AN UNBOUNDED SPECTRAL MAPPING THEOREM

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(Received 21 December 1965, revised 24 February 1967)

1. Introduction

Recall that the *spectrum*, $\sigma(T)$, of a linear operator T in a complex Banach space is the set of complex numbers λ such that $T-\lambda I$ does not have a densely defined bounded inverse. It is known [7, § 5.1] that $\sigma(T)$ is a closed subset of the complex plane C. If T is not bounded, $\sigma(T)$ may be empty or the whole of C. If $\sigma(T) \neq C$ and T is closed the spectral mapping theorem,

$$\sigma(p(T)) = p(\sigma(T)), \tag{1}$$

is valid for complex polynomials $p(z)$ [7, § 5.7]. Also, if T is closed and $\lambda \notin \sigma(T)$, $(T-\lambda I)^{-1}$ is everywhere defined.

In the case when T is a bounded normal operator on a complex Hilbert space $\mathfrak{H}$ ($TT^* = T^*T$) I proved recently by elementary methods [1, Theorem 2] that, for a complex polynomial $p(z, \bar{z})$ in z and $\bar{z}$,

$$\sigma[p(T, T^*)] = \{p(\lambda, \bar{\lambda}) : \lambda \in \sigma(T)\}. \tag{2}$$

An *unbounded normal* operator on $\mathfrak{H}$ is a densely defined closed operator T such that $TT^* = T^*T$. The purpose of this paper is to give the best generalisation of (2) for such operators. Precisely, we shall show that for an unbounded normal operator T,

$$\sigma[p(T, T^*)] = \text{cl}\,\{p(\lambda, \bar{\lambda}) : \lambda \in \sigma(T)\}; \tag{3}$$

(cl denotes closure) and that (3) is false if the closure operation is omitted from the right hand side. We shall write $p[\sigma(T)]$ for the set $\{p(\lambda, \bar{\lambda}) : \lambda \in \sigma(T)\}$ in future.

In the case when p is a polynomial in a single variable z it is not difficult to show that $p[\sigma(T)]$ is closed. In this case then, (3) is a generalisation of (1), for unbounded normal operators, because no assumption is made about $\sigma(T)$ in the proof of (3).

No use of the spectral theorem is made in this paper. This is done deliberately in the hope that the spectral theorem for unbounded normal operators may follow from (3) as in the bounded case it follows from (2)

https://doi.org/10.1017/S1446788700004663 Published online by Cambridge University Press

[1]. (Such a proof would need to be short as the null space projections we shall use are already sufficient for a deduction of the spectral theorem [3]).

2. Preliminaries

Henceforth we assume that T is a normal, but not necessarily bounded, operator on a complex Hilbert space $\mathfrak{H}$. We use the notation of [2, § 2]. Standard results about unbounded operators will be used without further reference. Most of what is required will be found summarised in [2, § 2] with further references. We require the polar decomposition

$$T = |T|U = U|T|$$

of T, with U unitary and $|T| = (T^*T)^{\frac{1}{2}}$, and the unique decomposition

$$A = A^+ - A^-$$

of a self adjoint operator A, with A^+ and A^- positive, self adjoint and $\mathfrak{R}(A^+) \subseteqq \mathfrak{N}(A^-)$, $\mathfrak{R}(A^-) \subseteqq \mathfrak{N}(A^+)$ [2, Theorems 22 and 12].

For each complex λ and non-negative r we define $E(\lambda, r)$ to be the (orthogonal) projection on $\mathfrak{N}[(|T-\lambda I|-rI)^+]$. It follows readily from [2, Lemmas 17, 18 and Theorem 23] that:

$$E(\lambda, r) \leqq E(\lambda, s) \quad (0 \leqq r \leqq s); \tag{4}$$

$$E(\lambda, r)x \to x \qquad (r \to \infty)(x \in \mathfrak{H}); \tag{5}$$

that $(T-\lambda I)E(\lambda, r)$ is a bounded normal operator with domain $\mathfrak{H}$ such that

$$||(T-\lambda I)E(\lambda, r)|| \leqq r; \tag{6}$$

and, writing $F(\lambda, r) = I - E(\lambda, r)$,

$$(|T-\lambda I|F(\lambda, r)x, F(\lambda, r)x) \geqq r||F(\lambda, r)x||^2 \quad (x \in \mathfrak{D}(T)). \tag{7}$$

