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SPECTRUM, NUMERICAL RANGE AND DAVIS-WIELANDT SHELL OF A NORMAL OPERATOR

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Abstract. We denote the numerical range of the normal operator T by $W(T)$. A characterization is given to the points in $W(T)$ that lie on the boundary. The collection of such boundary points together with the interior of the the convex hull of the spectrum of T will then be the set $W(T)$. Moreover, it is shown that such boundary points reveal a lot of information about the normal operator. For instance, such a boundary point always associates with an invariant (reducing) subspace of the normal operator. It follows that a normal operator acting on a separable Hilbert space cannot have a closed strictly convex set as its numerical range. Similar results are obtained for the Davis-Wielandt shell of a normal operator. One can deduce additional information of the normal operator by studying the boundary of its Davis-Wielandt shell. Further extension of the result to the joint numerical range of commuting operators is discussed.

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1. Introduction. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on the Hilbert space $\mathcal{H}$. We identify $\mathcal{B}(\mathcal{H})$ with the algebra M_n of $n \times n$ complex matrices if $\mathcal{H}$ has dimension n. The *numerical range* of $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \langle x, x\rangle = 1\},$$

which is useful for studying operators; see [5–7]. In particular, the geometrical properties of $W(T)$ often provide useful information about the algebraic and analytic properties of T. For instance, $W(T) = \{\mu\}$ if and only if $T = \mu I$; $W(T) \subseteq \mathbb{R}$ if and only if $T = T^*$; $W(T)$ has no interior point if and only if there are $a, b \in \mathbb{C}$ with $a \neq 0$ such that $aT + bI$ is self-adjoint. Moreover, there are nice connections between $W(T)$ and the spectrum $\sigma(T)$ of T. For example, the closure of $W(T)$, denoted by $\mathbf{cl}\,(W(T))$, always contains $\sigma(T)$. If T is normal, then $\mathbf{cl}\,(W(T)) = \mathbf{conv}\,\sigma(T)$, where $\mathbf{conv}\,S$ denotes the convex hull of the set S. Hence, $\mathbf{cl}\,(W(T))$ is completely determined by $\sigma(T)$ for a normal operator T. However, one can easily find examples of normal operators A and B with the same spectrum such that $W(A) \neq W(B)$.

EXAMPLE 1.1. Let $A = \mathrm{diag}\,(1, 1/2, 1/3, \ldots)$, $B = \mathrm{diag}\,(0, 1, 1/2, 1/3, \ldots)$ be two diagonal operators acting on ℓ_2. Then, $W(A) = (0, 1] \neq [0, 1] = W(B)$ and $\sigma(A) = \sigma(B) = \{1/n : n \geq 1\} \cup \{0\}$.

For two normal operators A and B with the same spectrum, we have $\mathbf{cl}\,(W(A)) = \mathbf{conv}\,\sigma(A) = \mathbf{conv}\,\sigma(B) = \mathbf{cl}\,(W(B))$. Thus, $W(A)$ and $W(B)$ can differ only by their boundaries $\partial W(A)$ and $\partial W(B)$. Hence, to describe the numerical range of a normal operator T, it suffices to determine which boundary points of $W(T)$ actually belong to $W(T)$. In this paper, a characterization is given to such boundary points. Moreover, we show that a point in $W(T) \cap \partial W(T)$ always leads to an orthogonal decomposition of the Hilbert space, and a corresponding decomposition of the operator T. It follows that a normal operator acting on a separable Hilbert space cannot have a closed strictly convex set as its numerical range. On the contrary, the numerical range of a non-normal matrix in M_2 is always a non-degenerate elliptical disk; see [**7**, Theorem 1.3.6].

Motivated by theoretical study and applications, researchers considered different generalizations of the numerical range; see for example [**5**, **6**] and [**7**, Chapter 1]. One of these generalizations is the *Davis-Wielandt shell* of $T \in \mathcal{B}(\mathcal{H})$ defined by

$$DW(T) = \{(\langle Tx, x\rangle, \langle Tx, Tx\rangle) : x \in \mathcal{H},\ \langle x, x\rangle = 1\};$$

see [**3**, **4**, **10**]. Evidently, the projection of the set $DW(T)$ on the first co-ordinate is the classical numerical range. So, $DW(T)$ captures more information about the operator T. For a normal operator $T \in \mathcal{B}(\mathcal{H})$, it is known that the closure of $DW(T)$ is the set

$$\mathbf{conv}\,\{(\lambda, |\lambda|^2) : \lambda \in \sigma(T)\};$$

see, for example [**9**, Theorem 2.1]. Thus, the interior of $DW(T)$ can be easily determined. However, the points in $DW(T)$ that lie on its boundary are not so well understood. We characterize such points and show that they lead to direct sum decomposition of T that cannot be detected by the geometrical features of $W(T)$. Inspired by some comments of the referee on an early version of this paper, we include a discussion of the extension of our results to the joint numerical range of commuting operators.

