Canad. J. Math. Vol. **53** (1), 2001 pp. 161–194

Classification of Simple Tracially AF C^*-Algebras

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Abstract. We prove that pre-classifiable (see 3.1) simple nuclear tracially AF C^*-algebras (TAF) are classified by their K-theory. As a consequence all simple, locally AH and TAF C^*-algebras are in fact AH algebras (it is known that there are locally AH algebras that are not AH). We also prove the following Rationalization Theorem. Let A and B be two unital separable nuclear simple TAF C^*-algebras with unique normalized traces satisfying the Universal Coefficient Theorem. If A and B have the same (ordered and scaled) K-theory and $K_0(A)_+$ is locally finitely generated, then $A \otimes Q \cong B \otimes Q$, where Q is the UHF-algebra with the rational K_0. Classification results (with restriction on K_0-theory) for the above C^*-algebras are also obtained. For example, we show that, if A and B are unital nuclear separable simple TAF C^*-algebras with the unique normalized trace satisfying the UCT and with $K_1(A) = K_1(B)$, and A and B have the same rational (scaled ordered) K_0, then $A \cong B$. Similar results are also obtained for some cases in which K_0 is non-divisible such as $K_0(A) = \mathbf{Z}[1/2]$.

Introduction

The purpose of this paper is to show that the pre-classifiable class of tracially AF C^*-algebras can be classified by their K-theoretic data. In recent years rapid progress has been made in classifying nuclear C^*-algebras, an ambitious program initiated by George Elliott (see [Ell1]) (we refer to Elliott's report on ICM 94 [Ell2] for more reference). For example, in the case of separable simple nuclear C^*-algebras of real rank zero and stable rank one, a large class of C^*-algebras has been classified using K-theoretic data (see [EG]). These C^*-algebras are direct limits of certain homogeneous C^*-algebras with slow dimensional growth. C^*-algebras in this class (denoted it by $\mathcal{C}_0$) also exhaust all possible K-theoretic data in the sense that every countable, weakly unperforated ordered graded group with Riesz property arises from K-theory of one of the C^*-algebras in $\mathcal{C}_0$. With these striking results, it becomes increasingly important to classify separable simple nuclear C^*-algebras of real rank zero and stable rank one without assuming that they are inductive limits of certain homogeneous C^*-algebras. One of the remarkable results in this direction can be found in [EE] where irrational rotation C^*-algebras are shown to be direct limits of circle algebras. Tracially AF C^*-algebras (TAF C^*-algebras) are introduced in [Ln6] in an attempt to replace the class of (simple) quasidiagonal C^*-algebras of real rank zero, stable rank one and with weakly unperforated K_0-group. It is shown in [Ln6] (by a result in [EG]) that every C^*-algebra in $\mathcal{C}_0$ is in fact TAF. Furthermore, it is shown that every unital simple TAF C^*-algebra A is quasidiagonal, has real rank zero, stable

Received by the editors November 20, 1998; revised April 10, 2000.
Research partially supported by NSF grant DMS 9801482.
AMS subject classification: 46L05, 46L35.

https://doi.org/10.4153/CJM-2001-007-8 Published online by Cambridge University Press

rank one and weakly unperforated $K_0(A)$. In [Ln7] we show that every separable nuclear quasidiagonal simple C^*-algebra of real rank zero, stable rank one, with weakly unperforated $K_0(A)$ and finitely many extremal normalized traces is TAF, if it is "locally" type I (for example, a direct limit of type I C^*-algebras). It would be of great interest and importance to classify separable simple nuclear TAF C^*-algebras. As the first step, we show in [Ln6] that every unital simple nuclear TAF C^*-algebra A satisfying the Universal Coefficient Theorem, with $K_1(A) = 0$ and $K_0(A) = \mathbf{Q}$ as an ordered group is isomorphic to a UHF-algebra. In this paper, we introduce a class of C^*-algebras that we call "pre-classifiable". We prove, using the results in [Ln5], that pre-classifiable TAF C^*-algebras are indeed classified by their K-theory. As an immediate consequence, we show that simple TAF C^*-algebras which are locally AH can be classified by their K-theoretic data. This allows us to drop the assumption that C^*-algebras are direct limits of homogeneous C^*-algebras and provide a generalization of the classification theorem in [EG]. We also show that other C^*-algebras can be shown to be pre-classifiable.

In order to be classified by their K-theoretic data, C^*-algebras are assumed to satisfy the Universal Coefficient Theorem (UCT). In this paper we also study the unital, separable, nuclear, simple TAF C^*-algebras with unique normalized traces. Assuming A and B are two such C^*-algebras with the same K-theory and assume that the positive cone of finitely subgroups of $K_0(A)$ is finitely generated, we show that $A \otimes Q \cong B \otimes Q$, where Q is the UHF-algebra with $K_0(Q) = \mathbf{Q}$ (see 6.7). This is what we call the Rationalization Theorem. Without using tensor product with Q, we obtain a classification theorem with certain restrictions on their K_0-theory (see Theorems 6.6 and 6.11). For example, if $K_0(A) = \mathbf{Q}$ and if A and B are separable nuclear simple TAF C^*-algebra with the UCT and the same (scaled ordered) K_*, then $A \cong B$.

The striking part of these classification results is that we do not assume that C^*-algebras have any special structure, such as direct limits of certain C^*-algebras of rather special forms or cross products of certain kinds. For example, if A is a unital separable simple nuclear TAF C^*-algebras with torsion free K_1 and $K_0 = \mathbf{Q}$, then, from our results, we know A has to be a direct limit of circle algebras. If $K_1(A) = \mathbf{Z}$ and $K_0(A) = \mathbf{Z}[1/2]$, then A is a Bunce-Dedden's algebra.

After this paper was first circulated we learned that Dadarlat and Eilers also obtained a result similar to ours in Section 6 (in the case that $K_0(A) = \mathbf{Q}$). Also they independently obtained a version of 5.9.

Acknowledgements This work is partially supported by a grant from the National Science Foundation. We have benefited from communications with N. C. Phillips. We would also like to thank Marius Dadarlat for making available to us his preprints [D3-5], and Claude Schochet for some e-mail correspondences.