As in the proof of [2, Theorem 23] it follows that for fixed λ the projections $E(\lambda, r)$ $(r \geqq 0)$ commute with each other and with $|T-\lambda I|$. (A stronger result than this is contained in [3, Lemma 1].) It also follows that:

$$\{(T-\lambda I)E(\lambda, r)\}^* = (T^*-\bar{\lambda}I)E(\lambda, r); \tag{8}$$

$$|(T-\lambda I)E(\lambda, r)| = |T-\lambda I|E(\lambda, r). \tag{9}$$

3. A counter-example

Suppose $\mathfrak{H}$ is separable and let (e_n) be a complete orthonormal sequence in $\mathfrak{H}$. Define T by

$$Tx = \sum_{n=1}^{\infty} (ni + 1/n)\xi_n e_n$$

for all $x = \sum_{n=1}^{\infty} \xi_n e_n$ such that the series for Tx is convergent. It can readily be seen that T is normal and unbounded and that

$$\sigma(T) = \{ni + 1/n : n = 1, 2, \cdots\}.$$

Now take $p(z, \bar{z}) = z + \bar{z}$. Then $\mathfrak{D}[p(T, T^*)] = \mathfrak{D}(T)$ and

$$p(T, T^*)x = 2 \sum_{n=1}^{\infty} (\xi_n/n)e_n \quad (x \in \mathfrak{D}(T)).$$

It follows that

$$\sigma[p(T, T^*)] = \{2/n : n = 1, 2, \cdots\} \cup \{0\}$$
$$\neq p[\sigma(T)].$$

This example shows that the closure operation cannot be omitted from (3). Note also that, in this example, $T + T^*$ is not closed.

4. An extension theorem

We have just seen that if $p(z, \bar{z})$ is a polynomial with complex coefficients $p(T, T^*)$ need not be a closed operator, and hence, certainly, not normal. We shall show that $p(T, T^*)$ has a closed extension and that this is always normal.

LEMMA 1. *If $p(z, \bar{z})$ has degree n then $p(T, T^*)$ has dense domain $\mathfrak{D}(T^n)$.*

PROOF. Because T is normal $\mathfrak{D}(T) = \mathfrak{D}(T^*)$ [2, Lemma 21 and Theorem 22, Corollary 2)]. Hence $\mathfrak{D}(T^r T^{*s}) = \mathfrak{D}(T^{r+s})$. It follows that $\mathfrak{D}[p(T, T^*)] = \mathfrak{D}(T^n)$. Finally $\mathfrak{D}(T^n) \supseteqq \mathfrak{D}(T^{*n}T^n) = \mathfrak{D}[(T^*T)^n]$ and, because T^*T is self adjoint, all its powers have dense domain.

Denote by $\bar{p}(z, \bar{z})$ the polynomial obtained from $p(z, \bar{z})$ by replacing all coefficients by their complex conjugates.

LEMMA 2. *The operator $\bar{p}(T^*, T)^*$ is the closure of $p(T, T^*)$ and is normal.*

PROOF. By Lemma 1, $\mathfrak{D}[\bar{p}(T^*, T)] = \mathfrak{D}(T^n)$. Hence, if x and y are in $\mathfrak{D}(T^n)$,

$$(\bar{p}(T^*, T)x, y) = (x, p(T, T^*)y).$$

Thus

$$p(T, T^*) \subseteqq \bar{p}(T^*, T)^*.$$

Write $E(0, r) = E(r)$ and suppose that $x \in \mathfrak{D}[\bar{p}(T^*, T)^*]$. It follows from (6) that $E(r)x \in \mathfrak{D}[p(T, T^*)]$ and, from (8), that

$$E(r)\bar{p}(T^*, T)^* \subseteqq \{\bar{p}(T^*, T)E(r)\}^*$$
$$= p(T, T^*)E(r).$$

Thus, by (5),

$$p(T, T^*)E(r)x = E(r)\bar{p}(T^*, T)^*x \to \bar{p}(T^*, T)^*x \qquad (r \to \infty).$$

Because $\bar{p}(T^*, T)^*$ is closed it follows that $\bar{p}(T^*, T)^*$ is the closure of $p(T, T^*)$; and hence, that $\bar{p}(T^*, T)^{**} = p(T, T^*)^*$.