In the following discussion, we denote $\mathbf{cl}\,(S)$ and ∂S as the closure and the boundary of a set S, respectively. Moreover, we use $\mathbf{int}\,(S)$ to denote the *relative interior* of S. For instance, if $\mathbf{cl}\,(S)$ is a line segment in $\mathbb{C}$, then $\mathbf{int}\,(S)$ will be the line segment obtained from $\mathbf{cl}\,(S)$ by removing its end points, although S has no interior points in $\mathbb{C}$. For $T \in \mathcal{B}(H)$, the *point spectrum* of $T \in \mathcal{B}(\mathcal{H})$ is denoted by $\sigma_p(T)$.

2. Numerical Ranges.

THEOREM 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then, $\mu \in W(T)$ is a boundary point if and only if $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that $T = T_1 \oplus T_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, with $\mu \in W(T_1) \subseteq \mathbf{L}$ for a straight line $\mathbf{L}$ and $W(T_2) \cap \mathbf{L} = \emptyset$.*

Proof. Let $\mu \in W(T)$ be a boundary point of $W(T)$. We may replace T by $aT + bI$ so that $\mu = 0$ and $\mathrm{Re}\,\nu \le 0$ for all $\nu \in W(T)$. Let $T = H + iG$, where H and G are self-adjoint. Since $W(H) = \{\mathrm{Re}\,\nu : \nu \in W(T)\}$, we see that $\langle Hx, x\rangle \le 0$ for any unit vector $x \in \mathcal{H}$. Thus, H is negative semidefinite. Let $\mathcal{H}_1$ be the kernel of H and $\mathcal{H}_2 = \mathcal{H}_1^{\perp}$. Then, $H = 0_{\mathcal{H}_1} \oplus H_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Since $HG = GH$, we see that $G = G_1 \oplus G_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Thus, $T = T_1 \oplus T_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Since $T_1 = iG_1$, $W(T_1) \subseteq i\mathbb{R}$, and $T_2 = H_2 + iG_2$ such that $W(H_2) \subseteq (-\infty, 0)$, it follows that $W(T_2) \cap i\mathbb{R} = \emptyset$.

Using the fact that $W(T_1 \oplus T_2) = \mathbf{conv}\{W(T_1) \cup W(T_2)\}$ (see, for example, [**7**, 1.2.10]), one can verify the converse. □

In Theorem 2.1, $W(T_1)$ may be a point or a line segment containing none, one or all of its end points; $W(T_2)$ may be an open set, a closed set, or neither.

EXAMPLE 2.2. We have $0 \in W(T) \cap \partial W(T)$ if $T = T_1 \oplus T_2 \in \mathcal{B}(\ell_2 \oplus \ell_2)$ for any choices of the following T_1 and T_2:

$T_1 = 0$ so that $W(T_1) = \{0\}$,

$T_1 = i(-I \oplus I)$ so that $W(T_1) = \{i\mu : \mu \in [-1, 1]\}$, or

$T_1 = i[\mathrm{diag}\,(1/2, 2/3, 3/4, \ldots) \oplus \mathrm{diag}\,(-1/2, -2/3, -3/4, \ldots)]$ so that $W(T_1) = \{i\mu : \mu \in (-1, 1)\}$;

$T_2 = \mathrm{diag}\,(e^{i2\pi/3}, e^{i4\pi/3}, -1/2)$ so that $W(T_2) = \mathbf{conv}\,\sigma(T_2)$,

$T_2 = e^{i2\pi/3}D \oplus e^{i4\pi/3}D \oplus (D - I)$ with $D = \mathrm{diag}\,(2/3, 3/4, 4/5, \ldots)$ so that $W(T_2) = \mathbf{int}\,(\mathbf{conv}\,\sigma(T_2)) = \mathbf{int}\,(\mathbf{conv}\,\{e^{i2\pi/3}, e^{i4\pi/3}, 0\})$, or

$T_2 = \mathrm{diag}\,(e^{i2\pi/3}, e^{i4\pi/3}) \oplus -\mathrm{diag}\,(1/3, 1/4, 1/5, \ldots)$ so that $W(T_2) = \mathbf{int}\,(\{e^{i2\pi/3}, e^{i4\pi/3}, 0\}) \cup \mathbf{conv}\,\{e^{i2\pi/3}, e^{i4\pi/3}\}$.

In connection to Theorem 2.1 and the above example, we give a detailed analysis of an operator A such that $W(A)$ is a subset of a straight line in $\mathbb{C}$ in the following. In particular, we give a description of $W(A)$ in terms of $\sigma(A)$ and $\sigma_p(A)$ and determine the algebraic structure of A. Note that the following proposition is valid for a general operator A.

PROPOSITION 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$ be such that $W(A) \subseteq \mathbf{L}$, where $\mathbf{L}$ is a straight line in $\mathbb{C}$. Then,*

$$W(A) = \mathbf{int}\,(\mathbf{conv}\,\sigma(A)) \cup \sigma_p(A)$$

and one of the following holds:

(a) *$A = \mu I$ and $W(A) = \{\mu\} \subseteq \mathbf{L}$.*

(b) *There are $a, b \in \mathbb{C}$ with $a \neq 0$ such that $\mathbf{cl}\,(W(A)) = a[-1, 1] + b \subseteq a\mathbb{R} + b$. In such case, an end point μ of the line segment $a[-1, 1] + b$ belongs to $W(A)$ if and only if $\mu \in \sigma_p(A)$.*

Proof. Suppose $W(A)$ is a subset of a line $\mathbf{L}$ in $\mathbb{C}$. Note that $W(A) = \{\mu\}$ if and only if $A = \mu I$. Assume that it is not the case. Then, there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that $\mathbf{cl}\,(W(A)) = a[-1, 1] + b \subseteq a\mathbb{R} + b$. Thus, $A = aS + bI$ such that $S = S^*$ with $\mathbf{cl}\,(W(S)) \subseteq [-1, 1]$. In particular, $\|S\| = 1$.