1 Preliminaries

1.1

We start with some conventions. Let A be a C^*-algebra.

(i) Let $a \in A$ be a positive element. We denote by $\mathrm{Her}(a)$ the hereditary C^*-subalgebra of A generated by a.
(ii) Let $\varepsilon > 0$, $\mathcal{F}$ and S be a subset of A. We write $x \in_\varepsilon S$, if there exists $y \in S$ such that $\|x - y\| < \varepsilon$, and write $\mathcal{F} \subset_\varepsilon S$, if $x \in_\varepsilon S$ for all $x \in \mathcal{F}$.
(iii) Let A and B be two C^*-algebras with A unital. Let $h_1, h_2 \colon B \to A$ be two linear maps and $\mathcal{F}$ be a subset of B. We write (for $\varepsilon > 0$)

$$h_1 \sim_\varepsilon h_2 \quad \text{on} \quad \mathcal{F}$$

if there exists a unitary $u \in A$ such that $\|u^* h_1(b) u - h_2(b)\| < \varepsilon$ for all $b \in \mathcal{F}$; and,

$$h_1 \approx_\varepsilon h_2 \quad \text{on} \quad \mathcal{F}$$

if $\|h_1(b) - h_2(b)\| < \varepsilon$ for all $b \in \mathcal{F}$.

Definition 1.2 (**[Ln6]**) Let A be a unital simple C^*-algebra. Recall that A is *tracially approximately finite dimensional* (TAF for brevity), if it satisfies the following: for any $\varepsilon > 0$, any finite subset $\mathcal{F}$ of A which contains a non-zero element x_1, and any nonzero $a \in A_+$ there exists a finite dimensional C^*-subalgebra $F \subset A$ with $p = 1_F$ such that

(1) $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$;
(2) $pxp \subset_\varepsilon F$ for all $x \in \mathcal{F}$
(3) $1 - p$ is unitarily equivalent to a projection in $\mathrm{Her}(a)$.

(For non-simple TAF C^*-algebras, see [Ln6]).

A TAF C^*-algebra is not in general an AF-algebra, but a "large" part of it is approximately finite dimensional. When (quasi) traces are good measurements for projections, *i.e.*, $t(p) < t(q)$ for all normalized (quasi) traces t implies that p is equivalent to a subprojection of q, (3) can be replaced by

(3′) $\tau(1 - p) < \sigma$ for all normalized quasi-traces on A. (Note that if A has real rank zero, stable rank one and weakly unperforated $K_0(A)$, then A has the "fundamental comparability" (see [BH]).)

Furthermore, in [Ln6], we prove the following theorem.

Theorem 1.3 (**3.4 and 3.6 in [Ln6]**) *Every unital simple TAF C^*-algebra A is quasidiagonal and has real rank zero, stable rank one and weakly unperforated $K_0(A)$.*

1.4

Let $A = \lim(A_n, \phi_n)$, where $A_n = \bigoplus_j^{m(n)} P_{n(j)} M_{n(j)}\big(C(X_{n(j)})\big) P_{n(j)}$ and $X_{n(j)}$ are finite connected CW complexes and $P_{n(j)}$ are projections in $M_{n(j)}\big(C(X_{n(j)})\big)$. When A is simple, A is said to have *slow dimension growth* if

$$\lim_{n\to\infty} \max_{1 \le j \le m(n)} \frac{\dim(X_{n(j)})}{\mathrm{rank}(P_{n(j)})} = 0.$$

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We denote by $\mathcal{C}_0$ the class of those simple unital C^*-algebras of the above form (with slow dimension growth) having real rank zero. The classification theorem in [EG] combination with the reduction theorem in [G1] and [D1] shows that the C^*-algebras in $\mathcal{C}_0$ are classified by their scaled ordered group $\big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big)$. It also shows that, given any countable weakly unperforated graded ordered group $\big(G_0, (G_0)_+, G_1\big)$ with the Riesz decomposition property, there exists a C^*-algebra $A \in \mathcal{C}_0$ such that $\big(K_0(A), K_0(A)_+, K_1(A)\big) = \big(G_0, (G_0)_+, G_1\big)$. In [Ln6], we show that every C^*-algebra in $\mathcal{C}_0$ is TAF.

In [Ln7] we show that every unital separable nuclear quasidiagonal simple C^*-algebra A of real rank zero, stable rank one with weakly unperforated $K_0(A)$ and with finitely many extremal normalized traces is TAF, if it is also locally type I (or a direct limit of type I C^*-algebras).

Definition 1.5 Let A and B be C^*-algebras, let $L\colon A \to B$ be a contractive completely positive linear morphism, let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a subset. L is said to be $\mathcal{F}$-ε-multiplicative, if

$$\|L(xy) - L(x)L(y)\| < \varepsilon$$

for all $x, y \in \mathcal{F}$.

Definition 1.6 Let A be a C^*-algebra. Denote by $\mathbf{P}(A)$ the set of all projections in $M_\infty\big((A \otimes C_n)^{\sim}\big)$, $M_\infty\big(C(S^1) \otimes (A \otimes C_n)^{\sim}\big)$, $n = 1, 2, \ldots$, where C_n is an abelian C^*-algebra so that $K_i(A \otimes C_n) = K_*(A; \mathbf{Z}/n\mathbf{Z})$ (see [Sc2]). In this paper we will often use the following six term exact sequence.

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(A, \mathbf{Z}/k\mathbf{Z}) & \longrightarrow & K_1(A) \\ \uparrow{\mathbf{k}} & & & & \downarrow{\mathbf{k}} \\ K_0(A) & \longleftarrow & K_1(A, \mathbf{Z}/k\mathbf{Z}) & \longleftarrow & K_1(A), \end{array}$$

where $\mathbf{k}(z) = kz$ for $z \in K_*(A)$ ([Sc2]). As in [DL2], we use the notation

$$\underline{K}(A) = \bigoplus_{1=0,1, n \in \mathbf{Z}_+} K_i(A; \mathbf{Z}/n\mathbf{Z}).$$

By $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big)$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect the direct sum decomposition and the so-call Bockstein operations (see [DL2]). $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big)_+$ is the subset of those maps α such that $\alpha\big(K_0(A)_+ \setminus \{0\}\big) \subset K_0(B)_+ \setminus \{0\}$. It follows from [DL2] that if A satisfies the Universal Coefficient Theorem, then $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big) = KL(A, B)$.