By (6) and (8) again,

$$\|p(T, T^*)(E(r)-E(s))x\| = \|\bar{p}(T^*, T)(E(r)-E(s))x\| \qquad (x \in \mathfrak{H}).$$

It follows from the preceding paragraph that

$$\mathfrak{D}[\bar{p}(T^*, T)^*] = \mathfrak{D}[p(T, T^*)^*],$$
$$\|\bar{p}(T^*, T)^*x\| = \|p(T, T^*)^*x\| \quad (x \in \mathfrak{D}[\bar{p}(T^*, T)^*]).$$

Thus [2, Lemma 21] $\bar{p}(T^*, T)^*$ is normal and the proof is complete.

Henceforth we shall write $P(T, T^*)$ for the closure of $p(T, T^*)$. Note that the proof of Lemma 2 shows that $E(r)$ commutes with $P(T, T^*)$.

Lemma 3. (i) $\mathfrak{N}[P(T, T^*)] = \mathrm{cl}\,\mathfrak{N}[p(T, T^*)]$:

(ii) $\sigma[P(T, T^*)] = \sigma[p(T, T^*)]$.

Proof. (i) Because a closed operator has a closed null space, we have

$$\mathrm{cl}\,\mathfrak{N}[p(T, T^*)] \subseteqq \mathfrak{N}[P(T, T^*)].$$

Conversely, suppose $x \in \mathfrak{N}[P(T, T^*)]$. Because $E(r)$ commutes with $P(T, T^*)$,

$$p(T, T^*)E(r)x = E(r)P(T, T^*)x = O.$$

Thus $E(r)x \in \mathfrak{N}[p(T, T^*)]$ and, by (5),

$$x = \lim_{r\to\infty} E(r)x \in \mathrm{cl}\,\mathfrak{N}[p(T, T^*)],$$

proving (i).

(ii) By (i) $P(T, T^*)$ is one to one if and only if $p(T, T^*)$ is one to one. Thus $P(T, T^*)$ has an inverse if and only if $p(T, T^*)$ has an inverse. If $p(T, T^*)$ has a bounded inverse

$$\|p(T, T^*)x\| \geqq m\|x\| \quad (x \in \mathfrak{D}[p(T, T^*)])$$

for some positive m. In this case an obvious limiting argument shows that

$$\|P(T, T^*)x\| \geqq m\|x\| \quad (x \in \mathfrak{D}[P(T, T^*)]).$$

Hence $P(T, T^*)$ has a bounded inverse if and only if $p(T, T^*)$ has a bounded inverse. Because $\mathfrak{N}[p(T, T^*)]$ is dense in $\mathfrak{N}[P(T, T^*)]$, $0 \in \sigma[p(T, T^*)]$ if and only if $0 \in \sigma[P(T, T^*)]$ and (ii) now follows.

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LEMMA 4. *If λ is complex, $\lambda \notin \sigma[p(T, T^*)]$ if and only if there is a positive number m such that*

$$||p(T, T^*)x - \lambda x|| \geqq m||x|| \quad (x \in \mathfrak{D}[p(T, T^*)]). \tag{10}$$

PROOF. If $\lambda \notin \sigma[p(T, T^*)]$ then (10) holds with

$$m = ||(p(T, T^*) - \lambda I)^{-1}||^{-1}.$$

Conversely, suppose that (10) holds. It follows that $p(T, T^*) - \lambda I$ has a bounded inverse. If $x \perp \mathfrak{R}[p(T, T^*) - \lambda I]$ then $x \perp \mathfrak{R}[P(T, T^*) - \lambda I]$ and, because $P(T, T^*) - \lambda I$ is normal,

$$x \in \mathfrak{N}[\{P(T, T^*) - \lambda I\}^*] = \mathfrak{N}[P(T, T^*) - \lambda I].$$

By Lemma 3(i) and (10),

$$x \in \operatorname{cl} \mathfrak{N}[p(T, T^*) - \lambda I] = \operatorname{cl}\{O\} = \{O\}.$$

Thus $\mathfrak{R}[p(T, T^*) - \lambda I]$ is dense in $\mathfrak{H}$ and $\lambda \notin \sigma[p(T, T^*)]$.