If the end point $a + b$ of $\mathbf{cl}\,(W(A))$ belongs to $W(A)$, then $1 \in W(S)$. So, there is a unit vector $x \in \mathcal{H}$ such that

$$1 = \langle Sx, x\rangle \leq \|Sx\|\|x\| \leq \|S\| \leq 1.$$

By the equality case of the Cauchy-Schwartz inequality, $Sx = x$, and thus $Ax = (a + b)x$. Thus, $a + b \in \sigma_p(A)$. Conversely, if $a + b \in \sigma_p(A)$, then $a + b \in W(A)$. Similarly, $-a + b \in W(A)$ if and only if $-a + b \in \sigma_p(A)$. □

The following corollary is immediate.

COROLLARY 2.4. *Suppose $A \in \mathcal{B}(\mathcal{H})$ is normal and $\mu \in W(A)$ is a boundary point. Then, there is a straight line $\mathbf{L}$ in $\mathbb{C}$ such that $W(A) \cap \mathbf{L} = \{\mu\}$ if and only if $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $A = \mu I_{\mathcal{H}_1} \oplus A_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ with $\mu \notin W(A_2)$.*

We present another example to illustrate our results and show that the set $W(T) \cap \partial W(T)$ cannot be determined by (and does not determine) $\sigma(T)$ and $\sigma_p(T)$ in general. The following corollary is useful for presenting the example:

COROLLARY 2.5. *Suppose* $A = d_1 I_{\mathcal{H}_1} \oplus d_2 I_{\mathcal{H}_2} \oplus \cdots \in \mathcal{B}(\mathcal{H})$ *such that* $\mathcal{H}$ *is an orthogonal sum of the closed subspaces* $\mathcal{H}_1, \mathcal{H}_2, \ldots$. *Then,*

$$W(A) = \mathbf{conv}\,\{d_n : n \geq 1\}.$$

Proof. The result follows from the inclusions

$$\mathbf{int}\,(W(A)) \subseteq \mathbf{conv}\,\{d_n : n \geq 1\}$$

$$\subseteq W(A) \subseteq \mathbf{cl}\,(W(A)) = \mathbf{cl}\,(\mathbf{conv}\,\{d_n : n \geq 1\})$$

and the description of $\partial(W(A)) \cap W(A)$ in Theorem 2.1. □

We are now ready to present the promised example. In particular, we construct normal operators $A, B, C \in \mathcal{B}(\mathcal{H})$ so that $\mathbf{cl}\,(W(A)) = \mathbf{cl}\,(W(B)) = \mathbf{cl}\,(W(C))$; A and C have different spectra and point spectra but $\partial W(A) \cap W(A) = \partial W(C) \cap W(C)$; B and C have the same spectrum and point spectrum but $\partial W(B) \cap W(B) \neq \partial W(C) \cap W(C)$.

EXAMPLE 2.6. Let $\{r_n : n \geq 1\}$ be a countable dense subset of the open interval $(0, 1)$ and $\{d_n : n \geq 1\}$ a countable dense subset of the interior of $\mathbf{conv}\,\{0, 1, i\}$. Let $A = [i] \oplus \mathrm{diag}\,(r_1, r_2, \ldots)$, $B = [i] \oplus \mathrm{diag}\,(d_1, d_2, \ldots)$ and $C = B \oplus M$, where M is the multiplication operator on $L_2([0, 1])$ defined by $M(f)(t) = t(f(t))$ for $t \in [0, 1]$. Then,

$$\mathbf{cl}\,(W(A)) = \mathbf{cl}\,(W(B)) = \mathbf{cl}\,(W(C)) = \mathbf{conv}\,\{0, 1, i\}.$$

Using Theorem 2.1, we have $\partial W(B) \cap W(B) = \{i\}$ and

$$\partial W(A) \cap W(A) = \{i\} \cup (0, 1) = \partial W(C) \cap W(C)$$

so that

$$\partial W(A) \cap W(A) = \partial W(C) \cap W(C) \neq \partial W(B) \cap W(B).$$

It is easy to check that

$$\sigma_p(A) = \{i\} \cup \{r_n : n \geq 1\}, \quad \sigma_p(B) = \sigma_p(C) = \{i\} \cup \{d_n : n \geq 1\},$$
$$\sigma(A) = \{i\} \cup [0, 1] \quad \text{and} \quad \sigma(B) = \sigma(C) = \mathbf{conv}\,\{0, 1, i\}.$$

COROLLARY 2.7. *Suppose* $\dim \mathcal{H}$ *is infinite and* $A \in \mathcal{B}(\mathcal{H})$ *is normal. If* A *is not unitarily reducible, then* $W(A) = \mathbf{int}\,(W(A))$. *In other words,* $W(A)$ *is a non-empty open set in* $\mathbb{C}$ *or* $W(A)$ *is a non-degenerate line segment without end points.*

Suppose S is a closed, bounded and convex subset of $\mathbb{C}$, with non-empty interior. We say that S is *strictly convex* if ∂S equals the set $\mathbf{Ext}\,(S)$ of extreme points of S.

