OUTER AUTOMORPHISMS OF HYPERCENTRAL p-GROUPS

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1. Introduction. In his celebrated paper [**3**] Gaschütz proved that any finite non-cyclic p-group always admits non-inner automorphisms of order a power of p. In particular this implies that, if G is a finite nilpotent group of order bigger than 2, then $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G) \neq 1$. Here, as usual, we denote by $\mathrm{Aut}(G)$ the full group of automorphisms of G while $\mathrm{Inn}(G)$ stands for the group of *inner* automorphisms, that is automorphisms induced by conjugation by elements of G. After Gaschütz proved this result, the following question was raised: "if G is an infinite nilpotent group, is it always true that $\mathrm{Out}(G) \neq 1$?"

This question was answered in the negative by Zalesskiĭ in [**8**] where he constructed a torsion-free nilpotent group of nilpotency class 2, without non-inner automorphisms. Hence, in the infinite case, the hypothesis of nilpotency is not sufficient to ensure the existence of automorphisms which are non-inner. Nevertheless these automorphisms exist if the infinite nilpotent group is a p-group, as was proved by Zalesskiĭ himself in [**9**]. But, after Buckley and Wiegold determined the cardinality of $\mathrm{Aut}(G)$ for G an infinite nilpotent p-group in [**1**], a sharper result was achieved by Menegazzo and Stonehewer [**4**]. They proved that, apart from a finite number of cases, an infinite nilpotent p-group has non-inner automorphisms of order a power of p, thus generalizing Gaschütz's theorem to the infinite case.

Hence, in the case of infinite groups, the hypothesis of being a p-group seems to play a decisive role in questions related to existence of non-inner automorphisms so that, at this point, it is natural to ask to what extent the nilpotency of the p-group G is needed to ensure $\mathrm{Out}(G) \neq 1$. Further investigations showed that, in a suitable setting, the hypothesis on the nilpotency of G can be dropped. In fact the following theorem was proved in [**6**].

THEOREM. *Let G be a locally finite p-group of cardinality $\aleph_0$. Then* $\mathrm{Aut}(G)$ *has cardinality* $2^{\aleph_0}$.

It is worth noting that M. Dixon obtained, by a clever examination of the proof of the above theorem, the following more precise result:

THEOREM [**2**]. *Suppose that G is an infinite countable locally finite p-group. If G is not divisible-by-finite, then* $\mathrm{Out}(G)$ *contains an uncountable elementary abelian p-subgroup.*

Is there any hope of extending the above results to p-groups of higher cardinality? The answer to this question is known to be negative. In his paper [**7**], Thomas showed that complete (that is $\zeta_1(G) = 1$ and $\mathrm{Out}(G) = 1$) uncountable groups do exist. So, if we want to find classes of locally finite p-groups which admit non-inner automorphisms, some extra hypotheses are needed. Since hypercentrality is a natural generalization of nilpotency, hypercentral p-groups seems to be a sensible class to investigate. The

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question of whether every hypercentral p-group has non-inner automorphisms was raised by M. Dixon during the 1993 meeting in Galway. Unfortunately hypercentral p-groups can have a fairly complicated structure so that, even though no examples of such groups admitting only inner automorphisms are known (at least to the author), a definitive result in this direction seems to be far out of reach. This note is meant as a first step in this investigation. What we can prove is the following result:

THEOREM 1. *Let G be a hypercentral p-group of height at most ω. If G is not cyclic of order* 2 *then* $\mathrm{Out}(G) \neq 1$.

However the arguments used in the proof of Theorem 1 can give some information on the size of $\mathrm{Out}(G)$. This is the subject of our Theorem 2.

THEOREM 2. *Let G be an infinite hypercentral p-group of height at most ω. Then* $\mathrm{Out}(G)$ *is uncountable.*

It would have been nice to be able to prove some similar result for groups of height $\omega + k$ (k a natural number) but the attempts we made in this direction were unsuccessful.

2. The results. Before we start proving our theorems, let us make some remarks.

Let G be any (infinite) group and assume that $\mathfrak{X} = \{N_i \mid i \in \mathbb{N}\}$ is an infinite set of normal subgroups of G, such that

(a) $N_i < N_j$, whenever $j < i$,

(b) $\bigcap_{i \in \mathbb{N}} N_i = 1$.

