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VB-Courant Algebroids, E-Courant Algebroids and Generalized Geometry

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Abstract. In this paper, we first discuss the relation between VB-Courant algebroids and E-Courant algebroids, and we construct some examples of E-Courant algebroids. Then we introduce the notion of a generalized complex structure on an E-Courant algebroid, unifying the usual generalized complex structures on even-dimensional manifolds and generalized contact structures on odd-dimensional manifolds. Moreover, we study generalized complex structures on an omni-Lie algebroid in detail. In particular, we show that generalized complex structures on an omni-Lie algebra $\mathrm{gl}(V) \oplus V$ correspond to complex Lie algebra structures on V.

1 Introduction

The theory of Courant algebroids was first introduced by Liu, Weinstein, and Xu [17] providing an extension of Drinfeld's double for Lie bialgebroids. The double of a Lie bialgebroid is a special Courant algebroid [17, 20]. Jacobi algebroids are natural extensions of Lie algebroids. Courant-Jacobi algebroids were considered by Grabowski and Marmo [7], and they can be viewed as generalizations of Courant algebroids. Both Courant algebroids and Courant–Jacobi algebroids have been extensively studied in the last decade, since these are crucial geometric tools in Poisson geometry and mathematical physics. It is known that they both belong to a more general framework, namely that of E-Courant algebroids. Indeed, E-Courant algebroids were introduced by Chen, Liu, and the second author in [5] as a differential geometric object encompassing Courant algebroids [17], Courant-Jacobi algebroids [7], omni-Lie algebroids [4], conformal Courant algebroids [2], and AV-Courant algebroids [14]. It turns out that E-Courant algebroids are related to more geometric structures such as VB-Courant algebroids [15].

The aim of this paper is two-fold. First, we illuminate the relationship between VB-Courant algebroids and E-Courant algebroids. Second, we study generalized complex structures on E-Courant algebroids. Recall that a generalized almost complex structure on a manifold M is an endomorphism $\mathcal{J}$ of the *generalized tangent bundle* $\mathbb{T}M := TM \oplus T^*M$ that preserves the natural pairing on $\mathbb{T}M$ and such that $\mathcal{J}^2 = -\mathrm{id}$. If, additionally, the $\sqrt{-1}$-eigenbundle of $\mathcal{J}$ in the complexification $\mathbb{T}M \otimes \mathbb{C}$ is involutive relative to the Dorfman (equivalently, the Courant) bracket, then $\mathcal{J}$ is said

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to be *integrable*, and $(M, \mathcal{J})$ is called a *generalized complex manifold*. See [3,6,8,9,22] for more details.

Given a vector bundle $E \xrightarrow{q} M$, we consider its gauge Lie algebroid $\mathfrak{D}E$, *i.e.,* the gauge Lie algebroid of the frame bundle $\mathcal{F}(E)$. It is known that $\mathfrak{D}E$ is a transitive Lie algebroid over M and the first jet bundle $\mathfrak{J}E$ is its E-dual bundle. In fact, $\mathfrak{ol}(E) = \mathfrak{D}E \oplus \mathfrak{J}E$ is called an *omni-Lie algebroid* [4], which is a generalization of Weinstein's concept of an omni-Lie algebra [25]. In particular, the line bundle case where E comes from a contact distribution brings us to the concept of a generalized contact bundle. To have a better grasp of the concept of a generalized contact bundle, we briefly review the line bundle approach to contact geometry. By definition, a contact structure on an odd-dimensional manifold M is a maximal non-integrable hyperplane distribution $H \subset TM$. In a dual way, any hyperplane distribution H on M can be regarded as a nowhere vanishing 1-form $\theta\colon TM \to L$ (its *structure form*) with values in the line bundle $L = TM/H$, such that $H = \ker\theta$. Replacing the tangent algebroid with the Atiyah algebroid of a line bundle in the definition of a generalized complex manifold, we obtain the notion of a *generalized contact bundle*. In this paper, we extend the concept of a generalized contact bundle to the context of E-Courant algebroids.

The paper is organized as follows. Section 2 contains basic definitions used in the sequel. Section 3 highlights the importance and naturality of the notion of E-Courant algebroids. Explicitly, the fat Courant algebroid associated with a VB-Courant algebroid (see the definition of a VB-Courant algebroid below) is an E-Courant algebroid. We observe the following facts:

- Given a crossed module of Lie algebras $(\mathfrak{m}, \mathfrak{g})$, we get an $\mathfrak{m}$-Courant algebroid $\mathrm{Hom}(\mathfrak{g}, \mathfrak{m}) \oplus \mathfrak{g}$, which was given in [13] as a generalization of an omni-Lie algebra.
- The omni-Lie algebroid $\mathfrak{ol}(E) = \mathfrak{D}E \oplus \mathfrak{J}E$ is the linearization of the VB-Courant algebroid $TE^* \oplus T^*E^*$. This generalizes the fact that an omni-Lie algebra is the linearization of the standard Courant algebroid.
- For a Courant algebroid $\mathcal{C}$, $T\mathcal{C}$ is a VB-Courant algebroid. The associated fat Courant algebroid $\mathfrak{J}\mathcal{C}$ is a T^*M-Courant algebroid. The fact that $\mathfrak{J}\mathcal{C}$ is a T^*M-Courant algebroid was first obtained in [5, Theorem 2.13].

In Section 4, we introduce generalized complex structures on E-Courant algebroids and provide examples. In Sections 5, we describe generalized complex structures on omni-Lie algebroids. In Section 6, we show that generalized complex structures on the omni-Lie algebra $\mathfrak{ol}(V)$ are in one-to-one correspondence with complex Lie algebra structures on V.

2 Preliminaries

Throughout the paper, M is a smooth manifold, d is the usual differential operator on forms, and $E \to M$ is a vector bundle. In this section, we recall the notions of E-Courant algebroids [5], omni-Lie algebroids [4], generalized complex structures [8,9], and generalized contact structures [23].

2.1 E-Courant Algebroids and Omni-Lie Algebroids

For a vector bundle $E \to M$, its gauge Lie algebroid $\mathfrak{D}E$ with the commutator bracket $[\cdot,\cdot]_{\mathfrak{D}}$ is just the gauge Lie algebroid of the frame bundle $\mathcal{F}(E)$, which is also called the *covariant differential operator bundle of* E (see [18, Example 3.3.4]). The corresponding Atiyah sequence is

$$0 \longrightarrow \mathrm{gl}(E) \xrightarrow{i} \mathfrak{D}E \xrightarrow{j} LM \longrightarrow 0. \tag{2.1}$$

In [4], the authors proved that the jet bundle $\mathfrak{J}E$ can be considered as an E-dual bundle of $\mathfrak{D}E$:

$$\mathfrak{J}E \cong \left\{ \nu \in \mathrm{Hom}(\mathfrak{D}E, E) \mid \nu(\Phi) = \Phi \circ \nu(\mathrm{id}_E) \text{ for all } \Phi \in \mathrm{gl}(E) \right\}. \tag{2.2}$$

Associated with the jet bundle $\mathfrak{J}E$, there is a jet sequence given by

$$0 \longrightarrow \mathrm{Hom}(TM, E) \xrightarrow{\mathbb{e}} \mathfrak{J}E \xrightarrow{\mathbb{p}} E \longrightarrow 0. \tag{2.3}$$

Define the operator $\mathbb{d}\colon \Gamma(E) \to \Gamma(\mathfrak{J}E)$ by

$$\mathbb{d}u(\mathfrak{d}) := \mathfrak{d}(u) \text{ for all } u \in \Gamma(E), \quad \mathfrak{d} \in \Gamma(\mathfrak{D}E).$$

An important formula that will be often used is

$$\mathbb{d}(fu) = \mathrm{d}f \otimes u + f\mathbb{d}u \quad \text{for all } u \in \Gamma(E),\ f \in C^\infty(M).$$

In fact, there is an E-valued pairing between $\mathfrak{J}E$ and $\mathfrak{D}E$ by setting

$$\langle \mu, \mathfrak{d} \rangle_E \triangleq \mathfrak{d}(u) \quad \text{for all } \mu \in (\mathfrak{J}E)_m,\ \mathfrak{d} \in (\mathfrak{D}E)_m, \tag{2.4}$$

where $u \in \Gamma(E)$ satisfies $\mu = [u]_m$. In particular, one has

$$\begin{aligned} \langle \mu, \Phi \rangle_E &= \Phi \circ \mathbb{p}(\mu) && \text{for all } \Phi \in \mathrm{gl}(E),\ \mu \in \mathfrak{J}E; \\ \langle \mathfrak{y}, \mathfrak{d} \rangle_E &= \mathfrak{y} \circ \mathbb{j}(\mathfrak{d}) && \text{for all } \mathfrak{y} \in \mathrm{Hom}(TM, E),\ \mathfrak{d} \in \mathfrak{D}E. \end{aligned}$$

For vector bundles P, Q over M and a bundle map $\rho\colon P \to Q$, we denote the induced E-dual bundle map by $\rho^\star$, *i.e.*,

$$\rho^\star\colon \mathrm{Hom}(Q, E) \longrightarrow \mathrm{Hom}(P, E), \quad \rho^\star(\nu)(k) = \nu(\rho(k)) \text{ for } k \in P, \nu \in \mathrm{Hom}(Q, E).$$

Definition 2.1 ([5]) An E-*Courant algebroid* is a quadruple $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}}, (\cdot,\cdot)_E, \rho)$, where $\mathcal{K}$ is a vector bundle over M such that $(\Gamma(\mathcal{K}), [\cdot,\cdot]_{\mathcal{K}})$ is a Leibniz algebra, $(\cdot,\cdot)_E\colon \mathcal{K} \otimes \mathcal{K} \to E$ a nondegenerate symmetric E-valued pairing that induces an embedding: $\mathcal{K} \hookrightarrow \mathrm{Hom}(\mathcal{K}, E)$ via $Y(X) = 2(X, Y)_E$, and $\rho\colon \mathcal{K} \to \mathfrak{D}E$ a bundle map called the anchor, such that for all $X, Y, Z \in \Gamma(\mathcal{K})$, the following properties hold:

(EC-1) $\rho[X, Y]_{\mathcal{K}} = [\rho(X), \rho(Y)]_{\mathfrak{D}}$;

(EC-2) $[X, X]_{\mathcal{K}} = \rho^\star \mathbb{d}(X, X)_E$;

(EC-3) $\rho(X)(Y, Z)_E = ([X, Y]_{\mathcal{K}}, Z)_E + (Y, [X, Z]_{\mathcal{K}})_E$;

(EC-4) $\rho^\star(\mathfrak{J}E) \subset \mathcal{K}$, *i.e.*, $(\rho^\star(\mu), X)_E = \frac{1}{2}\mu(\rho(X))$ for all $\mu \in \mathfrak{J}E$;

(EC-5) $\rho \circ \rho^\star = 0$.