COROLLARY 2.8. *Let* $A \in \mathcal{B}(\mathcal{H})$ *be normal and* $E = W(A) \cap \mathbf{Ext}\,(\mathbf{cl}\,(W(A)))$ *be uncountable. Then,* $\mathcal{H}$ *is nonseparable and every point in* E *is an eigenvalue of* A. *In particular, if* $W(A) = \mathbf{cl}\,(W(A))$ *is strictly convex with non-empty interior, then* $\mathcal{H}$ *is nonseparable and every boundary point of* $W(A)$ *is an eigenvalue.*

COROLLARY 2.9. *Let S be a bounded and convex subset of $\mathbb{C}$. Then, there exist a separable Hilbert space $\mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$ such that $S = W(A)$ if and only if $S \cap$ $\mathbf{Ext}\,(\mathbf{cl}\,(S))$ is countable.*

Proof. Suppose S is a bounded convex set such that $S \cap \mathbf{Ext}\,(\mathbf{cl}\,(S))$ is countable. Let $A = \operatorname{diag}(d_1, d_2, \ldots)$ such that $\{d_n : n \geq 1\}$ is the union of $S \cap \mathbf{Ext}\,(\mathbf{cl}\,(S))$ and a countable dense set of the relative interior of S, then $W(A) = S$. The converse follows from Corollary 2.8. □

3. Davis-Wielandt Shells. In this section, we characterize $DW(T) \cap \partial DW(T)$ for normal $T \in \mathcal{B}(\mathcal{H})$. In our discussion, we always identify $\mathbb{C} \times \mathbb{R}$ with $\mathbb{R}^3$.

THEOREM 3.1. *Suppose $T \in \mathcal{B}(\mathcal{H})$ is a normal operator. Then, $DW(T)$ and $\mathbf{conv}\,\{(\xi, |\xi|^2) : \xi \in \sigma(A)\}$ have the same interior. A point $(\mu, r) \in DW(T)$ is a boundary point if and only if $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$ with $T = T_1 \oplus T_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ such that $(\mu, r) \in DW(T_1) \subseteq \mathbf{P}$ for a plane $\mathbf{P}$ in $\mathbb{C} \times \mathbb{R}$ and $DW(T_2) \cap \mathbf{P} = \emptyset$.*

Proof. Let $T = H + iG$ be such that $H = H^*$ and $G = G^*$. Then, $DW(T)$ can be identified with the joint numerical range

$$W(H, G, T^*T) = \{(\langle Hx, x\rangle, \langle Gx, x\rangle, \langle T^*Tx, x\rangle) : x \in \mathcal{H}, \langle x, x\rangle = 1\} \subseteq \mathbb{R}^3.$$

Let $x \in \mathcal{B}(\mathcal{H})$ be a unit vector such that

$$(\mu_1, \mu_2, r) = (\langle Hx, x\rangle, \langle Gx, x\rangle, \langle T^*Tx, x\rangle)$$

is a boundary point of $W(H, G, T^*T)$. Let $\mathbf{P}$ be a support plane of $DW(T)$ passing through (μ_1, μ_2, r). Then, there are real numbers a, b, c, d such that

$$a\nu_1 + b\nu_2 + c\tilde{r} - d \leq a\mu_1 + b\mu_2 + cr - d = 0$$

for all $(\nu_1, \nu_2, \tilde{r}) \in W(H, G, T^*T)$. As a result, the operator $\tilde{T} = aH + bG + cT^*T - dI$ is negative semidefinite with a non-zero kernel. Let $\mathcal{H}_1$ be the kernel of $\tilde{T}$. Then, $\tilde{T} = \tilde{T}_1 \oplus \tilde{T}_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_1^\perp)$ such that $\langle \tilde{T}_2 y, y\rangle < 0$ for any unit vector y. Note that $\tilde{T}$ commutes with H, G. It follows that $H = H_1 \oplus H_2$ and $G = G_1 \oplus G_2$ acting on $\mathcal{H}_1 \oplus \mathcal{H}_1^\perp$ so that $T^*T = T_1^*T_1 \oplus T_2^*T_2$ for $T_1 = H_1 + iG_1$ and $T_2 = H_2 + iG_2$. Clearly, $W(H_1, G_1, T_1^*T_1) \subseteq \mathbf{P}$ and $W(H_2, G_2, T_2^*T_2)$ are contained in one of the half space determined by $\mathbf{P}$. Identifying $DW(T_j) = W(H_j, G_j, T_j^*T_j)$ for $j = 1, 2$, we get the desired conclusion on $DW(T)$.

It is easy to verify the sufficiency of the theorem. □

By Theorem 3.1, the study of points in $DW(T) \cap \partial DW(T)$ for a normal operator T reduces to the study of points in $DW(T_1)$ such that $DW(T_1)$ is a subset of a plane in $\mathbb{C} \times \mathbb{R}$. In the following, we give a detailed analysis of an operator A for which $DW(A)$ is a subset of a plane in $\mathbb{C} \times \mathbb{R}$. In particular, we give a description of $DW(A)$ in terms of $\sigma(A)$ and $\sigma_p(A)$.