Then there is a unique topology τ on G such that (G, τ) is a topological group and $\mathfrak{X}$ is a base for the filter of neighborhoods of 1. The group (G, τ) is Hausdorff and its topological completion $\hat{G}$ is isomorphic with $\varprojlim G/N_i$. Of course any infinite subset of $\mathfrak{X}$ defines the same topology on G so that, if $\mathfrak{K} = \{M_i \mid i \in \mathbb{N}\}$ is such a subset, we have

$$\varprojlim G/N_i \simeq \varprojlim G/M_i.$$

As usual $\zeta_n(G)$ will indicate the nth term of the upper central series of the group G while $\gamma_k(G)$ stands for the kth term of the lower central series.

We are now in a position to give the proof of our main result.

Proof of Theorem 1. Since the result is true when G is finite, we may assume $|G| \geq \aleph_0$. Let $C_n = C_G(\zeta_n(G))$. Notice that, since G/C_n stabilizes the chain

$$\zeta_1(G) \leq \zeta_2(G) \leq \ldots \zeta_n(G),$$

it is a nilpotent group of class smaller than n. Assume for a moment that there exists n such that $C_n = C_{n+k}$ for all $k \in \mathbb{N}$. Recall, that, in our setting, we have

$$G = \zeta_\omega(G) = \bigcup_{n \in \mathbb{N}} \zeta_n(G)$$

so that C_n is the centre of G. Thus G turns out to be nilpotent and, in this case, it is well known that $\mathrm{Out}(G) \neq 1$ (see [**9**]). For this reason we shall suppose, from now onward, that G is not a nilpotent group. There is therefore a subset

$$\mathfrak{X} = \{B_i \mid i \in \mathbb{N}\}$$

of $\{C_n \mid n \in \mathbb{N}\}$ such that $B_i < B_j$ whenever $i > j$. Each of the B_i is the centralizer of a suitable element of the ascending central series of G, say $B_i = C_G(\zeta_{n_i}(G))$. Set $\hat{G} = \varprojlim G/B_i$. We have that $\hat{G}$ is the completion of $G/\zeta_1(G)$ in the topology defined by the subgroups $\{B_n/\zeta_i(G) \mid n \in \mathbb{N}\}$. For every element $\hat{g} = (g_iB_i)_{i\in\mathbb{N}} \in \hat{G}$ we define the map

$$\phi(\hat{g}): G \to G$$

$$x \mapsto x^{g_i} \quad \text{if} \quad x \in \zeta_{n_i}(G).$$

It is readily seen that $\phi(\hat{g})$ is well defined and that it is actually an automorphism of G. We want now to show that some of the $\phi(\hat{g})$ are non-inner. Let τ be an inner automorphism induced by the element g. Since g is contained in $\zeta_i(G)$ for some i, we have $[G, \tau] \le \zeta_i(G)$. Hence, to prove our claim, it will be sufficient to find $\phi(\hat{g})$ such that $[G, \phi(\hat{g})]$ is not contained in any of the $\zeta_i(G)$. From now on let $H_i = \zeta_{n_i}(G)$.

Let g_1 be any element in $G\backslash B_1$ and assume that g_1 belongs to H_{n_1}. Define, for $n > 1$, the set

$$\mathscr{G}_{1,n} = \{x \in B_1\backslash B_n \mid \text{there exists an element } a \in H_n \text{ such that } [a, x] \notin H_{n_1}\}.$$

If $\mathscr{G}_{1,n} = \varnothing$ for all $n \in \mathbb{N}$ we have $[B_1\backslash B_n, H_n] \le H_{n_1}$. But $[B_n, H_n] = 1$ so that $[B_1, H_n] \le H_{n_1}$ for all n. Since G is the union of the H_n we obtain $[B_1, G] \le H_{n_1}$. We recall now that H_{n_1} is a term of the upper central series so that $[B_{1},{}_cG] = 1$, for a suitable integer c. Moreover G/B_1 is nilpotent of class s, say; hence B_1 contains $\gamma_s(G)$. This implies that G is nilpotent of class at most $c + s$. Since G is assumed to be non-nilpotent, we end up with an element $x \in B_1\backslash B_{l_2}$ satisfying the following property

there is an element $a_2 \in H_{l_2}$ *such that* $[a_2, x] \in H_{n_2}\backslash H_{n_1}$,

for suitable integers l_2, n_2. We set $g_2 = g_1x$ and note that g_2 also satisfies the above condition. Moreover $g_1g_2^{-1} \in B_1$.