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Obviously, a Courant algebroid is an E-Courant algebroid, where $E = M \times \mathbb{R}$, the trivial line bundle. Similar to the proof for Courant algebroids ([20, Lemma 2.6.2]), we have the following lemma.

Lemma 2.2 *For an* E*-Courant algebroid* $\mathcal{K}$*, one has*

$$[X, \rho^\star \mathrm{d}u]_{\mathcal{K}} = 2\rho^\star \mathrm{d}\,(X, \rho^\star \mathrm{d}u)_E, \quad [\rho^\star \mathrm{d}u, X]_{\mathcal{K}} = 0 \ \textit{for all}\ X \in \Gamma(\mathcal{K}),\ u \in \Gamma(E).$$

An omni-Lie algebroid, which was introduced in [4], is a very interesting example of E-Courant algebroids. Let us recall it briefly. There is an E-valued pairing $(\cdot,\cdot)_E$ on $\mathfrak{D}E \oplus \mathfrak{J}E$ defined by

$$(2.5)\qquad (\mathfrak{d} + \mu, \mathfrak{t} + \nu)_E = \tfrac{1}{2}(\langle \mu, \mathfrak{t}\rangle_E + \langle \nu, \mathfrak{d}\rangle_E) \quad \text{for all}\ \mathfrak{d} + \mu,\ \mathfrak{t} + \nu \in \mathfrak{D}E \oplus \mathfrak{J}E.$$

Furthermore, $\Gamma(\mathfrak{J}E)$ is invariant under the Lie derivative $\mathfrak{L}_{\mathfrak{d}}$ for any $\mathfrak{d} \in \Gamma(\mathfrak{D}E)$ that is defined by the Leibniz rule:

$$\langle \mathfrak{L}_{\mathfrak{d}}\mu, \mathfrak{d}'\rangle_E \triangleq \mathfrak{d}\,\langle \mu, \mathfrak{d}'\rangle_E - \langle \mu, [\mathfrak{d}, \mathfrak{d}']_{\mathfrak{D}}\rangle_E \quad \text{for all}\ \mu \in \Gamma(\mathfrak{J}E),\ \mathfrak{d}' \in \Gamma(\mathfrak{D}E).$$

On the section space $\Gamma(\mathfrak{D}E \oplus \mathfrak{J}E)$, we can define a bracket as follows:

$$(2.6)\qquad [\![\mathfrak{d} + \mu, \mathfrak{r} + \nu]\!] \triangleq [\mathfrak{d}, \mathfrak{r}]_{\mathfrak{D}} + \mathfrak{L}_{\mathfrak{d}}\nu - \mathfrak{L}_{\mathfrak{r}}\mu + \mathrm{d}\mu(\mathfrak{r}).$$

Definition 2.3 ([4]) The quadruple $(\mathfrak{D}E \oplus \mathfrak{J}E, [\![\cdot,\cdot]\!], (\cdot,\cdot)_E, \rho)$ is called an *omni-Lie algebroid*, where ρ is the projection from $\mathfrak{D}E \oplus \mathfrak{J}E$ to $\mathfrak{D}E$, $(\cdot,\cdot)_E$ and $[\![\cdot,\cdot]\!]$ are given by (2.5) and (2.6), respectively.

We will denote an omni-Lie algebroid by $\mathfrak{ol}(E)$.

2.2 Generalized Complex Structures and Generalized Contact Structures

The notion of a Courant algebroid was introduced in [17]. A Courant algebroid is a quadruple $(\mathcal{C}, [\![\cdot,\cdot]\!], (\cdot,\cdot)_+, \rho)$, where $\mathcal{C}$ is a vector bundle over M, $[\![\cdot,\cdot]\!]$ a bracket operation on $\Gamma(\mathcal{C})$, $(\cdot,\cdot)_+$ a nondegenerate symmetric bilinear form on $\mathcal{C}$, and $\rho\colon \mathcal{C} \to TM$ a bundle map called the anchor, such that some compatibility conditions are satisfied. See [20] for more details. Consider the generalized tangent bundle

$$\mathbb{T}M := TM \oplus T^*M.$$

On its section space $\Gamma(\mathbb{T}M)$, there is a Dorfman bracket

$$(2.7)\qquad [\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X\eta - i_Y \mathrm{d}\xi \quad \text{for all}\ X + \xi,\ Y + \eta \in \Gamma(\mathbb{T}M).$$

Furthermore, there is a canonical nondegenerate symmetric bilinear form on $\mathbb{T}M$:

$$(2.8)\qquad (X + \xi, Y + \eta)_+ = \frac{1}{2}\big(\eta(X) + \xi(Y)\big).$$

We call $(\mathbb{T}M, [\![\cdot,\cdot]\!], (\cdot,\cdot)_+, \mathrm{pr}_{TM})$ the standard Courant algebroid.

Definition 2.4 A *generalized complex structure* on a manifold M is a bundle map $\mathcal{J}\colon \mathbb{T}M \to \mathbb{T}M$ satisfying the algebraic properties:

$$\mathcal{J}^2 = -\mathrm{id} \quad \text{and} \quad (\mathcal{J}(u), \mathcal{J}(v))_+ = (u, v)_+$$

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and the integrability condition

$$[\![\mathcal{J}(u),\mathcal{J}(v)]\!]-[\![u,v]\!]-\mathcal{J}\big([\![\mathcal{J}(u),v]\!]+[\![u,\mathcal{J}(v)]\!]\big)=0\quad\text{for all } u,\ v\in\Gamma(\mathbb{T}M).$$

Here, $(\cdot,\cdot)_+$ and $[\![\cdot,\cdot]\!]$ are given by (2.8) and (2.7), respectively.

See [8, 9] for more details. Note that only even-dimensional manifolds can have generalized complex structures. In [23], the authors give the odd-dimensional analogue of the concept of a generalized complex structures extending the definition given in [10]. We now recall the definition of a generalized contact bundle from [23]. A *generalized contact bundle* is a line bundle $L\to M$ equipped with a *generalized contact structure*, *i.e.*, a vector bundle endomorphism $\mathcal{I}:\mathfrak{D}L\oplus\mathfrak{J}L\to\mathfrak{D}L\oplus\mathfrak{J}L$ such that

- $\mathcal{I}$ is *almost complex*, *i.e.*, $\mathcal{I}^2=-\mathrm{id}$;
- $\mathcal{I}$ is *skew-symmetric*, *i.e.*,
$$(\mathcal{I}\alpha,\beta)_L+(\alpha,\mathcal{I}\beta)_L=0\quad\text{for all }\alpha,\beta\in\Gamma(\mathfrak{D}L\oplus\mathfrak{J}L),$$
- $\mathcal{I}$ is *integrable*, *i.e.*,
$$[\![\mathcal{I}\alpha,\mathcal{I}\beta]\!]-[\![\alpha,\beta]\!]-\mathcal{I}[\![\mathcal{I}\alpha,\beta]\!]-\mathcal{I}[\![\alpha,\mathcal{I}\beta]\!]=0\quad\text{for all }\alpha,\beta\in\Gamma(\mathfrak{D}L\oplus\mathfrak{J}L).$$

Let $(L\to M,\mathcal{I})$ be a generalized contact bundle. Using the direct sum $\mathfrak{ol}(L)=\mathfrak{D}L\oplus\mathfrak{J}L$ and the definition, one can see that

$$\mathcal{I}=\begin{pmatrix}\phi & J^\sharp\\ \omega_\flat & -\phi^\dagger\end{pmatrix},$$

where J is a Jacobi bi-derivation, ϕ is an endomorphism of $\mathfrak{D}L$ compatible with J, and the 2-form $\omega:\wedge^2\mathfrak{D}L\to L$ and its associated vector bundle morphism $\omega_\flat:\mathfrak{D}L\to\mathfrak{J}L$ satisfy additional compatibility conditions [23].

3 VB-Courant Algebroids and E-Courant Algebroids

In this section, we highlight the relation between VB-Courant algebroids and E-Courant algebroids and give more examples of E-Courant algebroids.

Denote a double vector bundle

$$\begin{array}{ccc} D & \xrightarrow{\pi_B} & B\\ {\scriptstyle\pi_A}\downarrow & & \downarrow{\scriptstyle q_B}\\ A & \xrightarrow[q_A]{} & M\end{array}$$

with core C by $(D;A,B;M)$. The space of sections $\Gamma_B(D)$ is generated as a $C^\infty(B)$-module by core sections $\Gamma^c_B(D)$ and linear sections $\Gamma^l_B(D)$. See [19] for more details. For a section $c:M\to C$, the corresponding *core section* $c^\dagger:B\to D$ is defined as

$$c^\dagger(b_m)=\widetilde{0}_{b_m}+_A\overline{c(m)}\quad\text{for all } m\in M,\ b_m\in B_m,$$

where $\bar{\cdot}$ means the inclusion $C\hookrightarrow D$. A section $\xi:B\to D$ is called *linear* if it is a bundle morphism from $B\to M$ to $D\to A$ over a section $a\in\Gamma(A)$. Given $\psi\in\Gamma(B^*\otimes C)$, there is a linear section $\widetilde{\psi}:B\to D$ over the zero section $0^A:M\to A$ given by

$$\widetilde{\psi}(b_m)=\widetilde{0}_{b_m}+_A\overline{\psi(b_m)}.$$

Note that $\Gamma^l_B(D)$ is locally free as a $C^\infty(M)$-module. Therefore, $\Gamma^l_B(D)$ is equal to $\Gamma(\widehat{A})$ for some vector bundle $\widehat{A} \to M$. Moreover, we have the following short exact sequence of vector bundles over M:

$$0 \longrightarrow B^* \otimes C \longrightarrow \widehat{A} \longrightarrow A \longrightarrow 0. \tag{3.1}$$

Example 3.1 Let E be a vector bundle over M.

(i) The tangent bundle $(TE; TM, E; M)$ is a double vector bundle with core E. Then $\widehat{A}$ is the gauge bundle $\mathfrak{D}E$ and the exact sequence (3.1) is exactly the Atiyah sequence (2.1).
(ii) The cotangent bundle $(T^*E; E^*, E; M)$ is a double vector bundle with core T^*M. In this case, $\widehat{A}$ is exactly the jet bundle $\mathfrak{J}E^*$ and the exact sequence (3.1) is indeed the jet sequence (2.3).