Given a projection $p \in \mathbf{P}(A)$, if $L\colon A \to B$ is an $\mathcal{F}$-δ-multiplicative contractive completely positive linear morphism with sufficiently large $\mathcal{F}$ and sufficiently small δ, then $\|(L \otimes \mathrm{id})(p) - p'\| < 1/4$ for some projection p'. We will define $[L](p) = [q']$

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in $\underline{K}(B)$. It is easy to see that this is well defined (see [Ln2]). Suppose that q is also in $\mathbf{P}(A)$ with $[q] = k[p]$ for some integer k, by adding sufficiently many elements (partial isometries) in $\mathcal{F}$, we can assume that $[L](q) = k[L](p)$. Let $\mathcal{P} \subset \mathbf{P}(A)$ be a finite subset. We say $[L]|_{\mathcal{P}}$ is well defined, if $[L](p)$ is well defined for every $p \in \mathcal{P}$ and if $[p'] = [p]$ and $p' \in \mathcal{P}$, $[L](p') = [L](p)$. We see that this is possible by making $\mathcal{F}$ sufficiently large and δ sufficiently small. In what follows, when we write $[L]|_{\mathcal{P}}$ we mean that $[L]$ is well defined on $\mathcal{P}$. To save natation, abusing the language, the reader should be aware that in later use, we will not distinguish $[L](p)$ from $[L]([p])$. Moreover, it is sometimes rather convenient to write $\alpha(p)$ and $\alpha|_{\mathcal{P}}$ instead of $\alpha([p])$ and $\alpha|_{[\mathcal{P}]}$, where $\alpha \in \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big)$.

1.7

Let A be a stably finite C^*-algebra with the tracial state space $T(A)$. Let $\rho_A \colon K_0(A) \to \mathrm{Aff}\big(T(A)\big)$ be defined by $\rho_A(x) = \tau(x)$ for all $\tau \in T(A)$. Note that if $p \in M_n(A)$, we defined $\tau(p) = (\tau \otimes \mathrm{Tr})(p)$, where Tr is the standard trace on M_n.

1.8

Fix a unital C^*-algebra A. Let $\mathcal{P} \subset \mathbf{P}(A)$ be a finite subset. We denote by $\mathcal{P}_0$ the subset of $\mathcal{P}$ consisting of projections in $M_\infty(A)$, by G the subgroup generated by $\mathcal{P}$, and by G_0 the subgroup of $K_0(A)$ generated by $\mathcal{P}_0$. Let B be another C^*-algebra and let $L \colon A \to B$ be an $\mathcal{F}$-ε-multiplicative completely positive linear contraction. We may assume that $[L]|_{\mathcal{P}}$ is well defined. Suppose that $G = \mathbf{Z}^n \oplus \mathbf{Z}/k_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/k_m\mathbf{Z}$. Let $g_1, g_2, \dots, g_n$ be free generators of $\mathbf{Z}^n$ and $t_i \in \mathbf{Z}/k_i\mathbf{Z}$ be the generator with order k_i, $i = 1, 2, \dots, m$. Since every element in $K_0(A)$ may be written as $[p_1] - [p_2]$ for projections $p_1, p_2 \in A \otimes M_l$, for some $l > 0$, with sufficiently large $\mathcal{F}$ and sufficiently small ε, one can define $[L](g_j)$ and $[L](t_i)$ (see 1.6). Moreover (with sufficiently large $\mathcal{F}$ and sufficiently small ε), the order of $[L](t_i)$ divides k_i. Then we can define a map $[L]|_G$ by defining $[L](\sum_i^n n_i g_i + \sum_j^m m_j t_j) = \sum_i^k n_i [L](g_i) + \sum_j^m m_j [L](t_j)$. Note, in general, $[L]|_{\mathcal{P}}$ may *not* coincide with $[L]|_G$ on $\mathcal{P}$. However, if $\mathcal{F}$ is large enough and ε is small enough, they coincide. In what follows, we say $[L]|_G$ is well-defined and write $[L]|_G$ if

(1) $[L]$ is well-defined on $\{g_1, g_2, \dots, g_n, t_1, \dots, t_m\}$ with the order of $[L](t_i)$ dividing k_i,
(2) $[L]|_{\mathcal{P}} = [L]|_G$ on $\mathcal{P}$.

Suppose that A is stably finite and $T(A)$ is the tracial state space. Let $G_0 \subset K_0(A)$ be a finitely generated subgroup of $K_0(A)$. Since im ρ_A is torsion free, we may write $G_0 = F \oplus E$, where $E \subset \ker \rho_A$ and $(\rho_A)|_F$ is injective.

Suppose that G_1 and G_2 are ordered groups, $F_1 \subset G_1$, $F_2 \subset G_2$ are subgroups and $\gamma \colon F_1 \to F_2$ is a homomorphism. We say γ is positive if $\gamma(F_1 \cap G_{1+}) \subset F_2 \cap G_{2+}$. With this convention, we say $[L]|_{G_0}$ is positive if $[L]_{G_0 \cap K_0(A)_+}$ is positive. Note if $[L]_{G_0}$ is positive, $\rho_B\big([L](E)\big) = 0$.

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2 Automorphisms of Simple TAF C^*-algebras and a Uniqueness Theorem

The following theorem was proved in [Ln5]. This is very important for this paper.