Assume now that we have already found elements $g_1, g_2, \ldots, g_{r-1}$ in G and integers n_i, l_i, $1 \le i < r$, such that

(1) $g_ig_{i+1}^{-1} \in B_{l_i}$, $1 \le i < r-1$,

(2) for each $1 \le i < r$, there exists $a_i \in H_{l_i}$ such that $[a_i, g_i] \in H_{n_i}\backslash H_{n_{i-1}}$.

As before we define, for $n > l_{r-1}$,

$$\mathscr{G}_{r-1,n} = \{x \in B_{l_{r-1}}\backslash B_n \mid \exists\, a \in H_n \text{ such that } [a, x] \notin H_{n_{r-1}}\}.$$

The same argument used above applies and, if $\mathscr{G}_{r,n} = \varnothing$ for all n, it turns out that $[B_{l_{r-1}}, G]$ is contained in some term of the upper central series. Since $B_{l_{r-1}}$ contains $\gamma_k(G)$ for a suitable integer k, G would be nilpotent, a contradiction. Thus we can find integers l_r, n_r and elements a_r, x, such that

(i) $a_r \in H_{l_r}$ and $x \in B_{l_{r-1}}\backslash B_{l_r}$,

(ii) $[a_r, x] \in H_{n_r}\backslash H_{n_{r-1}}$.

If we set $g_r = g_{r-1}x$ it is easy to prove that the elements $g_1, \ldots, g_r$ and the integers $n_i, l_i, 1 \le i \le r$, satisfy conditions (1) and (2). Continuing this process we eventually find two sequences of integers, $\{l_i \mid i \in \mathbb{N}\}$ and $\{n_i \mid i \in \mathbb{N}\}$, and two infinite subsets of G, $\{a_i \mid i \in \mathbb{N}\}$ and $\{g_i \mid i \in \mathbb{N}\}$, satisfying the following

(1) $g_ig_{i+1}^{-1} \in B_{l_i}$, for all $i \in \mathbb{N}$,

(2) $a_i \in H_{l_i}$ and $[a_i, g_i] \in H_{n_i}\backslash H_{n_{i-1}}$.

Consider now the element $\hat{g} = (g_i B_{l_i})$ in $\mathrm{Cr}_{i \in \mathbb{N}}\, G/B_{l_i}$, the cartesian product of the groups G/B_{l_i}. Condition (1) ensures that $\hat{g}$ is actually an element of $\varprojlim G/B_{l_i}$ thus we can define the automorphism $\phi(\hat{g})$ of G. We shall show that $[G, \phi(\hat{g})]$ cannot be contained in any of the H_n. Fix an index n and let i be any index such that $n_{i-1} > n$. The image of the element a_i under the action of $\phi(\hat{g})$ is $a_i^{g_i}$. Hence $[a_i, \phi(\hat{g})] = [a_i, g_i] \notin H_{n_{i-1}}$ and, a fortiori $[a_i, \phi(\hat{g})] \notin H_n$. As we pointed out before, this is sufficient to show that $\phi(\hat{g})$ is non-inner.

Now we know that $\mathrm{Out}(G)$ is not trivial when G is hypercentral of height at most ω (and not cyclic of order 2) so we can concentrate on the study of its cardinality.

Proof of Theorem 2. Again let $\hat{G} = \varprojlim G/B_i$. The first fact we want to point out is that if $\hat{g} = (g_i B_i) \neq \hat{h} = (h_i B_i)$ are different elements of $\hat{G}$, then the automorphisms $\phi(\hat{g})$ and $\phi(\hat{h})$ are different too. If $\phi(\hat{g}) = \phi(\hat{h})$, we would have $x^{g_i} = x^{h_i}$ for all the elements x in H_i. Thus $g_i B_i = h_i B_i$ and, since this holds for all indices i, this means $\hat{g} = \hat{h}$.

Without loss of generality, we may assume that all the sets $\mathcal{G}_{n,n+1}$ are not empty since, as pointed out in the introduction, every infinite subset of $\{B_i/\zeta_1(G) \mid i \in \mathbb{N}\}$ gives rise to the same completion of $G/\zeta_1(G)$.