Definition 3.2 ([15]) A VB-*Courant algebroid* is a metric double vector bundle

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle q_B} \\ A & \xrightarrow[q_A]{} & M, \end{array}$$

with core C such that $\mathbb{E} \to B$ is a Courant algebroid and the following conditions are satisfied:

(i) The anchor map $\Theta\colon \mathbb{E} \to TB$ is linear; that is,

$$\Theta\colon (\mathbb{E}; A, B; M) \longrightarrow (TB; TM, B; M)$$

is a morphism of double vector bundles.
(ii) The Courant bracket is linear; that is,

$$[\![\Gamma^l_B(\mathbb{E}), \Gamma^l_B(\mathbb{E})]\!] \subseteq \Gamma^l_B(\mathbb{E}), \quad [\![\Gamma^l_B(\mathbb{E}), \Gamma^c_B(\mathbb{E})]\!] \subseteq \Gamma^c_B(\mathbb{E}), \quad [\![\Gamma^c_B(\mathbb{E}), \Gamma^c_B(\mathbb{E})]\!] = 0.$$

For a VB-Courant algebroid $\mathbb{E}$, we have the exact sequence (3.1). Note that the restriction of the pairing on $\mathbb{E}$ to linear sections of $\mathbb{E}$ defines a nondegenerate pairing on $\widehat{A}$ with values in B^*, which is guaranteed by the metric double vector bundle structure; see [11]. Coupled with the fact that the Courant bracket is closed on linear sections, one gets the following result.

Proposition 3.3 ([11]) *The vector bundle $\widehat{A}$ inherits a Courant algebroid structure with the pairing taking values in B^*, which is called the* fat *Courant algebroid of this* VB-*Courant algebroid.*

Alternatively, we have the following proposition.

Proposition 3.4 *For a* VB-*Courant algebroid* $(\mathbb{E}; A, B; M)$, *its associated fat Courant algebroid is a B^*-Courant algebroid.*

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Example 3.5 (Standard VB-Courant algebroid over a vector bundle) For a vector bundle E, there is a standard VB-Courant algebroid

$$\begin{array}{ccc} TE^* \oplus T^*E^* & \longrightarrow & E^* \\ \downarrow & & \downarrow \\ TM \oplus E & \longrightarrow & M \end{array}$$

with base E^* and core $E^* \oplus T^*M \to M$. The corresponding exact sequence is given by

$$0 \longrightarrow \mathrm{gl}(E) \oplus T^*M \otimes E \longrightarrow \widehat{A} \longrightarrow TM \oplus E \longrightarrow 0.$$

Actually, by Example 3.1, the corresponding fat Courant algebroid $\widehat{A}$ here is exactly the omni-Lie algebroid $\mathfrak{ol}(E) = \mathfrak{D}E \oplus \mathfrak{J}E$. So the omni-Lie algebroid is the linearization of the standard VB-Courant algebroid.

Example 3.6 (Tangent VB-Courant algebroid) The tangent bundle $T\mathcal{C}$ of a Courant algebroid $\mathcal{C} \to M$

$$\begin{array}{ccc} T\mathcal{C} & \longrightarrow & TM \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & M \end{array}$$

carries a VB-Courant algebroid structure with base TM and core $\mathcal{C} \to M$. The associated exact sequence is

$$0 \longrightarrow T^*M \otimes \mathcal{C} \longrightarrow \widehat{\mathcal{C}} \longrightarrow \mathcal{C} \longrightarrow 0.$$

Actually, the fat Courant algebroid $\widehat{\mathcal{C}}$ is $\mathfrak{J}\mathcal{C}$, which is a T^*M-Courant algebroid by Proposition 3.4. So we get that on the jet bundle of a Courant algebroid, there is a T^*M-Courant algebroid structure. This result was first given in [5].

A crossed module of Lie algebras consists of a pair of Lie algebras $(\mathfrak{m}, \mathfrak{g})$, an action $\triangleright$ of $\mathfrak{g}$ on $\mathfrak{m}$ and a Lie algebra morphism $\phi: \mathfrak{m} \to \mathfrak{g}$ such that

$$\phi(\xi) \triangleright \eta = [\xi, \eta]_{\mathfrak{m}}, \quad \phi(x \triangleright \xi) = [x, \phi(\xi)]_{\mathfrak{g}},$$

for all $x \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{m}$.

Given a crossed module, there is an action $\rho: \mathfrak{g} \ltimes \mathfrak{g}^* \to \mathfrak{X}(\mathfrak{m}^*)$ of the natural quadratic Lie algebra $\mathfrak{g} \ltimes \mathfrak{g}^*$ on $\mathfrak{m}^*$ given by

$$\rho(u + \alpha) = u \triangleright \cdot + \phi^* \alpha,$$

where $u \triangleright \cdot \in \mathrm{gl}(\mathfrak{m})$ is viewed as a linear vector field on $\mathfrak{m}^*$ and $\phi^* \alpha \in \mathfrak{m}^*$ is viewed as a constant vector field on $\mathfrak{m}^*$. Note that this action is coisotropic. We get the action Courant algebroid [16] $(\mathfrak{g} \ltimes \mathfrak{g}^*) \times \mathfrak{m}^*$ over $\mathfrak{m}^*$ with the anchor given by ρ and the Dorfman bracket given by

$$[e_1, e_2] = \mathcal{L}_{\rho(e_1)} e_2 - \mathcal{L}_{\rho(e_2)} e_1 + [e_1, e_2]_{\mathfrak{g} \ltimes \mathfrak{g}^*} + \rho^* \langle \mathrm{d} e_1, e_2 \rangle. \tag{3.2}$$

for any $e_1, e_2 \in \Gamma((\mathfrak{g} \ltimes \mathfrak{g}^*) \times \mathfrak{m}^*)$. Here, $\mathrm{d}e_1 \in \Omega^1(\mathfrak{m}^*, \mathfrak{g} \ltimes \mathfrak{g}^*)$ is given by Lie derivatives $(\mathrm{d}e_1)(X) = \mathcal{L}_X e_1$ for $X \in \mathfrak{X}(\mathfrak{m}^*)$. Moreover, it is a VB-Courant algebroid

$$\begin{array}{ccc} (\mathfrak{g} \ltimes \mathfrak{g}^*) \times \mathfrak{m}^* & \longrightarrow & \mathfrak{m}^* \\ \downarrow & & \downarrow \\ \mathfrak{g} & \longrightarrow & * \end{array}$$

with base $\mathfrak{m}^*$ and core $\mathfrak{g}^*$. See [15] for details. The associated exact sequence is

$$0 \longrightarrow \mathfrak{m} \otimes \mathfrak{g}^* \cong \mathrm{Hom}(\mathfrak{g}, \mathfrak{m}) \longrightarrow \widehat{A} \longrightarrow \mathfrak{g} \longrightarrow 0.$$

Since the double vector bundle is trivial, we have $\widehat{A} = \mathrm{Hom}(\mathfrak{g}, \mathfrak{m}) \oplus \mathfrak{g}$.

Moreover, applying (3.2), we get the Dorfman bracket on $\mathrm{Hom}(\mathfrak{g}, \mathfrak{m}) \oplus \mathfrak{g}$.

Proposition 3.7 *With the above notation,* $(\mathrm{Hom}(\mathfrak{g}, \mathfrak{m}) \oplus \mathfrak{g}, [\cdot, \cdot], (\cdot, \cdot)_{\mathfrak{m}}, \rho = 0)$ *is an* $\mathfrak{m}$*-Courant algebroid, where the pairing* $(\cdot, \cdot)_{\mathfrak{m}}$ *is given by*

$$(A + u, B + v)_{\mathfrak{m}} = \tfrac{1}{2}(Av + Bu),$$

and the Dorfman bracket is given by

$$\begin{aligned} [u, v] &= [u, v]_{\mathfrak{g}}; \\ [A, B] &= A \circ \phi \circ B - B \circ \phi \circ A; \\ [A, v] &= A \circ \mathrm{ad}^0_v - \mathrm{ad}^1_v \circ A + \cdot \triangleright Av + \phi(Av); \\ [v, A] &= \mathrm{ad}^1_v \circ A - A \circ \mathrm{ad}^0_v \end{aligned}$$

for all $A, B \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{m}), u, v \in \mathfrak{g}$. *Here,* $\mathrm{ad}^0_v \in \mathrm{gl}(\mathfrak{g})$ *and* $\mathrm{ad}^1_v \in \mathrm{gl}(\mathfrak{m})$ *are given by* $\mathrm{ad}^0_v(u) = [v, u]_{\mathfrak{g}}$ *and* $\mathrm{ad}^1_v(a) = v \triangleright a$*, respectively, and* $\cdot \triangleright Av \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{m})$ *is defined by* $(\cdot \triangleright Av)(u) = u \triangleright Av$.

Proof By (3.2), it is obvious that $[u, v] = [u, v]_{\mathfrak{g}}$. For $A, B \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{m}), v \in \mathfrak{g}$, applying (3.2), we find

$$[A, B] = \mathcal{L}_{\rho(A)} B - \mathcal{L}_{\rho(B)} A = \rho(A)B - \rho(B)A = A \circ \phi \circ B - B \circ \phi \circ A.$$

Observe that $\mathcal{L}_{\rho(v)} A = \mathrm{ad}^1_v(A) = \mathrm{ad}^1_v \circ A$ and $[A, v]_{\mathfrak{g} \ltimes \mathfrak{g}^*} = -(\mathrm{ad}^0)^*_v A = A \circ \mathrm{ad}^0_v$. We have

$$\begin{aligned} [A, v] &= \mathcal{L}_{\rho(A)} v - \mathcal{L}_{\rho(v)} A + [A, v]_{\mathfrak{g} \ltimes \mathfrak{g}^*} + \rho^* \langle \mathrm{d}A, v \rangle \\ &= 0 - \mathrm{ad}^1_v \circ A + A \circ \mathrm{ad}^0_v + \cdot \triangleright Av + \phi(Av), \end{aligned}$$

where we have used

$$\rho^* \langle \mathrm{d}A, v \rangle (u + B) = \rho(u + B)(Av) = u \triangleright Av + B(\phi(Av)).$$

Finally, we have

$$\begin{aligned} [v, A] &= \mathcal{L}_{\rho(v)} A - \mathcal{L}_{\rho(A)} v + [v, A]_{\mathfrak{g} \ltimes \mathfrak{g}^*} + \rho^* \langle \mathrm{d}v, A \rangle \\ &= \mathrm{ad}^1_v \circ A + 0 - A \circ \mathrm{ad}^0_v + 0. \end{aligned}$$

This completes the proof. ∎

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Remark 3.8 This bracket can be viewed as a generalization of an omni-Lie algebra. See [13, Example 5.2] for more details.