***Theorem 2.1* (A)** *Let A be a unital separable nuclear simple C^*-algebra satisfying the UCT. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ there exist a positive number $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{P} \subset \mathbf{P}(A)$ and an integer $n > 0$ satisfying the following: for any unital C^*-algebra B with real rank zero, stable rank one and weakly unperforated $K_0(B)$, if $\phi, \psi, \sigma\colon A \to B$ are three $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps with*

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}$$

and σ is unital, then there is a unitary $u \in M_{n+1}(B)$ such that

$$\big\| u^* \operatorname{diag}\big(\phi(a), \sigma(a), \dots, \sigma(a)\big)\, u - \operatorname{diag}\big(\psi(a), \sigma(a), \dots, \sigma(a)\big) \big\| < \varepsilon$$

for all $a \in \mathcal{F}$, where $\sigma(a)$ repeats n times.

***Lemma 2.2* (6.10 in [Ln6])** *Let A be a nuclear simple unital TAF C^*-algebra. For any $\varepsilon > 0$, and any finite subset $\mathcal{F} \subset A$ and any positive integer $n > 0$, there exist projections p_1 and p_2 such that $(1 - p_1)A(1 - p_1) = M_n(p_2Ap_2)$ with $p_1 \preceq p_2$, and there are unital $\mathcal{F}$-ε-multiplicative contractive completely positive linear maps $\phi\colon A \to F_1$, where F_1 is a finite dimensional C^*-subalgebra of p_2Ap_2, such that*

(a) $\|[p_i, x]\| < \varepsilon$ *for all* $x \in \mathcal{F}$;
(b) $\Big\| x - \Big(p_1xp_1 \oplus \operatorname{diag}\big(\phi(x), \phi(x), \dots, \phi(x)\big) \Big) \Big\| < \varepsilon$ *for all* $x \in \mathcal{F}$ *(where* $\phi(x)$ *repeats n times).*

Theorem 2.3 *Let A be a separable unital nuclear simple TAF C^*-algebra satisfying the UCT. Then, for any $\varepsilon > 0$, and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$, a finite subset $\mathcal{P} \subset P(A)$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any unital C^*-algebra B of real rank zero and stable rank one, and any two $\mathcal{G}$-δ-multiplicative morphisms $L_1, L_2\colon A \to B$ with*

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}},$$

there exists a unitary $U \in B$ such that

$$\operatorname{ad}(U) \circ L_1 \approx_\varepsilon L_2 \quad \text{on} \quad \mathcal{F}.$$

Proof Let $\mathcal{G}_1 = \mathcal{G}(\mathcal{F}, \varepsilon)$, $\delta_1 = \delta(\mathcal{F}, \varepsilon)$ and $\mathcal{P}_1 = \mathcal{P}(\mathcal{F}, \varepsilon)$ be as required in Theorem A. Also let n be the integer in Theorem A. Choose a finite subset $\mathcal{G}_2$ and a positive number $\eta < \delta_1/2$ satisfying the following: if $H_i\colon A \to B$ are both $\mathcal{G}_2$-η-multiplicative with

$$H_1 \approx_\eta H_2 \quad \text{on} \quad \mathcal{G}_2$$

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then $[H_1]|_{\mathcal{P}_1} = [H_2]|_{\mathcal{P}_1}$. Let $\delta_2 = \min(\delta_1/2, \eta/4)$.

Since A is nuclear, from the above lemma, there exist a finite dimensional C^*-subalgebra F of A with $p_2 = 1_F$ and $\mathcal{G}_2$-δ_2-multiplicative morphism $\phi\colon A \to F$ such that

$$\mathrm{id}_A \approx_{\delta_2} p_1xp_1 \oplus \mathrm{diag}\big(\phi(x), \phi(x), \dots, \phi(x)\big) \quad \text{on} \quad \mathcal{G}_2,$$

where ϕ repeats n times, and $(1 - p_1)A(1 - p_1) = M_n(p_2Ap_2)$. Moreover, η is so small that $[\psi]|_{\mathcal{P}_1}$ and $[\phi]|_{\mathcal{P}_1}$ are well defined and $[\psi]|_{\mathcal{P}_1} \oplus [\phi]|_{\mathcal{P}_1} = [\mathrm{id}_A]|_{\mathcal{P}}$, where $\psi(x) = p_1xp_1$ for all x.

Fix F. Choose a large $\mathcal{G}_3$, a small $\delta_3 > 0$ and a finite subset $\mathcal{Q} \subset \mathbf{P}(F)$ so that 6.8 in [Ln5] applies, *i.e.*, if $\Lambda_i\colon F \to B$, Λ_i are $\mathcal{G}_3$-δ_3-multiplicative morphisms and $[\Lambda_i]|_{\mathcal{Q}}$ are well defined and are the same, then

$$\Lambda_1 \sim_{\eta/4} \Lambda_2 \quad \text{on} \quad \phi(\mathcal{G}_1).$$

Now let $\delta = \min(\delta_1/2, \delta_2, \delta_3/2)$, $\mathcal{G} \supset \phi(\mathcal{G}_1) \cup \psi(\mathcal{G}_1) \cup \phi(\mathcal{G}_2) \cup \psi(\mathcal{G}_2) \cup \mathcal{G}_3$ and choose $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{Q}$ (note that $\mathbf{P}(F) \subset \mathbf{P}(A)$). We also assume that 1_A, $1_A - p$, and $\mathrm{diag}(0, 0, \dots, p_2, 0, \dots, 0)$ (i-th place is p_2) are in $\mathcal{G}$.

Suppose that $L_i\colon A \to p_2Bp_2$ is as in the theorem with the above δ, $\mathcal{G}$ and $\mathcal{P}$.