Note that $DW(A) \subseteq \mathbf{conv}\,\mathcal{P}$ for any $A \in \mathcal{B}(\mathcal{H})$, where

$$\mathcal{P} = \{(\xi, |\xi|^2) : \xi \in \mathbb{C}\} \tag{1}$$

is a paraboloid. Also, observe that if $A, A' \in \mathcal{B}(\mathcal{H})$ with $A' = \alpha A + \beta I$, where $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$, then

$$DW(A') = \{(\alpha\mu + \beta, |\alpha|^2 r + 2\mathrm{Re}\,(\alpha\bar{\beta}\mu) + |\beta|^2) : (\mu, r) \in DW(A)\}. \tag{2}$$

So, $DW(A')$ is the image of $DW(A)$ under a real bijective affine transform. Clearly, there is also a one-to-one correspondence between $\sigma_p(A')$ and $\sigma_p(A)$. Moreover, the affine transform will establish a one-to-one correspondence between the boundary points of $DW(A')$ and those of $DW(A)$. Hence, replacing A by A' will not affect the hypothesis and conclusion of the results in the following discussion.

Theorem 3.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be normal. Then, $DW(A)$ is a subset of a plane in $\mathbb{C} \times \mathbb{R}$ if and only if one of the following holds:*

(a) *$A = \mu I$ so that $DW(A) = \{(\mu, |\mu|^2)\}$ is a singleton.*

(b) *$\mathcal{H}$ has a closed subspace $\mathcal{H}_1$ such that $A = \mu_1 I_{\mathcal{H}_1} \oplus \mu_2 \mathcal{H}_{\mathcal{H}_1^\perp} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_1^\perp)$ and $DW(A) = \mathbf{conv}\,\{(\mu_1, |\mu_1|^2), (\mu_2, |\mu_2|^2)\}$.*

(c) *$\sigma(A)$ has more than two elements and there are $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ such that $\alpha A + \beta I$ is a self-adjoint operator and $DW(A)$ is contained in a plane parallel to the line $\{(0, s) : s \in \mathbb{R}\}$ in $\mathbb{C} \times \mathbb{R}$.*

(d) *$\sigma(A)$ has more than two elements and there are $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ such that $\alpha A + \beta I$ is a unitary operator and $DW(A)$ is contained in a plane not parallel to the line $\{(0, s) : s \in \mathbb{R}\}$ in $\mathbb{C} \times \mathbb{R}$.*

In all the cases (a) – (d) *we have*

$$DW(A) = \mathbf{int}\,(\mathbf{conv}\,\{(\mu, |\mu|^2) : \mu \in \sigma(A)\}) \cup \mathbf{conv}\,\{(\xi, |\xi|^2) : \xi \in \sigma_p(A)\}.$$

Proof. Suppose (a) – (c) hold. Then,

$$DW(A) \subseteq \mathbf{cl}\,(DW(A)) = \mathbf{conv}\,\{(\mu, \mu^2) : \mu \in \sigma(A)\}$$

is a subset of a plane in $\mathbb{C} \times \mathbb{R}$ parallel to the line $\{(0, s) : s \in \mathbb{R}\}$ in $\mathbb{C} \times \mathbb{R}$. Suppose (d) holds. Then, the operator $A' = \alpha A + \beta I$ satisfies $\|A'x\| = 1$ for all unit vectors $x \in \mathcal{B}(\mathcal{H})$. Thus, $DW(A')$ is a subset of a plane parallel to the complex plane in $\mathbb{C} \times \mathbb{R}$. Since $\alpha \neq 0$ and $\sigma(A') = \sigma(\alpha A + \beta I)$ has at least three elements not in a line, it follows from (2) that $DW(A)$ is a subset of a plane not parallel to the line $\{(0, s) : s \in \mathbb{R}\}$ in $\mathbb{C} \times \mathbb{R}$.

Suppose $DW(A)$ is a subset of a line or $DW(A)$ is a subset of a plane parallel to the line $\{(0, s) : s \in \mathbb{R}\}$ in $\mathbb{C} \times \mathbb{R}$. Then, the projection of $DW(A)$ to the first co-ordinate will be $W(A)$ and is a subset of a straight line in $\mathbb{C}$. Then there exist $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ such that $\alpha A + \beta I$ is self-adjoint. It follows that (a), (b) or (c) holds depending on $\sigma(A)$ has one, two or more elements.