More generally, since the category of Lie 2-algebroids and the category of VB-Courant algebroids are equivalent (see [15]), we get an E-Courant algebroid from a Lie 2-algebroid. This construction first appeared in [11, Corollary 6.9]. Explicitly, let $(A_0 \oplus A_{-1}, \rho_{A_0}, l_1, l_2 = l_2^0 + l_2^1, l_3)$ be a Lie 2-algebroid. Then we have an A_{-1}-Courant algebroid structure on

$$\mathrm{Hom}(A_0, A_{-1}) \oplus A_0,$$

where the pairing is given by

$$(D + u, D' + v)_{A_{-1}} = \frac{1}{2}(Dv + D'u)$$

for $D, D' \in \Gamma(\mathrm{Hom}(A_0, A_{-1}))$ and $u, v \in \Gamma(A_0)$, the anchor is

$$\rho\colon \mathrm{Hom}(A_0, A_{-1}) \oplus A_0 \to \mathfrak{D}A_{-1}, \quad \rho(D + u) = D \circ l_1 + l_2^1(u, \cdot),$$

and the Dorfman bracket is given by

$$\begin{aligned}
[u, v] &= l_2^0(u, v) + l_3(u, v, \cdot),\\
[D, D'] &= D \circ l_1 \circ D' - D' \circ l_1 \circ D,\\
[D, v] &= -l_2^1\big(v, D(\cdot)\big) + D\big(l_2^0(v, \cdot)\big) + l_2^1\big(\cdot, D(v)\big) + l_1(D(v)),\\
[v, D] &= l_2^1\big(v, D(\cdot)\big) - D\big(l_2^0(v, \cdot)\big).
\end{aligned}$$

4 Generalized Complex Structures on E-Courant Algebroids

In this section, we introduce the notion of a generalized complex structure on an E-Courant algebroid. We will see that it unifies the usual generalized complex structure on an even-dimensional manifold and the generalized contact structure on an odd-dimensional manifold.

Definition 4.1 A bundle map $\mathcal{J}\colon \mathcal{K} \to \mathcal{K}$ is called a *generalized almost complex structure* on an E-Courant algebroid $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}}, (\cdot,\cdot)_E, \rho)$ if it satisfies the algebraic properties

$$\mathcal{J}^2 = -1 \quad \text{and} \quad (\mathcal{J}(X), \mathcal{J}(Y))_E = (X, Y)_E. \tag{4.1}$$

Furthermore, $\mathcal{J}$ is called a *generalized complex structure* if the following integrability condition is satisfied:

$$[\mathcal{J}(X), \mathcal{J}(Y)]_{\mathcal{K}} - [X, Y]_{\mathcal{K}} - \mathcal{J}\big([\mathcal{J}(X), Y]_{\mathcal{K}} + [X, \mathcal{J}(Y)]_{\mathcal{K}}\big) = 0, \tag{4.2}$$

for all $X, Y \in \Gamma(\mathcal{K})$.

Proposition 4.2 *Let $\mathcal{J}\colon \mathcal{K} \to \mathcal{K}$ be a generalized almost complex structure on an* E-*Courant algebroid $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}}, (\cdot,\cdot)_E, \rho)$. Then we have $\mathcal{J}^\star|_{\mathcal{K}} = -\mathcal{J}$.*

Proof By (4.1), for all $X, Y \in \Gamma(\mathcal{K})$, we have

$$\mathcal{J}^\star(\mathcal{J}(Y))(X) = \mathcal{J}(Y)(\mathcal{J}(X)) = 2\,(\mathcal{J}(X), \mathcal{J}(Y))_E = 2\,(X, Y)_E = Y(X).$$

Since $X \in \Gamma(\mathcal{K})$ is arbitrary, we have

$$\mathcal{J}^\star(\mathcal{J}(Y)) = Y \quad \text{for all } Y \in \Gamma(\mathcal{K}).$$

For any $Z \in \Gamma(\mathcal{K})$, let $Y = -\mathcal{J}(Z)$. By (4.1), we have $Z = \mathcal{J}(Y)$. Then we have

$$\mathcal{J}^\star(Z) = \mathcal{J}^\star(\mathcal{J}(Y)) = Y = -\mathcal{J}(Z),$$

which implies that $\mathcal{J}^\star|_{\mathcal{K}} = -\mathcal{J}$. ∎

Remark 4.3 Generalized complex structures on an E-Courant algebroid $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}}, (\cdot,\cdot)_E, \rho)$ are in one-to-one correspondence with Dirac sub-bundles $S \subset \mathcal{K} \otimes \mathbb{C}$ such that $\mathcal{K} \otimes \mathbb{C} = S \oplus \overline{S}$. By a Dirac sub-bundle of $\mathcal{K}$, we mean a sub-bundle $S \subset \mathcal{K}$ that is closed under the bracket $[\cdot,\cdot]_{\mathcal{K}}$ and satisfies $S = S^\perp$. The pair $(S, \overline{S})$ is an E-Lie bialgebroid in the sense of [5].

Remark 4.4 Obviously, the notion of a generalized contact bundle associated with L, which was introduced in [23], is a special case of Definition 4.1, where E is the line bundle L. In particular, if E is the trivial line bundle $L^\circ = M \times \mathbb{R}$, we have

$$\mathfrak{D}L^\circ = TM \oplus \mathbb{R}, \quad \mathfrak{J}L^\circ = T^*M \oplus \mathbb{R}.$$

Therefore, $\mathcal{E}^1(M) = \mathfrak{D}L^\circ \oplus \mathfrak{J}L^\circ$. Thus, a generalized complex structure on an E-Courant algebroid unifies generalized complex structures on even-dimensional manifolds and generalized contact bundles on odd-dimensional manifolds

Example 4.5 Consider the E-Courant algebroid $A^* \otimes E \oplus A$ given in [5, Example 2.9] for any Lie algebroid $(A, [\cdot,\cdot]_A, a)$ and an A-module E. Twisted by a 3-cocycle $\Theta \in \Gamma(\wedge^3 A^*, E)$, one obtains the AV-Courant algebroid introduced in [14] by Li-Bland. Consider $\mathcal{J}$ of the form $\mathcal{J}_D = \begin{pmatrix} -R_D & 0 \\ 0 & D \end{pmatrix}$, where $D \in \mathrm{gl}(A)$ and $R_D: A^* \otimes E \to A^* \otimes E$ is given by $R_D(\phi) = \phi \circ D$. We get that $\mathcal{J}$ is a generalized complex structure on the E-Courant algebroid $A^* \otimes E \oplus A$ if and only if D is a Nijenhuis operator on the Lie algebroid A and $D^2 = -1$.

Actually, $D^2 = -1$ ensures that condition (4.1) holds. The Dorfman bracket on $\mathcal{K} = A^* \otimes E \oplus A$ is given by

$$[u + \Phi, v + \Psi]_{\mathcal{K}} = [u, v]_A + \mathcal{L}_u \Psi - \mathcal{L}_v \Phi + \rho^\star \mathbb{d}\Phi(v)$$

for all $u, v \in \Gamma(A)$, $\Phi, \Psi \in \Gamma(A^* \otimes E)$, where $\rho^\star: \mathfrak{J}E \to A^* \otimes E$ is the dual of the A-action $\rho: A \to \mathfrak{D}E$ on E. Then it is straightforward to see that the integrability condition (4.2) holds if and only if D is a Nijenhuis operator on A.

Any generalized complex structure on a Courant algebroid induces a Poisson structure on the base manifold (see *e.g.,* [1]). Similarly, any generalized complex structure on an E-Courant algebroid induces a Lie algebroid or a local Lie algebra structure ([12]) on E .

Theorem 4.6 *Let $\mathcal{J}: \mathcal{K} \to \mathcal{K}$ be a generalized complex structure on an E-Courant algebroid $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}}, (\cdot,\cdot)_E, \rho)$. Define a bracket operation $[\cdot,\cdot]_E: \Gamma(E) \wedge \Gamma(E) \to \Gamma(E)$ by*

$$[u, v]_E \triangleq 2(\mathcal{J}\rho^\star \mathbb{d}u, \rho^\star \mathbb{d}v)_E = (\rho \circ \mathcal{J} \circ \rho^\star)(\mathbb{d}u)(v) \quad \textit{for all } u, v \in \Gamma(E). \tag{4.3}$$

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Then $(E, [\cdot,\cdot]_E, \mathbb{j}\circ\rho\circ\mathcal{J}\circ\rho^\star\circ\mathbb{d})$ *is a Lie algebroid when* $\mathrm{rank}(E)\geq 2$ *and* $(E, [\cdot,\cdot]_E)$ *is a local Lie algebra when* $\mathrm{rank}(E)=1$.

Proof The bracket is obviously skew-symmetric. By the integrability of $\mathcal{J}$, we have

$$[\mathcal{J}(\rho^\star \mathbb{d}u), \mathcal{J}(\rho^\star \mathbb{d}v)]_{\mathcal{K}} - [\rho^\star \mathbb{d}u, \rho^\star \mathbb{d}v]_{\mathcal{K}} - \mathcal{J}\big([\mathcal{J}(\rho^\star \mathbb{d}u), \rho^\star \mathbb{d}v]_{\mathcal{K}} + [\rho^\star \mathbb{d}u, \mathcal{J}(\rho^\star \mathbb{d}v)]_{\mathcal{K}}\big) = 0.$$

Pairing with $\rho^\star \mathbb{d}w$ for $w\in\Gamma(E)$, by (EC-3) in Definition 2.1 and the first equation in Lemma 2.2, we have

$$\begin{aligned}
&([\mathcal{J}(\rho^\star \mathbb{d}u), \mathcal{J}(\rho^\star \mathbb{d}v)]_{\mathcal{K}}, \rho^\star \mathbb{d}w)_E\\
&\quad= \rho(\mathcal{J}\rho^\star \mathbb{d}u)\,(\mathcal{J}\rho^\star \mathbb{d}v, \rho^\star \mathbb{d}w)_E - (\mathcal{J}\rho^\star \mathbb{d}v, [\mathcal{J}\rho^\star \mathbb{d}u, \rho^\star \mathbb{d}w]_{\mathcal{K}})_E\\
&\quad= 2\left(\rho^\star \mathbb{d}\,(\mathcal{J}\rho^\star \mathbb{d}v, \rho^\star \mathbb{d}w)_E, \mathcal{J}\rho^\star \mathbb{d}u\right)_E - 2\left(\mathcal{J}\rho^\star \mathbb{d}v, \rho^\star \mathbb{d}\,(\mathcal{J}\rho^\star \mathbb{d}u, \rho^\star \mathbb{d}w)_E\right)_E\\
&\quad= \frac{1}{2}\big[u, [v,w]_E\big]_E - \frac{1}{2}\big[v, [u,w]_E\big]_E.
\end{aligned}$$

By (EC-1) and (EC-5) in Definition 2.1, we have

$$\big([\rho^\star \mathbb{d}u, \rho^\star \mathbb{d}v]_{\mathcal{K}}, \rho^\star \mathbb{d}w\big)_E = 0.$$