Assume, for the moment, B_n/B_{n+1} is non-cyclic for infinitely many n. By the previous remark we may suppose that this actually holds for all n.

The group B_1/B_2 has at least three non-identity elements. We want to show that there are at least two distinct cosets of B_2 in B_1 intersecting $\mathcal{G}_{1,2}$. Pick $x \in \mathcal{G}_{1,2}$ so that $xB_2 \cap \mathcal{G}_{1,2} \neq \varnothing$. Let yB_2 be any other coset distinct from B_2, xB_2 and $x^{-1}B_2$. If $yB_2 \cap \mathcal{G}_{1,2} \neq \varnothing$ there is nothing to prove. Otherwise $yB_2 \cap \mathcal{G}_{1,2} = \varnothing$ but then the element xy belongs to $\mathcal{G}_{1,2}$ and $xB_2 \neq xyB_2$. Obviously this argument works for all the sets $\mathcal{G}_{n,n+1}$ so that we can select two elements x_n, y_n in each $\mathcal{G}_{n,n+1}$, with the property that $x_n y_n^{-1} \notin B_{n+1}$. For every element $\epsilon \in \{0,1\}^{\mathbb{N}}$ we construct a sequence $\mathcal{S}(\epsilon)$ of elements in G in the following way:

(i) $g_1(\epsilon)$ is any element of G,

(ii) $g_n(\epsilon) = g_{n-1}(\epsilon)x_n$ if $\epsilon(n) = 0$, and $g_n(\epsilon) = g_{n-1}(\epsilon)y_n$ if $\epsilon(n) = 1$.

It is easily seen that $\mathfrak{S} = \{(g_i(\epsilon)B_i) \mid \epsilon \in \{0,1\}^{\mathbb{N}}\}$ is an uncountable subset of $\hat{G}$ whose elements induce non-inner automorphisms of G.

The only case we have to deal with is, therefore, B_n/B_{n+1} non-cyclic for only finitely many n. As above, we may assume B_n/B_{n+1} is cyclic for all $n \in \mathbb{N}$.

Let, for each $n \in \mathbb{N}$, $i(n)$ be defined as

$$i(n) = \min\{r > n \mid B_n/B_r \text{ is not cyclic}\}$$

if $\{r > n \mid B_n/B_r \text{ is not cyclic}\}$ is not empty, or $i(n) = n$ otherwise. Define $m_1 = 1$, $m_2 = i(1)$ and, by induction, $m_{k+1} = i(m_k)$. The set $\{m_k \mid k \in \mathbb{N}\}$ is infinite. Otherwise there exists n such that B_n/B_r is cyclic for all $r > n$. Thus $B_n/\zeta_1(G)$ can be embedded in an infinite pro-cyclic pro-p-group (the group of p-adic integers), and this cannot happen since such a group is torsion-free, while B_n is a p-group. Using the subsequence $\{B_{m_k}/\zeta_1(G) \mid k \in \mathbb{N}\}$ instead of $\{B_n/\zeta_1(G) \mid n \in \mathbb{N}\}$ and the first part of this proof, we get the claim.

Finally we point out that, in a particular situation, something can be said about the existence of non-inner automorphisms of p-power order. The next corollary is really a straightforward consequence of the previous theorems.

COROLLARY. *With the same hypotheses as Theorem* 2, *if* $G/\zeta_1(G)$ *has finite exponent, then* $\mathrm{Out}(G)$ *has an uncountable normal p-subgroup.*

Proof. $\hat{G}$ has the same exponent as $G/\zeta_1(G)$, and since the B_n are characteristic subgroups of G, $\hat{G}$ is normal in $\mathrm{Aut}(G)$.

Unfortunately we could not prove the above corollary for all hypercentral p-groups of height ω, nor were we able to produce any example of such a group G, for which $\hat{G}/(G/\zeta_1(G))$ is torsion-free. What is true is that the result is already false for p-groups of height $\omega+1$ even if we ask only for the existence of one non-inner p-automorphism (some examples are contained in Section 3 of [**5**]). For this reason it would be interesting to know whether or not the hypothesis on the boundedness of the exponent of $G/\zeta_1(G)$ in the above Corollary could be relaxed.

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