Now, suppose $DW(A)$ is not a subset of a line, and $DW(A) \subseteq \mathbf{P}$, where $\mathbf{P}$ is not parallel to the line $\{(0, s) : s \in \mathbb{R}\}$ in $\mathbb{C} \times \mathbb{R}$. Then there exist b, c and $d \in \mathbb{R}$ such that for all $(\mu_1 + i\mu_2, r) \in DW(A)$, we have

$$r + 2(b\mu_1 + c\mu_2) = d.$$

Since $r \geq \mu_1^2 + \mu_2^2$, we have,

$$d + (b^2 + c^2) = \left(r - \left(\mu_1^2 + \mu_2^2\right)\right) + (b + \mu_1)^2 + (c + \mu_2)^2 \geq 0.$$

If $d + (b^2 + c^2) = 0$, then $DW(A')$ consists of one point $(-b - ic, b^2 + c^2)$ so that A' is a scalar operator, which is a contradiction. Hence, $d + (b^2 + c^2) > 0$. Let $\alpha = 1/\sqrt{d + (b^2 + c^2)}$ and $\beta = (b + ic)/\sqrt{d + (b^2 + c^2)}$. Then for every $(\mu_1 + i\mu_2, r) \in DW(A)$, we have

$$|\alpha|^2 r + 2\mathrm{Re}\,(\alpha\bar{\beta}\mu) + |\beta|^2 = \frac{1}{d + (b^2 + c^2)}\left(r + 2(b\mu_1 + c\mu_2) + b^2 + c^2\right) = 1.$$

Therefore, for $A' = \alpha A + \beta I$, we have

$$DW(A') \subseteq \{(\xi, 1) : \xi \in \mathbb{C}\} = \mathbf{P}', \tag{3}$$

that is, $\|A'x\|^2 = 1$ for all unit vector $x \in \mathcal{H}_1$. Since A is normal and so is A', it follows that A' is unitary.

Finally, we consider the equality

$$DW(A) = \mathbf{int}\,(\mathbf{conv}\,\{(\mu, |\mu|^2) : \mu \in \sigma(A)\}) \cup \mathbf{conv}\,\{(\xi, |\xi|^2) : \xi \in \sigma_p(A)\}.$$

Clearly, the equality is valid if (a) or (b) holds. The "$\supseteq$" inclusion is clear. To prove the reverse inclusion, we establish the following:

Claim. If

$$(\mu, r) \in DW(A) \setminus \mathbf{int}\,(\mathbf{cl}\,(DW(A))),$$

then

$$(\mu, r) \in \mathbf{conv}\,\{(\xi, |\xi|^2) : \xi \in \sigma_p(A)\}.$$

Suppose (c) holds. We may replace A by $\alpha A + \beta I$ and assume that A is self-adjoint. Then

$$DW(A) \subseteq \mathbf{conv}\,\{(\mu, |\mu|^2) : \mu \in \sigma(A)\}$$

is a convex lamina in $\mathbb{R} \times [0, \infty)$. If c and d are the maximum and minimum of $\sigma(A)$, then the upper edge of the lamina equals $\mathbf{conv}\,\{(c, |c|^2), (d, |d|^2)\}$. The points on this set may or may not lie in $DW(A)$ depending on whether $c, d \in \sigma_p(A)$. Similarly, we have to examine the lower edges or boundary curve of the lamina.

To establish the claim in this case, let $x \in \mathcal{H}$ be a unit vector such that $(\langle Ax, x\rangle, \|Ax\|^2) = (\mu, r) \notin \mathbf{int}\,(\mathbf{cl}\,(DW(A)))$. If $r = \mu^2$, then by the Cauchy-Schwartz inequality, $Ax = \mu x$ and hence $\mu \in \sigma_p(A)$. Suppose $r \neq \mu^2$. Let $\mathbf{L}$ be a support line of $DW(A)$ passing through (μ, r) and suppose $\mathbf{L}$ intersects the parabola $P = \{(s, s^2) : s \in \mathbb{R}\}$ at $(\mu_1, |\mu_1|^2)$ and $(\mu_2, |\mu_2|^2)$. Clearly, μ_1, μ_2, μ are all distinct. We may replace A by $A - (\mu_1 + \mu_2)I/2$ and assume that $\mu_1 + \mu_2 = 0$. We may further assume that $|\mu_1| = 1$. Otherwise, replace A by $A/|\mu_1|$. Thus, we may assume that $\mathbf{L} = \{(\xi, 1) : \xi \in \mathbb{R}\}$ is an upper edge or a lower edge of the convex lamina $DW(A)$ with $(\mu, r) = (\mu, 1) \in \mathbf{L}$. Consequently, 1 is either the maximum or the minimum of $\sigma(A^*A)$.

Let $\mathcal{H}_0$ be the kernel of $A^*A - I$. Since $(\langle Ax, x\rangle, \|Ax\|^2) = (\mu, 1)$, we see that $x \in \mathcal{H}_0$. Since A is self-adjoint, we can further decompose $\mathcal{H}_0$ into the direct sum of $\mathcal{H}_1$ and $\mathcal{H}_2$, which are the kernel of $A - I$ and $A + I$, respectively. Note that neither $\mathcal{H}_1$ nor $\mathcal{H}_2$ can be a zero space, otherwise, we cannot have $x \in \mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that

$\langle Ax, x\rangle = \mu$. Thus, A can be written as $I_{\mathcal{H}_1} \oplus -I_{\mathcal{H}_2} \oplus A_0 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_0^\perp)$. Then

$$\begin{aligned}(\mu, r) \in DW(I_{\mathcal{H}_1} \oplus -I_{\mathcal{H}_2}) \\ &= \mathbf{conv}\,\{(1, 1), (-1, 1)\} \\ &\subseteq \mathbf{conv}\,\{(\xi, |\xi|^2) : \xi \in \sigma_p(A)\}.\end{aligned}$$