Finally, using Lemma 2.2, we have

$$\begin{aligned}
&\big([\mathcal{J}(\rho^\star \mathbb{d}u), \rho^\star \mathbb{d}v]_{\mathcal{K}} + [\rho^\star \mathbb{d}u, \mathcal{J}(\rho^\star \mathbb{d}v)]_{\mathcal{K}}, \mathcal{J}\rho^\star \mathbb{d}w\big)_E\\
&\quad= 2\big(\rho^\star \mathbb{d}\left(\mathcal{J}\rho^\star \mathbb{d}u, \rho^t \star \mathbb{d}v\right)_E, \mathcal{J}\rho^\star \mathbb{d}w\big)_E + 0\\
&\quad= \frac{1}{2}\big[w, [u,v]_E\big]_E.
\end{aligned}$$

Thus, we get the Jacobi identity for $[\cdot,\cdot]_E$. To see the Leibniz rule, by definition, we have

$$[u, fv]_E = f[u,v]_E + \mathbb{j}\rho\mathcal{J}\rho^\star\mathbb{d}(u)(f)v.$$

So it is a Lie algebroid structure if and only if $\mathbb{j}\circ\rho\circ\mathcal{J}\circ\rho^\star\circ\mathbb{d}\colon E\to TM$ is a bundle map, which is always true when $\mathrm{rank}(E)\geq 2$ (see the proof of [4, Theorem 3.11]). ∎

5 Generalized Complex Structures on Omni-Lie Algebroids

In this section, we study generalized complex structures on the omni-Lie algebroid $\mathfrak{ol}(E)$. We view $\mathfrak{ol}(E)$ as a sub-bundle of $\mathrm{Hom}(\mathfrak{ol}(E), E)$ by the nondegenerate E-valued pairing $(\cdot,\cdot)_E$, *i.e.*,

$$e_2(e_1) \triangleq 2\,(e_1, e_2)_E \quad \text{for all } e_1, e_2\in\Gamma(\mathfrak{ol}(E)).$$

By Proposition 4.2, we have the following corollary.

Corollary 5.1 *A bundle map* $\mathcal{J}\colon \mathfrak{ol}(E)\to\mathfrak{ol}(E)$ *is a generalized almost complex structure on the omni-Lie algebroid* $\mathfrak{ol}(E)$ *if and only if the following conditions are satisfied:*

$$\mathcal{J}^2 = -\mathrm{id}, \quad \mathcal{J}^\star|_{\mathfrak{ol}(E)} = -\mathcal{J}.$$

Since $\mathfrak{ol}(E)$ is the direct sum of $\mathfrak{D}E$ and $\mathfrak{J}E$, we can write a generalized almost complex structure $\mathcal{J}$ in the form of a matrix. To do that requires some preparation.

Vector bundles $\mathrm{Hom}(\wedge^k\mathfrak{D}E, E)_{\mathfrak{J}E}$ and $\mathrm{Hom}(\wedge^k\mathfrak{J}E, E)_{\mathfrak{D}E}$ are introduced in [5, 21] to study deformations of omni-Lie algebroids and deformations of Lie algebroids respectively. More precisely, we have

$$\mathrm{Hom}(\wedge^k\mathfrak{D}E, E)_{\mathfrak{J}E} \triangleq \left\{\mu \in \mathrm{Hom}(\wedge^k\mathfrak{D}E, E) \mid \mathrm{Im}(\mu_\natural) \subset \mathfrak{J}E\right\}, \qquad (k \geq 2),$$
$$\mathrm{Hom}(\wedge^k\mathfrak{J}E, E)_{\mathfrak{D}E} \triangleq \left\{\mathfrak{d} \in \mathrm{Hom}(\wedge^k\mathfrak{J}E, E) \mid \mathrm{Im}(\mathfrak{d}^\sharp) \subset \mathfrak{D}E\right\}, \qquad (k \geq 2),$$

in which $\mu_\natural\colon \wedge^{k-1}\mathfrak{D}E \to \mathrm{Hom}(\mathfrak{D}E, E)$ is given by

$$\mu_\natural(\mathfrak{d}_1, \dots, \mathfrak{d}_{k-1})(\mathfrak{d}_k) = \mu(\mathfrak{d}_1, \dots, \mathfrak{d}_{k-1}, \mathfrak{d}_k) \quad \text{for } \mathfrak{d}_1, \dots, \mathfrak{d}_k \in \mathfrak{D}E,$$

and $\mathfrak{d}^\sharp$ is defined similarly. By (2.2), for any $\mu \in \mathrm{Hom}(\wedge^k\mathfrak{D}E, E)_{\mathfrak{J}E}$, we have

$$\mu(\mathfrak{d}_1, \dots, \mathfrak{d}_{k-1}, \Phi) = \Phi \circ \mu(\mathfrak{d}_1, \dots, \mathfrak{d}_{k-1}, \mathrm{id}_E). \tag{5.1}$$

Furthermore, $(\Gamma(\mathrm{Hom}(\wedge^\bullet\mathfrak{D}E, E)_{\mathfrak{J}E}), \mathbb{d})$ is a subcomplex of $(\Gamma(\mathrm{Hom}(\wedge^\bullet\mathfrak{D}E, E), \mathbb{d})$, where $\mathbb{d}$ is the coboundary operator of the gauge Lie algebroid $\mathfrak{D}E$ with the obvious action on E.

Proposition 5.2 *Any generalized almost complex structure $\mathcal{J}$ on the omni-Lie algebroid $\mathfrak{ol}(E)$ must be of the form*

$$\mathcal{J} = \begin{pmatrix} N & \pi^\sharp \\ \sigma_\natural & -N^\star \end{pmatrix}, \tag{5.2}$$

where $N\colon \mathfrak{D}E \to \mathfrak{D}E$ is a bundle map satisfying

$$N^\star(\mathfrak{J}E) \subset \mathfrak{J}E, \quad \pi \in \Gamma\big(\mathrm{Hom}(\wedge^2\mathfrak{J}E, E)_{\mathfrak{D}E}\big), \quad \sigma \in \Gamma\big(\mathrm{Hom}(\wedge^2\mathfrak{D}E, E)_{\mathfrak{J}E}\big)$$

such that the following conditions hold:

$$\pi^\sharp \circ \sigma_\natural + N^2 = -\mathrm{id}, \quad N \circ \pi^\sharp = \pi^\sharp \circ N^\star, \quad \sigma_\natural \circ N = N^\star \circ \sigma_\natural.$$

Proof By Corollary 5.1, for any generalized almost complex structure $\mathcal{J}$, we have $\mathcal{J}^\star|_{\mathfrak{ol}(E)} = -\mathcal{J}$. Thus, $\mathcal{J}$ must be of the form

$$\mathcal{J} = \begin{pmatrix} N & \phi \\ \psi & -N^\star \end{pmatrix},$$

where $N\colon \mathfrak{D}E \to \mathfrak{D}E$ is a bundle map satisfying $N^\star(\mathfrak{J}E) \subset \mathfrak{J}E$, $\phi\colon \mathfrak{J}E \to \mathfrak{D}E$ and $\psi\colon \mathfrak{D}E \to \mathfrak{J}E$ are bundle maps satisfying

$$-(\phi(\mu), \nu)_E = (\mu, \phi(\nu))_E, \qquad -(\psi(\mathfrak{d}), \mathfrak{t})_E = (\mathfrak{d}, \psi(\mathfrak{t}))_E.$$

Therefore, we have $\phi = \pi^\sharp$ for some $\pi \in \Gamma(\mathrm{Hom}(\wedge^2\mathfrak{J}E, E)_{\mathfrak{D}E})$, and $\psi = \sigma_\natural$ for some $\sigma \in \Gamma(\mathrm{Hom}(\wedge^2\mathfrak{D}E, E)_{\mathfrak{J}E})$. This finishes the proof of the first part. As for the second part, it is straightforward to see that the conditions follow from the fact that $\mathcal{J}^2 = -\mathrm{id}$. ∎

Remark 5.3 A line bundle L satisfies $\mathfrak{J}L = \mathrm{Hom}(\mathfrak{D}L, L)$ and $\mathfrak{D}L = \mathrm{Hom}(\mathfrak{J}L, L)$. Therefore, the condition $N^\star(\mathfrak{J}L) \subset \mathfrak{J}L$ always holds.

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Theorem 5.4 *A generalized almost complex structure $\mathcal{J}$ given by (5.2) is a generalized complex structure on the omni-Lie algebroid $\mathfrak{ol}(E)$ if and only if the following hold:*

(i) *π satisfies the equation*

$$\pi^\sharp([\mu,\nu]_\pi) = [\pi^\sharp(\mu),\pi^\sharp(\nu)]_{\mathfrak{D}} \quad \text{for all } \mu,\nu\in\Gamma(\mathfrak{J}E), \tag{5.3}$$

where the bracket $[\cdot,\cdot]_\pi$ on $\Gamma(\mathfrak{J}E)$ is defined by

$$[\mu,\nu]_\pi \triangleq \mathfrak{L}_{\pi^\sharp(\mu)}\nu - \mathfrak{L}_{\pi^\sharp(\nu)}\mu - \mathbb{d}\left\langle\pi^\sharp(\mu),\nu\right\rangle_E. \tag{5.4}$$

(ii) *π and N are related by the formula*

$$N^\star([\mu,\nu]_\pi) = \mathfrak{L}_{\pi^\sharp(\mu)}(N^\star(\nu)) - \mathfrak{L}_{\pi^\sharp(\nu)}(N^\star(\mu)) - \mathbb{d}\pi(N^\star(\mu),\nu). \tag{5.5}$$

(iii) *N satisfies the condition*

$$T(N)(\mathfrak{d},\mathfrak{t}) = \pi^\sharp(i_{\mathfrak{d}\wedge\mathfrak{t}}\mathbb{d}\sigma) \quad \text{for all } \mathfrak{d},\mathfrak{t}\in\Gamma(\mathfrak{D}E), \tag{5.6}$$

where $T(N)$ is the Nijenhuis tensor of N defined by

$$T(N)(\mathfrak{d},\mathfrak{t}) = [N(\mathfrak{d}),N(\mathfrak{t})]_{\mathfrak{D}} - N([N(\mathfrak{d}),\mathfrak{t}]_{\mathfrak{D}} + [\mathfrak{d},N(\mathfrak{t})]_{\mathfrak{D}} - N[\mathfrak{d},\mathfrak{t}]_{\mathfrak{D}}).$$

(iv) *N and σ are related by the following condition*

$$\mathbb{d}\sigma(N(\mathfrak{d}),\mathfrak{t},\mathfrak{k}) + \mathbb{d}\sigma(\mathfrak{d},N(\mathfrak{t}),\mathfrak{k}) + \mathbb{d}\sigma(\mathfrak{d},\mathfrak{t},N(\mathfrak{k})) = \mathbb{d}\sigma_N(\mathfrak{d},\mathfrak{t},\mathfrak{k}), \tag{5.7}$$

for all $\mathfrak{d},\mathfrak{t},\mathfrak{k}\in\Gamma(\mathfrak{D}E)$, where $\sigma_N\in\Gamma(\mathrm{Hom}(\wedge^2\mathfrak{D}E,E)_{\mathfrak{J}E})$ is defined by $\sigma_N(\mathfrak{d},\mathfrak{t}) = \sigma(N(\mathfrak{d}),\mathfrak{t})$.