We state as a corollary.

LEMMA 4′. *$\lambda \in \sigma[p(T, T^*)]$ if and only if there is a sequence (x_n) of elements of $\mathfrak{D}[p(T, T^*)]$ such that $||x_n|| = 1$ for all n and*

$$p(T, T^*)x_n - \lambda x_n \to O \qquad (n \to \infty).$$

REMARK. Lemmas 4 or 4′ applied to $p(z, \bar{z}) = z$ give the well known result [5, X 11.9.10 Exercise 13], [6, § 31 Theorem 2] that a normal operator has no residual spectrum.

5. The spectral mapping theorem

Two more lemmas are needed for the proof of (3). The first is stronger than [4, V.11.3.3], but is, presumably, well known. The second is more general than is strictly necessary for our purposes.

LEMMA 5. *If S is normal, $\lambda \notin \sigma(S)$ and $m = d(\lambda, \sigma(S))$ $(= \inf\{|\lambda - \mu| : \mu \in \sigma(S)\})$, then*

$$||(S - \lambda I)^{-1}|| = m^{-1}.$$

PROOF. $(S - \lambda I)^{-1}$ is everywhere defined (because S is closed), bounded and normal. If S is bounded

$$\sigma[(S - \lambda I)^{-1}] = \{(\mu - \lambda)^{-1} : \mu \in \sigma(S)\}$$

and if S is not bounded

$$\sigma[(S - \lambda I)^{-1}] = \{(\mu - \lambda)^{-1} : \mu \in \sigma(S)\} \cup \{0\}.$$

In either case, by [1, Theorem 1],

$$\begin{aligned}||(S-\lambda I)^{-1}|| &= \sup\{|\alpha| : \alpha \in \sigma[(S-\lambda I)^{-1}]\}\\ &= \sup\{|\mu-\lambda|^{-1} : \mu \in \sigma(S)\}\\ &= m^{-1}.\end{aligned}$$

COROLLARY. *Under the above conditions,*

$$||(S-\lambda I)x|| \geqq m||x|| \qquad (x \in \mathfrak{D}(S)).$$

NOTE. The formulae for $\sigma[(S-\lambda I)^{-1}]$ follow from [4, VII. 3.11 and VII 9.5] or [7, Theorem 5.71-A]. Alternatively, and preferably, they can be proved directly without using any operational calculus.

For complex λ and $r \geqq 0$ let

$$D(\lambda, r) = \{z \in C : |z-\lambda| \leqq r\}, \quad T(\lambda, r) = TE(\lambda, r).$$

LEMMA 6. *If* $r > s \geqq 0$,

$$\sigma(T) \cap D(\lambda, s) \subseteqq \sigma[T(\lambda, r)] \subseteqq [\sigma(T) \cap D(\lambda, r)] \cup \{0\}.$$

PROOF. Suppose $\mu \in \sigma(T) \cap D(\lambda, s)$. By Lemma 4′ there is a sequence (x_n) in $\mathfrak{D}(T)$ such that $||x_n|| = 1$ for all n and $Tx_n-\mu x_n \to O$ $(n \to \infty)$. By (7),

$$\begin{aligned}||(T-\lambda I)F(\lambda, r)x_n||\,||F(\lambda, r)x_n|| &= ||\,|T-\lambda I|F(\lambda, r)x_n||\,||F(\lambda, r)x_n||\\ &\geqq (|T-\lambda I|F(\lambda, r)x_n, F(\lambda, r)x_n)\\ &\geqq r||F(\lambda, r)x_n||^2.\end{aligned}$$

Now,

$$\begin{aligned}||(T-\lambda I)F(\lambda, r)x_n|| &\leqq |\mu-\lambda|\,||F(\lambda, r)x_n||+||(T-\mu I)F(\lambda, r)x_n||\\ &\leqq s||F(\lambda, r)x_n||+||F(\lambda, r)(T-\mu I)x_n||\\ &\leqq s||F(\lambda, r)x_n||+||(T-\mu I)x_n||,\end{aligned}$$

so that

$$r||F(\lambda, r)x_n||^2 \leqq s||F(\lambda, r)x_n||^2+||(T-\mu I)x_n||\,||F(\lambda, r)x_n||.$$