Put $L_i'(x) = L_i\big(\psi(x)\big)$, $i = 1, 2$. Then

$$L_i \overset{\eta/4}{\approx} \mathrm{diag}(L_i', L_i \circ \phi, L_i \circ \phi, \dots, L_i \circ \phi, L_i \circ \phi)$$

on $\mathcal{G}$. Note that $L_i|_F$ are $\mathcal{G}_2$-δ_2-multiplicative. We also have

$$[L_i]|_{\mathcal{P}_1} = [\mathrm{diag}(L_i', L_i \circ \phi, L_i \circ \phi, \dots, L_i \circ \phi, L_i \circ \phi)]_{\mathcal{P}_1}.$$

Furthermore, we have

$$[L_1 \mid F]|_{\mathcal{Q}} = [L_2 \mid F]|_{\mathcal{Q}}.$$

By applying 6.8 in [Ln5], we obtain a unitary $V \in B$ such that

$$\mathrm{ad}(V) \circ L_1 \sim_{\eta/4} L_2 \quad \text{on} \quad \phi(\mathcal{G}_1).$$

Note that $\mathrm{diag}(0, 0, \dots, p_2, 0, \dots, 0)$ are in $\mathcal{G}$. So we have

$$L_2 \sim_{\eta/2} \mathrm{diag}(L_2', L_1 \circ \phi, L_1 \circ \phi, \dots, L_1 \circ \phi, L_1 \circ \phi) \quad \text{on} \quad \mathcal{G}_1.$$

As a consequence (by the choice of η),

$$[L_2]|_{\mathcal{P}_1} = [\mathrm{diag}(L_2', L_1 \circ \phi, L_1 \circ \phi, \dots, L_1 \circ \phi, L_1 \circ \phi)]|_{\mathcal{P}_1}.$$

Since we also have $[L_1]|_{\mathcal{P}_1} = [L_2]|_{\mathcal{P}_1}$, we obtain that

$$[L_1']|_{\mathcal{P}_1} = [L_2']|_{\mathcal{P}_1}.$$

Now Theorem A provides a unitary $U \in B$ such that

$$\mathrm{ad}(U) \circ L_2 \approx_{\varepsilon} L_1 \quad \text{on} \quad \mathcal{F}. \qquad \blacksquare$$

Theorem 2.4 *Let A be a unital nuclear simple TAF C^*-algebra which satisfies the UCT. Then an automorphism $\alpha\colon A \to A$ is approximately inner if and only if $[\alpha] = [\mathrm{id}_A]$ in $KL(A, A)$.*

Proof The "only if" part follows from 4.5 in [Ln5] (see also [R]). Now assume that $[\alpha] = [\mathrm{id}_A]$ in $KL(A, A)$. Then, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, we have

$$[\alpha]|_{\mathcal{P}} = [\mathrm{id}_A]|_{\mathcal{P}}.$$

It follows from 2.3, for any finite subset $\mathcal{F} \subset A$ and any $\varepsilon > 0$, that there exists a unitary $U \in A$ such that

$$\mathrm{ad}(U) \circ \alpha \approx_{\varepsilon} \mathrm{id}_A \quad \text{on} \quad \mathcal{F}.$$

This implies that α is approximately inner. ■

3 Classification of Nuclear Simple C^*-algebras of TAF by their K-Theory

Definition 3.1 A nuclear C^*-algebra A is said to satisfy *condition* (K) if for any unital simple TAF C^*-algebra B, any $\alpha \in \mathrm{Hom}_{\Lambda}\big(\underline{K}(A), \underline{K}(B)\big)_+$ and any finite subset $\mathcal{P} \subset \mathbf{P}(A)$ with $[1_A] \subset \mathcal{P}$, there exists a sequence of completely positive linear contractions $L_n\colon A \to B \otimes \mathcal{K}$ such that

$$\|L_n(a)L_n(b) - L_n(ab)\| \to 0$$

for all $a, b \in A$ and

$$[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

A nuclear separable simple C^*-algebra is said to be pre-classifiable if it satisfies the UCT and if for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a C^*-subalgebra A_1 of A which satisfies condition (K) such that

$$\mathcal{F} \subset_{\varepsilon} A_1.$$

Lemma 3.2 *Let A_0 be a C^*-subalgebra of a C^*-algebra A and let B be a C^*-algebra.*

(1) Suppose that B is a finite dimensional C^*-algebra and $\phi\colon A_0 \to B$ is a unital completely positive linear map. Then there exists a unital completely positive contraction $\psi\colon A \to B$ such that $\psi|_{A_0} = \phi$. (See [Pa, 5.2 and 5.3].)

(2) Suppose that there is a completely positive linear map $\phi\colon A_0 \to B$. Then, if either A_0 or B is nuclear, for any $\varepsilon > 0$, and any finite subset $\mathcal{F} \subset A_0$, there is a completely positive linear map $\psi\colon A \to B$ such that $\|\psi\| = \|\phi\|$ and

$$\|\phi(a) - \psi(a)\| < \varepsilon$$

for all $a \in \mathcal{F}$.

Proof Part (1) is a corollary of 5.2 in [Pa]. We note by 5.3 in [Pa], ϕ is a contraction. Let $B = M_{n_1} \oplus \cdots \oplus M_{n_k}$ and e_i be a unit of M_{n_i}. Let $\phi_i(a) = e_i\phi(a)e_i$. Then $\phi_i\colon A_0 \to M_{n_i}$ are unital and completely positive. By 5.2 in [Pa], there is a unital completely positive map $\psi_i\colon A \to M_{n_i}$ such that $(\psi_i)|_{A_0} = \phi_i$. Define $\psi(a) = \bigoplus_i \psi_i(a)$. Then $\psi\colon A \to B$ is unital and completely positive. By 5.3 in [Pa], it is a contraction.

Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset A_0$. Let $\mathcal{G} = \phi(\mathcal{F})$. Since B is nuclear, there are completely positive contractions $L_1\colon B \to M_n$ and $L_2\colon M_n \to B$ for some positive integer n such that

$$\|b - L_1 \circ L_2(b)\| < \varepsilon$$

for all $b \in \mathcal{G}$. Consider the completely positive map $h_1 = L_1 \circ \phi\colon A_0 \to M_n$. By 5.2 in [Pa], there exists a completely positive map $\psi_1\colon A \to M_n$ such that $\|\psi_1\| = \|\phi\|$ and $\psi_1|_{A_0} = L_1 \circ \phi$. Define $\psi = L_2 \circ \psi_1$. Then clearly ψ satisfies the requirements. The proof when A_0 is nuclear is the same. But we first approximately extend id_{A_0} to a completely positive linear contraction from A to A_0. ■

Lemma 3.3 *Let A be a separable C^*-algebra such that there exists a sequence $\{A_n\}$ C^*-subalgebras which satisfy condition (K) with the following property: for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a C^*-subalgebra A_k such that*

$$\mathcal{F} \subset_\varepsilon A_k.$$

Then A satisfies condition (K).