Finally, suppose (d) holds. We may replace A by $\alpha A + \beta I$ and assume that A is unitary. Hence, $DW(A) \subseteq \{(\mu, 1) : \mu \in W(A)\}$, $W(A)$ is a subset of the closed unit disk and $\sigma(A)$ is a subset of the unit circle in $\mathbb{C}$. Suppose $(\mu, r) \notin \mathbf{int}\,(\mathbf{cl}\,(DW(A)))$. Then, there is a supporting line $\mathbf{L}$ on $W(A)$ passing through μ. By Theorem 2.1, $A = A_1 \oplus A_2$, with $\mu \in W(A_1)$. Note that $DW(A_1) \subseteq DW(A) \subseteq \{(\nu, 1) : \nu \in W(A)\}$. Thus, $DW(A_1)$ is a subset of a line segment passing through $(\mu, 1)$. From the result in (b), we see that $(\mu, 1) \in \mathbf{conv}\,\{(\nu, 1) : \nu \in \sigma_p(A_1)\} \subseteq \mathbf{conv}\,\{(\xi, |\xi|^2) : \xi \in \sigma_p(A)\}$. □

Similar to Corollary 2.5, we have the following corollary for the Davis-Wielandt shell:

COROLLARY 3.3. *Suppose $A = d_1 I_{\mathcal{H}_1} \oplus d_2 I_{\mathcal{H}_2} \oplus \cdots \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{H}$ is an orthogonal sum of the closed subspaces $\mathcal{H}_1, \mathcal{H}_2, \ldots$. Then*

$$DW(A) = \mathbf{conv}\,\{(d_n, |d_n|^2) : n \geq 1\}.$$

We can use the operators in Example 2.6 to illustrate our results on Davis-Wielandt shells.

EXAMPLE 3.4. Let A, B, C be defined as in Example 2.6. Then,

$$\partial DW(A) \cap DW(A) = \left(\cup_{n\geq 1}\mathbf{conv}\left\{(i, 1), \left(r_n, r_n^2\right)\right\}\right) \cup \mathbf{conv}\left\{\left(r_n, r_n^2\right) : n \geq 1\right\},$$

$$\partial DW(B) \cap DW(B) = \{(i, 1)\} \cup \left\{\left(d_n, d_n^2\right) : n \geq 1\right\},$$

and

$$\partial DW(C) \cap DW(C) = \{(i, 1)\} \cup \left\{\left(d_n, d_n^2\right) : n \geq 1\right\} \cup \{(\mu, r) : \mu \in (0, 1), \mu^2 < r < \mu\}.$$

By Corollary 3.3, we have

$$DW(X) = \mathbf{conv}\,\{(\mu, |\mu|^2) : \mu \in \sigma_p(X)\} \qquad \text{for } X = A, B, C,$$

and

$$\begin{aligned}&DW(C) \\ &= \mathbf{conv}\,\{DW(B) \cup DW(M)\} \\ &= \mathbf{conv}\,\{(\mu, |\mu|^2) : \mu \in \sigma_p(C)\} \cup \{(\mu, r) : \mu \in (0, 1), \mu^2 < r < \mu\}.\end{aligned}$$

Recall that $\partial W(A) \cap W(A) = \partial W(C) \cap W(C) = \{i\} \cup (0, 1)$. It is clear that the boundary structure of $DW(A)$ can provide more information of A than $W(A)$. In particular, we have

$$\sigma(A) = \{\mu \in \mathbb{C} : (\mu, |\mu|^2) \in \partial DW(A)\}$$

and

$$\sigma_p(A) = \{\mu \in \mathbb{C} : (\mu, |\mu|^2) \in DW(A)\}.$$

Note that the analog of Corollary 2.9 does not hold for the Davis-Wielandt shell. In particular, the operator C in the above example acts on a separable Hilbert space and $(DW(C))$ has uncountably many extreme point lying in $DW(C)$.

4. Joint numerical ranges. Inspired by the comments of the referee on an early version of the paper, we see that our results on the numerical range and the Davis-Wielandt shell can be further extended to the *joint numerical range* $W(A_1, \ldots, A_m)$ of mutually commuting operators $A_1, \ldots, A_m \in \mathcal{B}(\mathcal{H})$ defined as the set of $(a_1, \ldots, a_m) \in \mathbb{C}^m$ with

$$a_j = \langle A_j x, x\rangle \quad \text{for } j = 1, \ldots, m,$$

for some unit vector $x \in \mathcal{H}$; see [**2**, **8**, **11**] and references there in. While $W(A)$ and $DW(A)$ are useful for studying an operator A, the joint numerical range $W(A_1, \ldots, A_m)$ is useful in studying the joint behavior of the operators $A_1, \ldots, A_m$. Suppose $A_j = H_j + iG_j$ for $H_j = H_j^*$ and $G_j = G_j^*$ for $j = 1, \ldots, m$, then $W(A_1, \ldots, A_m) \subseteq \mathbb{C}^m$ can be identified with $W(H_1, G_1, \ldots, H_m, G_m) \subseteq \mathbb{R}^{2m}$. So, we can focus on the joint numerical ranges of self-adjoint operators $A_1, \ldots, A_m \in \mathcal{B}(\mathcal{H})$. Define the *joint approximate point spectrum* $\sigma_\pi(A_1, \ldots, A_m)$ to be the set of points $(a_1, \ldots, a_m)$ such that $\sum_{j=1}^m \|(A_j - a_j I)x_n\| \to 0$ for a sequence $\{x_n\}$ of unit vector in $\mathcal{H}$. It is known that