Because $r > s$ and $(T-\mu I)x_n \to O$ it follows that

$$x_n-E(\lambda, r)x_n = F(\lambda, r)x_n \to O \qquad (n \to \infty).$$

Thus we may, and do, assume that $x_n = E(\lambda, r)x_n$ for each n. Then

$$\begin{aligned}T(\lambda, r)x_n-\mu x_n &= E(\lambda, r)(T-\mu I)x_n\\ &\to O \qquad (n \to \infty);\end{aligned}$$

$$\mu \in \sigma[T(\lambda, r)], \text{ and } \sigma(T) \cap D(\lambda, s) \subseteqq \sigma[T(\lambda, r)].$$

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Now suppose that $\mu \in \sigma[T(\lambda, r)]$ and that $\mu \neq 0$. By Lemma 4′ there is a sequence (x_n) in $\mathfrak{H}$ such that $||x_n|| = 1$ for all n and $T(\lambda, r)x_n - \mu x_n \to O$ $(n \to \infty)$. Then

$$T(\lambda, r)x_n - \mu E(\lambda, r)x_n = E(\lambda, r)(T(\lambda, r)x_n - \mu x_n) \to O \qquad (n \to \infty).$$

Hence $\mu(x_n - E(\lambda, r)x_n) \to O$ $(n \to \infty)$ and, because $\mu \neq 0$, we may assume $x_n = E(\lambda, r)x_n$ for all n. In this case, $x_n \in \mathfrak{D}(T)$ for all n,

$$Tx_n - \mu x_n = T(\lambda, r)x_n - \mu x_n \to O \qquad (n \to \infty)$$

and $\mu \in \sigma(T)$. Also,

$$(T - \lambda I)E(\lambda, r)x_n - (\mu - \lambda)x_n = Tx_n - \mu x_n$$

so that $\mu - \lambda \in \sigma[(T - \lambda I)E(\lambda, r)]$ and, by (6),

$$|\mu - \lambda| \leqq ||(T - \lambda I)E(\lambda, r)|| \leqq r.$$

Thus $\mu \in D(\lambda, r)$ and

$$\sigma[T(\lambda, r)] \subseteqq [\sigma(T) \cap D(\lambda, r)] \cup \{0\}$$

as required.

For a related result see [5, X 2.6]. The operator T defined on a separable space with complete orthonormal sequence (e_n) by

$$T \sum \xi_n e_n = \sum (1 + 1/n)\xi_n e_n$$

shows that we cannot have $s = r$ in Lemma 6 (consider $s = 1$).

LEMMA 7. $p[\sigma(T) \subseteqq \sigma[p(T, T^*)]$.

PROOF. Let $\lambda \in \sigma(T)$ and take $r > |\lambda|$. As in the proof of Lemma 6 we obtain a sequence (x_n) in $\mathfrak{H}$ such that $x_n = E(0, r)x_n$ and $||x_n|| = 1$ for all n, and $Tx_n - \lambda x_n \to O$ $(n \to \infty)$. Thus $x_n \in \mathfrak{D}[p(T, T^*)]$ for all n, $T(0, r)x_n - \lambda x_n \to O$ $(n \to \infty)$ and, as in the proof of [1, Lemma 3],

$$p(T, T^*)x_n - p(\lambda, \bar{\lambda})x_n = p(T(0, r), T(0, r)^*)x_n - p(\lambda, \bar{\lambda})x_n \to O \qquad (n \to \infty).$$

By Lemma 4′, $p(\lambda, \bar{\lambda}) \in \sigma[p(T, T^*)]$ and

$$p[\sigma(T)] \subseteqq \sigma[p(T, T^*)].$$

THEOREM. $\sigma[p(T, T^*)] = \mathrm{cl}\, p[\sigma(T)]$.

