha} \quad \text{for } v \in T_2.$$

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It is a task of medium difficulty to prove that the group K is a HN-group, and we leave this to the reader.

If P is the maximal perfect normal subgroup of K, we see that $K/PC_K(P)$ has Fitting length 3, also $K/C_K(Z(P))$ has Fitting length 2. So the existence of nontrivial perfect normal subgroups in a HN-group does not lead to further restrictions on the Fitting length of the soluble quotients, and the bound in Lemma 1.4 is attained. (The reader will have noticed that K is a twisted wreath product with factors isomorphic to $\langle x, T_1 \rangle$ and to $\langle z \rangle$.)

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