Proof Let $\alpha \in \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big)_+$. Given any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, we may assume that $\mathcal{P} \subset M_m(A \otimes C)$ for some abelian C^*-algebra (see 1.6). The restriction of α on $K_0(A \otimes C)$ is an element in $\mathrm{Hom}\big(K_0(A \otimes C), K_0(B \otimes C)\big)_+$.

Let $\mathcal{P} = \{p_1, p_2, \dots, p_n\}$. Clearly, there is $k_0 > 0$ such that, for each $k > k_0$, there are projections $q_1^{(k)}, q_2^{(k)}, \dots, q_n^{(k)} \in M_m(A_k \otimes C)$ such that

$$\|q_i^{(k)} - p_i\| < 1/2^{k+1}.$$

(So $[q_i] = [p_i]$ in $K_0(A \otimes C)$.) Let $j_k\colon A_k \to A$ be the embedding. Denote by α' an element in $KK(A, B)$ with image α in $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big)$. Denote $\gamma \in \mathrm{Hom}_\Lambda\big(\underline{K}(A_k), \underline{K}(B)\big)$ for the image of $j_k \times \alpha'$ in $KK(A_k, B)$. Then $\gamma(q_i^{(k)}) = \alpha(p_i)$, $i = 1, 2, \dots, n$. By the assumption, we have a sequence of finite subsets $\mathcal{F}_k \subset A_k$ such that the union $\bigcup_k \mathcal{F}_k$ is dense in the unit ball. Since each A_k satisfies condition (K), there are completely positive contractions $\phi_k\colon A_k \to B$ which are $1/2^k$-$\mathcal{F}_k$-multiplicative such that

$$[\phi_k](q_i^{(k)}) = \gamma(q_i^{(k)}) = \alpha(p_i).$$

Since A is nuclear, by the above lemma, there are completely positive contractions $L_k\colon A \to B$ such that

$$\|L_k(a) - \phi_k(a)\| < 1/2^{k+1}$$

for all $a \in \mathcal{F}_k$. It is easy to see that, for large k,

$$[L_k]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

■

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Lemma 3.4 *Let A be a unital pre-classifiable C^*-algebra and B a unital C^*-algebra with the cancellation. Then for any $\alpha \in KL(A,B)_+$ with $\alpha([1_A]) = [1_B]$, any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, any finite subset $\mathcal{G} \subset A$ and any $\delta > 0$, there exists a $\mathcal{G}$-δ-multiplicative morphism $L\colon A \to B$ such that*

$$[L]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

Proof For any $\mathcal{G}_1$ and $\delta_1 > 0$, since A is pre-classifiable, by 3.3, there is a $\mathcal{G}_1$-δ_1-multiplicative contractive completely positive linear morphism $L\colon A \to B \otimes \mathcal{K}$ such that $[L]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$. We may assume that $1_A \in \mathcal{G}_1$. There is a projection $q \in B \otimes M_n$ which is close to $a = L(1_A)$. So without loss of generality, we may assume that $L = qLq$. Furthermore, since $a = L(1_A)$ is close to q, we may assume now that $L(1_A)$ is invertible in $q(B \otimes M_n)q$. So by replacing L by $|a|^{-1/2}L|a|^{-1/2}$, we may further assume that $L(1_A) = q$ is a projection. So, with $[1_A] \in \mathcal{P}$, we may assume that $[L(1_A)] = [1_B]$, since $\alpha([1_A]) = [1_B]$. Since B has cancellation, there exists a unitary $W \in B \otimes M_{2n}$ such that

$$W^* L(1_A) W = 1_B.$$

Now $\operatorname{ad}(W) \circ L$ meets the requirement of the lemma. ∎

3.5

Let $A = \lim_{n\to\infty}(A_n, \phi_n)$ and $B = \lim_{n\to\infty}(B_n, \psi_n)$ be two direct limits of C^*-algebras, where $\phi_n\colon A_n \to A_{n+1}$ and $\psi_n\colon B_n \to B_{n+1}$ are connecting homomorphisms, respectively. We also denote by $\psi_{n,\infty}$ the homomorphism from B_n to B induced by the direct limit system. Suppose that we have the following not necessarily commutative diagram

$$\begin{array}{ccccccc}
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & \cdots B \\
\downarrow{\scriptstyle L_1} & \nearrow{\scriptstyle \Lambda_1} & \downarrow{\scriptstyle L_2} & \nearrow{\scriptstyle \Lambda_2} & \downarrow{\scriptstyle L_3} & & \\
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots A
\end{array}$$

where each L_i and Λ_i are contractive completely positive linear morphisms. Suppose further that there are finite subsets $\mathcal{F}_n \subset B_n$ and $\mathcal{G}_n \subset A_n$ with $\psi_n(\mathcal{F}_n) \cup \Lambda_n(\mathcal{G}_n) \subset \mathcal{F}_{n+1}$ such that $\bigcup_{n=1}^{\infty} \psi_{n,\infty}(\mathcal{F}_n)$ is dense in B, and with $\phi_n(\mathcal{G}_n) \cup L_n(\mathcal{F}_n) \subset \mathcal{G}_{n+1}$ such that $\bigcup_{n=1}^{\infty} \phi_{n,\infty}(\mathcal{G}_n)$ is dense in A, respectively, and suppose that there are decreasing sequences of positive numbers $\{\varepsilon_n\}$, $\{e_n\}$ and $\{\delta_n\}$ with $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, $\sum_{n=1}^{\infty} e_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$ such that L_n are $\mathcal{F}_n$-δ_n-multiplicative, Λ_n are $\mathcal{G}_n$-δ_n-multiplicative,

$$\psi_n \approx_{\varepsilon_n} \Lambda_n \circ L_n \text{ on } \mathcal{F}_n \quad \text{and} \quad \phi_n \approx_{e_n} L_{n+1} \circ \Lambda_n \text{ on } \mathcal{G}_n.$$

Then the diagram is (two-sided) approximately intertwining. A now standard argument of Elliott (see 2.1, 2.2 and 2.3 in [Ell1], also see [Th]) shows that there is an isomorphism $h\colon B \to A$.