PROOF. By Lemma 7, $p[\sigma(T)] \subseteqq \sigma[p(T, T^*)]$ and, because $\sigma[p(T, T^*)]$ is closed,

$$\mathrm{cl}\, p[\sigma(T)] \subseteqq \sigma[p(T, T^*)]. \tag{11}$$

Conversely, suppose that $\lambda \notin \mathrm{cl}\, p[\sigma(T)]$. We consider two cases.

CASE 1, $\lambda \neq p(0, 0)$. In this case the distance m from λ to $\{p(0, 0)\} \cup \mathrm{cl}\, p[\sigma(T)]$ is positive. For $r > 0$, by (2) and Lemma 6,

$$\begin{aligned}\sigma[p(T(0, r), T(0, r)^*)] &= p[\sigma(T(0, r))]\\ &\subseteqq \{p(0, 0)\} \cup p[\sigma(T) \cap D(0, r)]\\ &\subseteqq \{p(0, 0)\} \cup \mathrm{cl}\, p[\sigma(T)].\end{aligned}$$

Hence, by the Corollary to Lemma 5,

$$\begin{aligned}||p(T, T^*)E(0, r)x-\lambda x|| &= ||p(T(0, r), T(0, r)^*)x-\lambda x||\\ &\geqq m||x|| \qquad (x \in \mathfrak{H}).\end{aligned}$$

Letting $r \to \infty$ we deduce,

$$||P(T, T^*)x-\lambda x|| \geqq m||x|| \qquad (x \in \mathfrak{D}[P(T, T^*)]).$$

Hence, by Lemma 4, $\lambda \notin \sigma[p(T, T^*)]$.

CASE 2, $\lambda = p(0, 0)$. Because $\lambda \notin \mathrm{cl}\, p[\sigma(T)]$, it follows that $0 \notin \sigma(T)$ so that T has a bounded inverse S with domain $\mathfrak{H}$. Because T is unbounded

$$\sigma(S) = \{\mu^{-1} : \mu \in \sigma(T)\} \cup \{0\}.$$

By [1, Theorem 1], $||S|| = \sup \{|\alpha| : \alpha \in \sigma(S)\}$ so, because $S \neq O$, it follows that $\sigma(T)$ is non-empty. Hence there exists α in $\sigma(T)$ and $\lambda \neq p(\alpha, \bar{\alpha})$. Write $T_1 = T-\alpha I$ and $q(z, \bar{z}) = p(z+\alpha, \bar{z}+\bar{\alpha})$. We have

$$\begin{aligned}\sigma(T_1) &= \sigma(T)-\alpha = \{\mu-\alpha : \mu \in \sigma(T)\},\\ q(T_1, T_1^*) &= p(T, T^*),\\ q(0, 0) &= p(\alpha, \bar{\alpha}) \neq \lambda,\end{aligned}$$

and,

$$\begin{aligned}q[\sigma(T_1)] &= \{p(\mu-\alpha+\alpha, \bar{\mu}-\bar{\alpha}+\bar{\alpha}) : \mu \in \sigma(T)\}\\ &= p[\sigma(T)].\end{aligned}$$

Because T_1 is normal, $\lambda \notin \mathrm{cl}\, q[\sigma(T_1)]$ and $\lambda \neq q(0, 0)$, Case 1 shows that

$$\lambda \notin \sigma[q(T_1, T_1^*)] = \sigma[p(T, T^*)].$$

These two cases show that

$$\sigma[p(T, T^*)] \subseteqq \mathrm{cl}\, p[\sigma(T)].$$

and, with (11) complete the proof of the theorem.

We close with a comment on [1]. The remark following [1, Theorem 2] called for an elementary proof that if A is bounded and normal and $0 \in \sigma(A)$ then the set,

$$F(A) = \{x \in H : ||A^m x|| \leqq ||x|| \qquad (m = 1, 2, \cdots)\}$$

contains non-zero elements, and claimed that this would simplify the proof that

$$\sigma[p(T, T^*)] \subseteqq p[\sigma(T)].$$

A proof that $F(A) \neq \{0\}$ can be given using the null space projections and (7). This method, however, seems to be no shorter than the proof given in [1]. In the unbounded case a proof that $\sigma[p(T, T^*)] \subseteqq \mathrm{cl}\, p[\sigma(T)]$ can be based on the subspaces $F(A)$ but the situation is more complicated and no appreciable simplification is obtained.

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