Proof Consider the integrability condition (4.2). In fact, there are two equations since $\Gamma(\mathfrak{ol}(E))$ has two components $\Gamma(\mathfrak{D}E)$ and $\Gamma(\mathfrak{J}E)$. First let $e_1=\mu$, $e_2=\nu$ be elements in $\Gamma(\mathfrak{J}E)$; then we have $\mathcal{J}(\mu)=\pi^\sharp(\mu)-N^\star(\mu)$, $\mathcal{J}(\nu)=\pi^\sharp(\nu)-N^\star(\nu)$ and $[\![\mu,\nu]\!]=0$. Therefore, we obtain

$$\begin{aligned}
&[\![\pi^\sharp(\mu)-N^\star(\mu),\pi^\sharp(\nu)-N^\star(\nu)]\!]\\
&\qquad\qquad -\mathcal{J}\big([\![\pi^\sharp(\mu)-N^\star(\mu),\nu]\!]+[\![\mu,\pi^\sharp(\nu)-N^\star(\nu)]\!]\big)\\
&\quad=[\pi^\sharp(\mu),\pi^\sharp(\nu)]_{\mathfrak{D}}-\pi^\sharp(\mathfrak{L}_{\pi^\sharp(\mu)}\nu-i_{\pi^\sharp(\nu)}\mathbb{d}\mu)+N^\star(\mathfrak{L}_{\pi^\sharp(\mu)}\nu-i_{\pi^\sharp(\nu)}\mathbb{d}\mu)\\
&\qquad-\mathfrak{L}_{\pi^\sharp(\mu)}N^\star(\nu)+i_{\pi^\sharp(\nu)}\mathbb{d}N^\star(\mu)=0.
\end{aligned}$$

Thus, we get conditions (5.3) and (5.5).

Then let $e_1=\mathfrak{d}\in\Gamma(\mathfrak{D}E)$ and $e_2=\mu\in\Gamma(\mathfrak{J}E)$; we have $\mathcal{J}(e_1)=N(\mathfrak{d})+\sigma_\natural(\mathfrak{d})$ and $\mathcal{J}(e_2)=\pi^\sharp(\mu)-N^\star(\mu)$. Therefore, we obtain

$$\begin{aligned}
&[\![N(\mathfrak{d})+\sigma_\natural(\mathfrak{d}),\pi^\sharp(\mu)-N^\star(\mu)]\!]-[\![\mathfrak{d},\mu]\!]\\
&\qquad\qquad -\mathcal{J}\big([\![N(\mathfrak{d})+\sigma_\natural(\mathfrak{d})),\mu]\!]+[\![\mathfrak{d},\pi^\sharp(\mu)-N^\star(\mu)]\!]\big)\\
&\quad=[N(\mathfrak{d}),\pi^\sharp(\mu)]_{\mathfrak{D}}-N[\mathfrak{d},\pi^\sharp(\mu)]_{\mathfrak{D}}-\pi^\sharp\big(\mathfrak{L}_{N(\mathfrak{d})}\mu-\mathfrak{L}_{\mathfrak{d}}N^\star(\mu)\big)\\
&\qquad+N^\star(\mathfrak{L}_{N(\mathfrak{d})}\mu-\mathfrak{L}_{\mathfrak{d}}N^\star(\mu))-\mathfrak{L}_{N(\mathfrak{d})}N^\star(\mu)-i_{\pi^\sharp(\mu)}\mathbb{d}\sigma_\natural(\mathfrak{d})\\
&\qquad-\mathfrak{L}_{\mathfrak{d}}\mu-\sigma_\natural[\mathfrak{d},\pi^\sharp(\mu)]_{\mathfrak{D}}=0.
\end{aligned}$$

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Thus, we have

$$[N(\mathfrak{d}),\pi^\sharp(\mu)]_{\mathfrak{D}} = N[\mathfrak{d},\pi^\sharp(\mu)]_{\mathfrak{D}} + \pi^\sharp(\mathfrak{L}_{N(\mathfrak{d})}\mu - \mathfrak{L}_{\mathfrak{d}}N^\star(\mu)), \tag{5.8}$$

$$N^\star(\mathfrak{L}_{N(\mathfrak{d})}\mu - \mathfrak{L}_{\mathfrak{d}}N^\star(\mu)) = \mathfrak{L}_{N(\mathfrak{d})}N^\star(\mu) + i_{\pi^\sharp(\mu)}\mathbb{d}\sigma_\natural(\mathfrak{d}) + \mathfrak{L}_{\mathfrak{d}}\mu + \sigma_\natural[\mathfrak{d},\pi^\sharp(\mu)]_{\mathfrak{D}}. \tag{5.9}$$

We claim that (5.8) is equivalent to (5.5). In fact, applying (5.8) to $\nu \in \Gamma(\mathfrak{J}E)$ and (5.5) to $\mathfrak{d} \in \Gamma(\mathfrak{D}E)$, we get the same equality.

Next let $e_1 = \mathfrak{d}$ and $e_2 = \mathfrak{t}$ be elements in $\Gamma(\mathfrak{D}E)$; we have $\mathcal{J}(e_1) = N(\mathfrak{d}) + \sigma_\natural(\mathfrak{d})$ and $\mathcal{J}(e_2) = N(\mathfrak{t}) + \sigma_\natural(\mathfrak{t})$. Therefore, we have

$$\begin{aligned}
&[[N(\mathfrak{d}) + \sigma_\natural(\mathfrak{d}), N(\mathfrak{t}) + \sigma_\natural(\mathfrak{t})]]\\
&\qquad - [\mathfrak{d},\mathfrak{t}]_{\mathfrak{D}} - \mathcal{J}([[N(\mathfrak{d}) + \sigma_\natural(\mathfrak{d}),\mathfrak{t}]] + [[\mathfrak{d}, N(\mathfrak{t}) + \sigma_\natural(\mathfrak{t})]])\\
&= [N(\mathfrak{d}),N(\mathfrak{t})]_{\mathfrak{D}} - [\mathfrak{d},\mathfrak{t}]_{\mathfrak{D}} - N([N(\mathfrak{d}),\mathfrak{t}]_{\mathfrak{D}} + [\mathfrak{d},N(\mathfrak{t})]_{\mathfrak{D}}) - \pi^\sharp(\mathfrak{L}_{\mathfrak{d}}\sigma_\natural(\mathfrak{t})\\
&\quad - i_{\mathfrak{t}}\mathbb{d}\sigma_\natural(\mathfrak{d})) + \mathfrak{L}_{N(\mathfrak{d})}\sigma_\natural(\mathfrak{t}) - i_{N(\mathfrak{t})}\mathbb{d}\sigma_\natural(\mathfrak{d}) - \sigma_\natural([N(\mathfrak{d}),\mathfrak{t}]_{\mathfrak{D}} + [\mathfrak{d},N(\mathfrak{t})]_{\mathfrak{D}})\\
&\quad + N^\star(\mathfrak{L}_{\mathfrak{d}}\sigma_\natural(\mathfrak{t}) - i_{\mathfrak{t}}\mathbb{d}\sigma_\natural(\mathfrak{d}))\\
&= 0.
\end{aligned}$$

Thus, we have

$$[N(\mathfrak{d}),N(\mathfrak{t})]_{\mathfrak{D}} - [\mathfrak{d},\mathfrak{t}]_{\mathfrak{D}} - N([N(\mathfrak{d}),\mathfrak{t}]_{\mathfrak{D}} + [\mathfrak{d},N(\mathfrak{t})]_{\mathfrak{D}}) = \pi^\sharp(\mathfrak{L}_{\mathfrak{d}}\sigma_\natural(\mathfrak{t}) - i_{\mathfrak{t}}\mathbb{d}\sigma_\natural(\mathfrak{d})), \tag{5.10}$$

$$\sigma_\natural([N(\mathfrak{d}),\mathfrak{t}]_{\mathfrak{D}} + [\mathfrak{d},N(\mathfrak{t})]_{\mathfrak{D}}) - \mathfrak{L}_{N(\mathfrak{d})}\sigma_\natural(\mathfrak{t}) + i_{N(\mathfrak{t})}\mathbb{d}\sigma_\natural(\mathfrak{d}) = N^\star(\mathfrak{L}_{\mathfrak{d}}\sigma_\natural(\mathfrak{t}) - i_{\mathfrak{t}}\mathbb{d}\sigma_\natural(\mathfrak{d})). \tag{5.11}$$

We claim that (5.9) and (5.10) are equivalent. In fact, applying (5.9) and (5.10) to $\mathfrak{t} \in \Gamma(\mathfrak{D}E)$ and $\mu \in \Gamma(\mathfrak{J}E)$, respectively, we get the same equality

$$\begin{aligned}
&\langle [N(\mathfrak{d}),N(\mathfrak{t})]_{\mathfrak{D}} - [\mathfrak{d},\mathfrak{t}]_{\mathfrak{D}} - N([N(\mathfrak{d}),\mathfrak{t}]_{\mathfrak{D}} + [\mathfrak{d},N(\mathfrak{t})]_{\mathfrak{D}}),\mu\rangle_E\\
&\quad = \mathfrak{d}\left\langle \pi^\sharp\sigma_\sharp(\mathfrak{t}),\mu\right\rangle_E + \left\langle \sigma_\sharp(\mathfrak{t}),[\mathfrak{d},\pi^\sharp\mu]_{\mathfrak{D}}\right\rangle_E + \mathfrak{t}\left\langle \sigma_\sharp(\mathfrak{d}),\pi^\sharp(\mu)\right\rangle_E - \pi^\sharp(\mu)\left\langle \sigma_\sharp(\mathfrak{d}),\mathfrak{t}\right\rangle_E\\
&\qquad - \left\langle \sigma_\sharp(\mathfrak{d}),[\mathfrak{t},\pi^\sharp(\mu)]_{\mathfrak{D}}\right\rangle_E.
\end{aligned}$$

By the equality $\pi^\sharp \circ \sigma_\natural + N^2 = -\mathrm{id}$ and (5.10), we have

$$[N(\mathfrak{d}),N(\mathfrak{t})]_{\mathfrak{D}} + N^2[\mathfrak{d},\mathfrak{t}]_{\mathfrak{D}} - N([N(\mathfrak{d}),\mathfrak{t}]_{\mathfrak{D}} + [\mathfrak{d},N(\mathfrak{t})]_{\mathfrak{D}}) = \pi^\sharp(\mathfrak{L}_{\mathfrak{d}}\sigma_\natural(\mathfrak{t}) - i_{\mathfrak{t}}\mathbb{d}\sigma_\natural(\mathfrak{d}) - \sigma_\natural[\mathfrak{d},\mathfrak{t}]_{\mathfrak{D}}),$$

which implies that $T(N)(\mathfrak{d},\mathfrak{t}) = \pi^\sharp(i_{\mathfrak{d}\wedge\mathfrak{t}}\mathbb{d}\sigma)$. Thus, (5.10) is equivalent to (5.6).