Let $L_1\colon A \to B$ be a $\mathcal{F}_1$-δ_1-multiplicative contractive completely positive linear morphism. We say L_1 is 1-$\mathcal{F}$-invertible, if, for any finite subset set $\mathcal{G}_1 \subset B$ and $d_1 > 0$,

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there exists a $\mathcal{G}_1$-d_1-multiplicative contractive completely positive linear morphism $\psi_1\colon B \to A$ such that

$$\psi_1 \circ L_1 \sim_{\delta_1} \mathrm{id}_A \quad \text{on} \quad \mathcal{F}_1.$$

We say L_1 is k-$\mathcal{F}$-invertible, if ψ_1 can be further chosen to be $(k-1)$-$\mathcal{G}_1$-invertible (as a morphism from B to A). We say L_1 is recursively $\mathcal{F}_1$-invertible, if it is k-$\mathcal{F}_1$-invertible for all integers $k > 0$.

Theorem 3.6 **(Elliott)** *Let A and B be two separable C^*-algebras. Suppose that there is a recursively $\mathcal{F}$-invertible $L\colon A \to B$ (for some finite subset $\mathcal{F} \subset A$). Then there is an isomorphism $h\colon A \to B$.*

Proof By the definition, one can construct an approximate interwining diagram:

$$\begin{array}{ccccccc}
A & \xrightarrow{\mathrm{id}_A} & A & \xrightarrow{\mathrm{id}_A} & A & \xrightarrow{\mathrm{id}_A} & \cdots A \\
\downarrow{\scriptstyle \phi_1} & \nearrow{\scriptstyle \psi_1} & \downarrow{\scriptstyle \phi_2} & \nearrow{\scriptstyle \psi_2} & \downarrow{\scriptstyle \phi_3} & & \\
B & \xrightarrow{\mathrm{id}_B} & B & \xrightarrow{\mathrm{id}_B} & B & \xrightarrow{\mathrm{id}_B} & \cdots B.
\end{array}$$

So by, 3.5, there exists an isomorphism $h\colon A \to B$. ∎

Theorem 3.7 *Let A and B be two unital separable nuclear simple pre-classifiable TAF C^*-algebras. Then A is isomorphic to B if and only if there exists an order isomorphism*

$$\gamma\colon \big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big) \to \big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big).$$

Proof By [RS], there exists $\alpha \in KK(A, B)$ such that α induces the isomorphism γ. Let $\beta \in KK(B, A)$ such that $\beta = \alpha^{-1}$. We also use α for the corresponding element in $KL(A, B)$. For any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, since A satisfies condition (K) and B is unital simple TAF C^*-algebra, by 3.4, for any finite subset $\mathcal{F}$ and any $\varepsilon > 0$, there exists a $\mathcal{F}$-ε-multiplicative completely positive contraction $L_1\colon A \to B$ such that

$$[L_1]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

Similarly, for any finite subset $\mathcal{Q} \subset \mathbf{P}(B)$, any finite subset $\mathcal{G} \subset B$ and $\delta > 0$, there exists a $\mathcal{G}$-δ-multiplicative completely positive contraction $\phi_1\colon B \to A$ such that

$$[\phi_1]|_{\mathcal{Q}} = \beta|_{\mathcal{Q}}.$$

With sufficiently large $\mathcal{Q}$, we have

$$[\phi_1 \circ L_1]|_{\mathcal{P}} = [\mathrm{id}_A]|_{\mathcal{P}}.$$

Thus, by 2.3, for any given finite subset $\mathcal{F}_1 \subset A$ and $\sigma > 0$, with sufficiently large $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{P}$, and sufficiently small $\varepsilon > 0$ and δ, there exists a unitary $u_1 \in A$ such that

$$\mathrm{ad}(u_1) \circ \phi_1 \circ L_1 \approx_\sigma \mathrm{id}_A \quad \text{on} \quad \mathcal{F}.$$

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Define $\psi_1 = \mathrm{ad}(u_1) \circ \phi_1$. Repeating the above, for any finite subset $\mathcal{F}_1 \subset A$ and $\eta > 0$, we obtain a $\mathcal{F}_1$-η-multiplicative completely positive contraction $L_2: A \to B$ such that

$$L_2 \circ \psi_1 \approx_{\sigma/2} \mathrm{id}_B \quad \text{on} \quad \mathcal{G}.$$

Since this process continues, we see that L_1 is recursively $\mathcal{F}$-invertible (and ψ_1 is recursively G-invertible). It follows from Theorem 3.6 that A is isomorphic to B. ∎

4 Pre-Classifiable C^*-algebras

In this section we will give some classes of C^*-algebras that are pre-classifiable.

Definition 4.1 A C^*-algebra C is said to be in the class $\mathcal{SA}$, if C is isomorphic to a finite direct sum of matrices over the unitization of $C_0(X_i) \otimes A$, where each X_i is a connected finite CW complex with one base point ξ excluded and A is a nuclear separable stably finite C^*-algebra satisfying the UCT and admits tracial states. Let $C = \big(C_0(X) \otimes A\tilde{}\big)$. Suppose that $p, q \in M_n(C)$ are two projections with $\pi(p) = \pi(q)$, where $\pi: C \to C/C_0(X) \otimes A \cong \mathbf{C}$ is the quotient map. Then, for each $t \in X$, $p(t)$ is homotopy to $p(\xi)$ which is identified with a projection in M_n, since X is connected. Therefore $p(t)$ is homotopy to $q(t)$ for every $t \in X$. In particular, for every trace τ on C, $\tau(p-q) = 0$. This implies that $K_0(C) = \mathbf{Z} \oplus G$, where $G = \ker \rho_C$ (see 1.7), and $K_0(C) \subset \mathbf{N} \oplus \ker \rho_C \cup \{0\}$.