$$\mathbf{cl}\ (W(A_1, \ldots, A_m)) = \mathbf{conv}\, \sigma_\pi(A_1, \ldots, A_m)$$

if $A_1, \ldots, A_m \in \mathcal{B}(\mathcal{H})$ are mutually commuting self-adjoint operators; see [**1**, Corollary 36.11] and [**11**].

Suppose $B_1, \ldots, B_m \in \mathcal{B}(\mathcal{H})$ are mutually commuting self-adjoint operators. If the real linear span of $I_{\mathcal{H}}, B_1, \ldots, B_m$ has dimension $k \leq m$, then $W(B_1, \ldots, B_m)$ is a subset of a $(k-1)$-dimensional hyperplane in $\mathbb{R}^m$, that is,

$$W(B_1, \ldots, B_m) \subseteq (b_1, \ldots, b_m) + \mathbf{V}$$

for a $(k-1)$-dimensional subspace $\mathbf{V}$ of $\mathbb{R}^m$. We can extend Theorem 2.1 and Theorem 3.1 (and their proofs) to the following:

THEOREM 4.1. *Suppose $A_1, \ldots, A_m \in \mathcal{B}(\mathcal{H})$ are mutually commuting self-adjoint operators. Then $(a_1, \ldots, a_m) \in W(A_1, \ldots, A_m) \cap \partial W(A_1, \ldots, A_m)$ if and only if $\mathcal{H}$ admits an orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $A_j = B_j \oplus C_j$ for $j = 1, \ldots, m$ such that $(a_1, \ldots, a_m) \in W(B_1, \ldots, B_m) \subseteq \mathbf{P}$ for a hyperplane in $\mathbb{R}^m$ and $W(C_1, \ldots, C_m) \cap \mathbf{P} = \emptyset$.*

Similar to the study in Sections 2 and 3, one may analyze the geometric structure of $W(B_1, \ldots, B_m)$ in connection to the algebraic structure of $B_1, \ldots, B_m$ in Theorem 4.1. If the boundary point $(a_1, \ldots, a_m)$ of $W(A_1, \ldots, A_m)$ lies in the relative interior of $W(B_1, \ldots, B_m)$, then not much can be said. Otherwise, we can apply the theorem again to further decompose B_j into the direct sum of two operators for $j = 1, \ldots, m$. If this procedure can be repeated until we have $(a_1, \ldots, a_m) \in W(\tilde{B}_1, \ldots, \tilde{B}_m)$ so that $W(\tilde{B}_1, \ldots, \tilde{B}_m)$ lies on a hyperplane of dimension 0 or 1, then we can apply Theorem 3.2 to conclude that each $\tilde{B}_j$ is a scalar operator, or $\tilde{B}_j = \mu_j I \oplus \nu_j I$ with $a_j \in (\mu_j, \nu_j)$

for all $j = 1, \ldots, m$. Of course, in the latter case, $(a_1, \ldots, a_m)$ is again in the relative interior of $W(\tilde{B}_1, \ldots, \tilde{B}_m)$. Summarizing the above discussion, we have the following:

PROPOSITION 4.2. *Under the hypotheses of Theorem 4.1. If* $(a_1, \ldots, a_m) \in W(A_1, \ldots, A_m)$ *is a boundary point, then* $B_1, \ldots, B_m$ *can be chosen so that one of the following holds:*

(a) $(a_1, \ldots, a_m)$ *is in the relative interior of* $W(B_1, \ldots, B_m)$.

(b) $B_j = a_j I$ *for* $j = 1, \ldots, m$. *This case holds if and only if* $(a_1, \ldots, a_m)$ *is an extreme point in* $W(A_1, \ldots, A_m)$.

Statement (b) of the above theorem is the main theorem in [**8**]. Similar to Corollary 2.9, we have the following:

COROLLARY 4.3. *Let* S *be a bounded and convex subset of* $\mathbb{R}^m$. *Then there exist a separable Hilbert space* $\mathcal{H}$ *and mutually commuting self-adjoint operators* $A_1, \ldots, A_m \in \mathcal{B}(\mathcal{H})$ *such that* $S = W(A_1, \ldots, A_m)$ *if and only if* $S \cap \mathbf{Ext}\,(\mathbf{cl}\,(S))$ *is countable.*

Note that one may sometimes use the joint numerical range to study $DW(A)$ as in our proof of Theorem 3.1. But one cannot just treat $DW(A)$ as a special case of the joint numerical range. For instance, one can extend Corollary 2.9 to the joint numerical range (Corollary 4.3) but not to the Davis-Wielandt shell (as noted at the end of Section 3). In this connection, it would be interesting to characterize those bounded convex sets in $\mathbb{R}^3$ that can be realized as $DW(A)$ for a normal operator A acting on a separable Hilbert space.

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