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Finally, we consider condition (5.11). Acting on an arbitrary $\mathfrak{k} \in \Gamma(\mathfrak{D}E)$, we have

$$\begin{aligned} N(\mathfrak{d})\langle\sigma_\natural(\mathfrak{t}),\mathfrak{k}\rangle_E - \langle\sigma_\natural(\mathfrak{t}),[N(\mathfrak{d}),\mathfrak{k}]_\mathfrak{D}\rangle_E + \langle\sigma_\natural(\mathfrak{k}),[N(\mathfrak{d}),\mathfrak{t}]_\mathfrak{D}\rangle_E - N(\mathfrak{t})\langle\sigma_\natural(\mathfrak{d}),\mathfrak{k}\rangle_E \\ + \mathfrak{k}\langle\sigma_\natural(\mathfrak{d}),\mathfrak{t}\rangle_E + \langle\sigma_\natural(\mathfrak{d}),[N(\mathfrak{t}),\mathfrak{k}]_\mathfrak{D}\rangle_E + \langle\sigma_\natural(\mathfrak{k}),[\mathfrak{d},N(\mathfrak{t})]_\mathfrak{D}\rangle_E \\ + \mathfrak{d}\langle\sigma_\natural(\mathfrak{t}),N(\mathfrak{k})\rangle_E - \langle\sigma_\natural(\mathfrak{t}),[\mathfrak{d},N(\mathfrak{k})]_\mathfrak{D}\rangle_E - \mathfrak{t}\langle\sigma_\natural(\mathfrak{d}),N(\mathfrak{k})\rangle_E \\ + N(\mathfrak{k})\langle\sigma_\natural(\mathfrak{d}),\mathfrak{t}\rangle_E + \langle\sigma_\natural(\mathfrak{d}),[\mathfrak{t},N(\mathfrak{k})]_\mathfrak{D}\rangle_E \\ = \mathbb{d}\sigma(N(\mathfrak{d}),\mathfrak{t},\mathfrak{k}) + \mathfrak{t}\sigma(N(\mathfrak{d}),\mathfrak{k}) - \mathfrak{k}\sigma(N(\mathfrak{d}),\mathfrak{t}) + \sigma([\mathfrak{t},\mathfrak{k}]_\mathfrak{D},N(\mathfrak{d})) \\ + \mathbb{d}\sigma(\mathfrak{d},N(\mathfrak{t}),\mathfrak{k}) - \mathfrak{d}\sigma(N(\mathfrak{t}),\mathfrak{k}) - \sigma([\mathfrak{d},\mathfrak{k}]_\mathfrak{D},N(\mathfrak{t})) \\ + \mathbb{d}\sigma(\mathfrak{d},\mathfrak{t},N(\mathfrak{k})) + \sigma([\mathfrak{d},\mathfrak{t}]_\mathfrak{D},N(\mathfrak{k})) \\ = 0. \end{aligned}$$

Note that the following equality holds:

$$\begin{aligned} \sigma(\mathfrak{d},N(\mathfrak{t})) &= -\langle\sigma_\natural(N(\mathfrak{t})),\mathfrak{d}\rangle_E = -\langle N^\star(\sigma_\natural(\mathfrak{t})),\mathfrak{d}\rangle_E \\ &= -\langle\sigma_\natural(\mathfrak{t}),N(\mathfrak{d})\rangle_E = \sigma(N(\mathfrak{d}),\mathfrak{t}). \end{aligned}$$

Therefore, we have

$$(i_N\mathbb{d}\sigma)(\mathfrak{d},\mathfrak{t},\mathfrak{k}) = \mathbb{d}\sigma_N(\mathfrak{d},\mathfrak{t},\mathfrak{k}),$$

which implies that (5.11) is equivalent to (5.7). ∎

Remark 5.5 Let $\mathcal{J} = \begin{pmatrix} N & \pi^\sharp \\ \sigma_\natural & -N^\star \end{pmatrix}$ be a generalized complex structure on the omni-Lie algebroid $\mathfrak{ol}(E)$. Then π satisfies (5.3). On one hand, in [4], the authors showed that such π will give rise to a Lie bracket $[\cdot,\cdot]_E$ on $\Gamma(E)$ via

$$[u,v]_E = \pi^\sharp(\mathbb{d}u)(v) \quad \text{for all } u,v \in \Gamma(E).$$

On the other hand, by Theorem 4.6, the generalized complex structure $\mathcal{J}$ will also induce a Lie algebroid structure on E by (4.3). By the equality

$$\pi^\sharp = \rho \circ \mathcal{J} \circ \rho^\star,$$

these two Lie algebroid structures on E are the same.

Remark 5.6 Recall that any $b \in \Gamma(\mathrm{Hom}(\wedge^2\mathfrak{D}E,E)_{\mathfrak{J}E})$ defines a transformation $e^b: \mathfrak{ol}(E) \to \mathfrak{ol}(E)$, defined by

$$e^b\begin{pmatrix}\mathfrak{d}\\ \mu\end{pmatrix} = \begin{pmatrix}\mathrm{id} & 0\\ b_\natural & \mathrm{id}\end{pmatrix}\begin{pmatrix}\mathfrak{d}\\ \mu\end{pmatrix} = \begin{pmatrix}c\mathfrak{d}\\ \mu + i_\mathfrak{d} b\end{pmatrix}.$$

Thus, e^b is an automorphism of the omni-Lie algebroid $\mathfrak{ol}(E)$ if and only if $\mathbb{d}b = 0$. In this case, e^b is called a *B-field transformation*. Actually, an automorphism of the omni-Lie algebroid $\mathfrak{ol}(E)$ is just the composition of an automorphism of the vector bundle E and a B-field transformation. In fact, B-field transformations map generalized complex structures on $\mathfrak{ol}(E)$ into new generalized complex structures as follows:

$$\mathcal{J}^b = \begin{pmatrix}\mathrm{id} & 0\\ b_\natural & \mathrm{id}\end{pmatrix} \circ \mathcal{J} \circ \begin{pmatrix}\mathrm{id} & 0\\ -b_\natural & \mathrm{id}\end{pmatrix}.$$

Example 5.7 Let $D: E \to E$ be a bundle map satisfying $D^2 = -\mathrm{id}$. Define $R_D: \mathfrak{D}E \to \mathfrak{D}E$ by $R_D(\mathfrak{d}) = \mathfrak{d} \circ D$ and $\widehat{D}: \mathfrak{J}E \to \mathfrak{J}E$ by $\widehat{D}(\mathbb{d}u) = \mathbb{d}(Du)$ for $u \in \Gamma(E)$. Then

$$\mathcal{J} = \begin{pmatrix} R_D & 0 \\ 0 & -\widehat{D} \end{pmatrix}$$

is a generalized complex structure on $\mathfrak{ol}(E)$. In fact, since

$$\langle R_D^\star(\mathbb{d}u), \mathfrak{d} \rangle_E = \langle \mathbb{d}u, \mathfrak{d} \circ D \rangle_E = \mathfrak{d}(D(u)) = \left\langle \widehat{D}(\mathbb{d}u), \mathfrak{d} \right\rangle_E,$$

we have $R_D^\star = \widehat{D}$. It is straightforward to check that the Nijenhuis tensor $T(R_D)$ vanishes, and the condition $D^2 = -\mathrm{id}$ ensures that $R_D^2 = -\mathrm{id}$.

Let $\pi \in \Gamma(\mathrm{Hom}(\wedge^2 \mathfrak{J}E, E)_{\mathfrak{D}E})$ and suppose that the induced map $\pi^\sharp: \mathfrak{J}E \to \mathfrak{D}E$ is an isomorphism of vector bundles. Then the rank of E is 1 or is equal to the dimension of M. We denote by $(\pi^\sharp)^{-1}$ the inverse of $\pi^\sharp$ and by π^{-1} the corresponding element in $\Gamma(\mathrm{Hom}(\wedge^2 \mathfrak{D}E, E)_{\mathfrak{J}E})$.

Lemma 5.8 *With the above notation, the following two statements are equivalent:*

(i) $\pi \in \Gamma(\mathrm{Hom}(\wedge^2 \mathfrak{J}E, E)_{\mathfrak{D}E})$ *satisfies* (5.3)*;*
(ii) π^{-1} *is closed,* i.e., $\mathbb{d}\pi^{-1} = 0$.

Proof The conclusion follows from the following equality:

$$\left\langle \pi^\sharp([\mu, \nu]_\pi) - [\pi^\sharp(\mu), \pi^\sharp(\nu)]_{\mathfrak{D}}, \gamma \right\rangle_E = -\mathbb{d}\pi^{-1}(\pi^\sharp(\mu), \pi^\sharp(\nu), \pi^\sharp(\gamma)),$$

for all $\mu,\ \nu,\ \gamma \in \Gamma(\mathfrak{J}E)$, which can be obtained by straightforward computations. ∎

Let $(E, [\cdot,\cdot]_E, a)$ be a Lie algebroid. Define $\pi^\sharp: \mathfrak{J}E \to \mathfrak{D}E$ by

$$\pi^\sharp(\mathbb{d}u)(\cdot) = [u, \cdot]_E \quad \text{for all } u \in \Gamma(E). \tag{5.12}$$

Then $\pi^\sharp$ satisfies (5.3). Furthermore, $(\mathfrak{J}E, [\cdot,\cdot]_\pi, \mathbb{j} \circ \pi^\sharp)$ is a Lie algebroid, where the bracket $[\cdot,\cdot]_\pi$ is given by (5.4). By Theorem 5.4 and Lemma 5.8, we have the following corollary.

Corollary 5.9 *Let* $(E, [\cdot,\cdot]_E, a)$ *be a Lie algebroid such that the induced map* $\pi^\sharp: \mathfrak{J}E \to \mathfrak{D}E$ *is an isomorphism. Then*

$$\mathcal{J} = \begin{pmatrix} 0 & \pi^\sharp \\ -(\pi^\sharp)^{-1} & 0 \end{pmatrix}$$

is a generalized complex structure on $\mathfrak{ol}(E)$*.*

Example 5.10 Let $(TM, [\cdot,\cdot]_{TM}, \mathrm{id})$ be the tangent Lie algebroid. Define $\pi^\sharp: \mathfrak{J}(TM) \to \mathfrak{D}(TM)$ by $\pi^\sharp(\mathbb{d}u) = [u, \cdot]_{TM}$. Then $\pi^\sharp$ is an isomorphism. See [4, Corollary 3.9] for details. Then

$$\mathcal{J} = \begin{pmatrix} 0 & \pi^\sharp \\ -(\pi^\sharp)^{-1} & 0 \end{pmatrix}$$

is a generalized complex structure on the omni-Lie algebroid $\mathfrak{ol}(TM)$.

Example 5.11 Let (M,ω) be a symplectic manifold and let $(T^*M,[\cdot,\cdot]_{\omega^{-1}},(\omega^\sharp)^{-1})$ be the associated natural Lie algebroid. Define $\pi^\sharp\colon\mathfrak{J}(T^*M)\to\mathfrak{D}(T^*M)$ by

$$\pi^\sharp(\mathbb{d}u)=[u,\cdot]_{\omega^{-1}},$$

which is an isomorphism (see [4, Corollary 3.10]). Then

$$\mathcal{J}=\begin{pmatrix}0 & \pi^\sharp\\ -(\pi^\sharp)^{-1} & 0\end{pmatrix}$$

is a generalized complex structure on the omni-Lie algebroid $\mathfrak{ol}(T^*M)$.

To conclude this section, we introduce the notion of an algebroid-Nijenhuis structure, which can give rise to generalized complex structures on the omni-Lie algebroid $\mathfrak{ol}(E)$.

Definition 5.12 Let $(E,[\cdot,\cdot]_E,a)$ be a Lie algebroid, let $N\colon\mathfrak{D}E\to\mathfrak{D}E$ be a Nijenhuis operator on the Lie algebroid $(\mathfrak{D}E,[\cdot,\cdot]_{\mathfrak{D}},\mathbb{j})$ satisfying $N^\star(\mathfrak{J}E)\subset\mathfrak{J}E$, and let $\pi\colon\mathfrak{J}E\to\mathfrak{D}E$ be given by (5.12). Then N and π are said to be compatible if

$$N\circ\pi^\sharp=\pi^\sharp\circ N^\star\quad\text{and}\quad C(\pi,N)=0,$$

where

$$C(\pi,N)(\mu,\nu)\triangleq[\mu,\nu]_{\pi_N}-\big([N^\star(\mu),\nu]_\pi+[\mu,N^\star(\nu)]_\pi-N^\star[\mu,\nu]_\pi\big),$$

for all $\mu,\nu\in\Gamma(\mathfrak{J}E)$. Here, $\pi_N\in\Gamma(\mathrm{Hom}(\wedge^2\mathfrak{J}E,E)_{\mathfrak{D}E})$ is given by

$$\pi_N(\mu,\nu)=\big\langle\nu,N\pi^\sharp(\mu)\big\rangle_E\quad\text{for all }\mu,\nu\in\Gamma(\mathfrak{J}E).$$

If N and π are compatible, we call the pair (π,N) an *algebroid-Nijenhuis* structure on the Lie algebroid $(E,[\cdot,\cdot]_E,a)$.

The following lemma is straightforward, so we omit the proof.

Lemma 5.13 *Let $(E,[\cdot,\cdot]_E,a)$ be a Lie algebroid, let π be given by (5.12), and let $N\colon\mathfrak{D}E\to\mathfrak{D}E$ be a Nijenhuis structure. Then (π,N) is an algebroid-Nijenhuis structure on the Lie algebroid $(E,[\cdot,\cdot]_E,a)$ if and only if $N\circ\pi^\sharp=\pi^\sharp\circ N^\star$ and*

$$N^\star[\mu,\nu]_\pi=\mathfrak{L}_{\pi(\mu)}N^\star(\nu)-\mathfrak{L}_{\pi(\nu)}N^\star(\mu)-\mathbb{d}\pi(N^\star(\mu),\nu).$$

By Theorem 5.4 and Lemma 5.13, we have the following theorem.

Theorem 5.14 *Let $(E,[\cdot,\cdot]_E,a)$ be a Lie algebroid, let π be given by (5.12), and let $N\colon\mathfrak{D}E\to\mathfrak{D}E$ be a Nijenhuis structure. Then the following statements are equivalent:*

(i) *(π,N) is an algebroid-Nijenhuis structure and $N^2=-\mathrm{id}$;*

(ii) *$\mathcal{J}=\begin{pmatrix}N & \pi^\sharp\\ 0 & -N^\star\end{pmatrix}$ is a generalized complex structure on the omni-Lie algebroid $\mathfrak{ol}(E)$.*

Remark 5.15 An interesting special case is that where $E=L$ is a line bundle. Then (π,N) becomes a *Jacobi-Nijenhuis structure* on M. Jacobi–Nijenhuis structures were studied by L. Vitagliano and the third author in [24]. In this case, π defines a Jacobi biderivation $\{\cdot,\cdot\}$ of L (*i.e.*, a skew-symmetric bracket that is a first order differential

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operator, hence a derivation, in each argument). Moreover, this bi-derivation is compatible with N in the sense that $\pi^\sharp \circ N^\star = N \circ \pi^\sharp$ and $C(\pi, N) = 0$. It defines a new Jacobi bi-derivation $\{\cdot,\cdot\}_N$. Furthermore, $(\{\cdot,\cdot\},\{\cdot,\cdot\}_N)$ is a *Jacobi bi-Hamiltonian* structure; *i.e.*, $\{\cdot,\cdot\}$, $\{\cdot,\cdot\}_N$ and $\{\cdot,\cdot\}+\{\cdot,\cdot\}_N$ are all Jacobi brackets.

6 Generalized Complex Structures on Omni-Lie Algebras

In this section, we consider the case where E reduces to a vector space V. Then we have

$$\mathfrak{D}V = \mathrm{gl}(V), \quad \mathfrak{J}V = V.$$

Furthermore, the pairing (2.4) reduces to

$$\langle A, u\rangle_V = Au \quad \text{for all } A \in \mathrm{gl}(V),\ u \in V.$$

Any $u \in V$ is a linear map from $\mathrm{gl}(V)$ to V,

$$u(A) = \langle A, u\rangle_V = Au.$$

Therefore, an omni-Lie algebroid reduces to an omni-Lie algebra, which was introduced by Weinstein in [25] to study the linearization of the standard Courant algebroid.

Definition 6.1 An omni-Lie algebra associated with V is a triple

$$\big(\,\mathrm{gl}(V) \oplus V,\ [\![\cdot,\cdot]\!],\ (\cdot,\cdot)_V\,\big),$$

where $(\cdot,\cdot)_V$ is a nondegenerate symmetric pairing given by

$$(A+u, B+v)_V = \tfrac{1}{2}(Av + Bu) \quad \text{for all } A, B \in \mathrm{gl}(V),\ u, v \in V,$$

and $[\![\cdot,\cdot]\!]$ is a bracket operation given by

$$[\![A+u, B+v]\!] = [A, B] + Av.$$

We will simply denote an omni-Lie algebra associated with a vector space V by $\mathfrak{ol}(V)$.

Lemma 6.2 *For any vector space V, we have*

$$\mathrm{Hom}(\wedge^2 \mathrm{gl}(V), V)_V = 0,$$
$$\mathrm{Hom}(\wedge^2 V, V)_{\mathrm{gl}(V)} = \mathrm{Hom}(\wedge^2 V, V).$$

Proof In fact, for any $\phi \in \mathrm{Hom}(\wedge^2 \mathrm{gl}(V), V)_V$ and $A, B \in \mathrm{gl}(V)$, by (5.1), we have

$$\phi(A \wedge B) = B \circ \phi(A \wedge \mathrm{id}_V) = -B \circ A \circ \phi(\mathrm{id}_V \wedge \mathrm{id}_V) = 0.$$

Therefore, $\phi = 0$, which implies that $\mathrm{Hom}(\wedge^2 \mathrm{gl}(V), V)_V = 0$. The second equality is obvious. ∎

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Proposition 6.3 *Any generalized almost complex structure* $\mathfrak{J}: \mathrm{gl}(V) \oplus V \to \mathrm{gl}(V) \oplus V$ *on the omni-Lie algebra* $\mathfrak{ol}(V)$ *is of the form*

$$\begin{pmatrix} -R_D & \pi^\sharp \\ 0 & D \end{pmatrix}, \tag{6.1}$$

where $\pi \in \mathrm{Hom}(\wedge^2 V, V)$*,* $D \in \mathrm{gl}(V)$ *satisfying* $D^2 = -\mathrm{id}_V$ *and* $\pi(Du, v) = \pi(u, Dv)$*, and* $R_D: \mathrm{gl}(V) \to \mathrm{gl}(V)$ *is the right multiplication,* i.e., $R_D(A) = A \circ D$.

Proof By Proposition 5.2 and Lemma 6.2, we can assume that a generalized almost complex structure is of the form $\begin{pmatrix} N & \pi^\sharp \\ 0 & -N^\star \end{pmatrix}$, where $N: \mathrm{gl}(V) \to \mathrm{gl}(V)$ satisfies $N^\star \in \mathrm{gl}(V)$ and $N^2 = -\mathrm{id}_{\mathrm{gl}(V)}$, and $\pi \in \mathrm{Hom}(\wedge^2 V, V)$. Let $D = -N^\star$; then we have

$$(Dv)(A) = -N^\star(v)(A) = -v(N(A)) = -N(A)v.$$

On the other hand, we have $(Dv)(A) = ADv$, which implies that $N(A) = -R_D(A)$. It is obvious that $N^2 = -\mathrm{id}_{\mathrm{gl}(V)}$ is equivalent to $D^2 = -\mathrm{id}_V$. The proof is complete. ∎

Theorem 6.4 *A generalized almost complex structure* $\mathfrak{J}: \mathrm{gl}(V) \oplus V \to \mathrm{gl}(V) \oplus V$ *on the omni-Lie algebra* $\mathfrak{ol}(V)$ *given by* (6.1) *is a generalized complex structure if and only if the following hold:*

(i) π *defines a Lie algebra structure* $[\cdot, \cdot]_\pi$ *on* V*;*
(ii) $D^2 = -\mathrm{id}_V$ *and* $D[u, v]_\pi = [u, Dv]_\pi$ *for* $u, v \in V$.

Thus, a generalized complex structure on the omni-Lie algebra $\mathfrak{ol}(V)$ *gives rise to a complex Lie algebra structure on* V.

Proof By Theorem 5.4, we have

$$[u, v]_\pi = \pi^\sharp(u)(v) = \pi(u, v).$$

Condition (5.3) implies that $[\cdot, \cdot]_\pi$ gives a Lie algebra structure on V. Condition (5.5) implies that $D[u, v]_\pi = [u, Dv]_\pi$. The other conditions are valid.

The conditions $D^2 = -\mathrm{id}_V$ and $D[u, v]_\pi = [u, Dv]_\pi$ say by definition that D is a complex Lie algebra structure on $(V, [\cdot, \cdot]_\pi)$. This finishes the proof. ∎

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