A C^*-algebra B is said to be in the class $\mathcal{LSA}$ (locally $\mathcal{SA}$), if for any finite subset $\mathcal{F} \subset B$ and any $\varepsilon > 0$, there exists a C^*-subalgebra $C \in \mathcal{SA}$ such that

$$\mathcal{F} \subset_\varepsilon C.$$

Lemma 4.2 *Let A be a unital C^*-algebra, B be a unital separable simple TAF C^*-algebra and F be a finite dimensional C^*-subalgebra of B. Let G be a subgroup generated by a finite subset of $\mathbf{P}(A)$. Suppose that there is a $\mathcal{F}$-δ-multiplicative contractive completely positive linear morphism $\psi: A \to F \subset B$ such that $[\psi]|_G$ is well defined. Then, for any $\varepsilon > 0$, there exist a finite dimensional C^*-subalgebra $C \subset B$ and a $\mathcal{F}$-δ-multiplicative contractive completely positive linear morphism $L: A \to C \subset B$ such that*

$$[L]|_{G \cap K_0(A, \mathbf{Z}/k\mathbf{Z})} = [\psi]|_{G \cap K_0(A, \mathbf{Z}/k\mathbf{Z})}, \quad \text{and} \quad \tau(1_C) < \varepsilon$$

for all tracial states τ in $T(B)$ and for all $k \geq 1$ so that $G \cap K_0(A, \mathbf{Z}/k\mathbf{Z}) \neq \varnothing$, where L and ψ are viewed as maps to B. Furthermore, if $[\psi]_{G \cap K_0(A)}$ is positive, so is $[L]|_{G \cap K_0(A)}$.

Proof We assume that $G \cap K_0(A, \mathbf{Z}/k\mathbf{Z}) = \varnothing$ for $k > K$. To save the notation, we first consider the case that $F = M_n$. Let $p = 1_F$ and $e_1, e_2, \ldots, e_n$ be mutually orthogonal minimal projections in F. By [BH], the image of ρ_B (see 1.7) is dense in $\mathrm{Aff}\big(T(B)\big)$. From this, it is easy to find projections $q_1, q_0 \leq e_1$ such that $q_1 q_0 = 0$, $[q_1] + K![q_0] = [e_1]$ and $\tau(q_1) < \varepsilon/n$ for all $\tau \in T(B)$. Therefore we obtain a C^*-subalgebra $C \subset B$ such that $C \cong M_n$ and its minimal projection are equivalent to q_1. In particular $\tau(1_C) < \varepsilon$ for all $\tau \in T(B)$. Let $\phi: F \to C$ be an isomorphism. Define $L = \phi \circ \psi$. Let $j_1: F \to B$ and $j_2: C \to B$ be embedding. By the choice

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of q_1, $[q_1]$ and $[e_1]$ have the same image in $K_0(B)/kK_0(B)$ for all $k \leq K$. Therefore $(j_1)_* = (j_2 \circ \phi)_*$ on $K_0(F, \mathbf{Z}/k\mathbf{Z})$ for all $k \leq K$. Since $K_1(M_n) = 0$, by the six-term exact sequence in 1.6, both $[L]$ and $[\psi]$ map $K_0(A, \mathbf{Z}/k\mathbf{Z})$ to $K_0(B)/kK_0(B)$ and factor through $K_0(F, \mathbf{Z}/k\mathbf{Z})$. Therefore

$$[L]|_{G\cap K_0(A,\mathbf{Z}/k\mathbf{Z})} = [\psi]|_{G\cap K_0(A,\mathbf{Z}/k\mathbf{Z})}, \quad i = 1, 2, \dots, K.$$

The general case that F is a finite direct sum of matrix algebras follows immediately. ■

Theorem 4.3 *Every unital separable simple C^*-algebra in $\mathcal{LSA}$ is pre-classifiable.*

Proof Following Definition 3.1, let A be a unital separable simple C^*-algebra in $\mathcal{LSA}$ and B be a unital separable simple TAF C^*-algebra. We first prove the following:

Claim There is a simple unital C^*-algebra $C \in \mathcal{C}_0$ such that $\big(K_0(C), K_0(C)_+, [1_C], K_1(C)\big) = \big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big)$ and there exists a unital embedding $j\colon C \to A$ which induces the identity maps on $K_*(C)$ $(= K_*(B))$.

Note that, by [Ln6], $\big(K_0(B), K_0(B)_+\big)$ is a countable weakly unperforated ordered group with the Riesz decomposition property. By 4.18 in [EG], there is a unital simple C^*-algebra $C \in \mathcal{C}_0$ (see 1.4) which is an inductive limit of $P_nC_nP_n$, where each C_n has the form $\bigoplus_{j=1}^{k(n)} M_{[j,k]}(X_{j,k})$ and $P_n \in C_n$ is a projection, where $X_{j,k}$ is of from S^1, T_{II} and T_{III} (see notation in Section 4 in [EG]) such that $\big(K_0(B), K_0(B)_+, K_1(B)\big) = \big(K_0(C), K_0(C)_+, K_1(C)\big)$ and in $K_0(B)$, $[1_C] = [1_B]$. The point is that $K_0(C_n)$ has no infinite cyclic part of infinitesimal elements.

To construct j, by the proof of 3.7 (using only one sided approximate intertwining), it suffices to show that each $P_nC_nP_n$ satisfies condition (K). By considering each summand, we may assume that $C_n = M_{l(n)}(X_{j(n)})$, where $X_{j(n)}$ is as above. Let $\alpha \in KK(C_n, B)_+$. Fix a finite subset $\mathcal{P} \in \mathbf{P}(P_nC_nP_n) = \mathbf{P}(C_n)$ and a finite subset $\mathcal{F} \subset P_nC_nP_n$. Since P_n is in fact a projection of C_n, it is clear that it suffices to have a $\mathcal{G}$-ε-multiplicative contractive completely positive linear morphism $L\colon C_n \to M_l(B)$ (for some l) with some sufficiently large $\mathcal{G}$ and sufficiently small ε such that $[L]|_G$ is well defined and $[L]|_G = \alpha|_G$, where G is the subgroup generated by $\mathcal{P}$.

It follows from [DL1] that one can construct a unital $\mathcal{G}$-$\varepsilon/2$-multiplicative contractive completely positive linear morphism $L_1\colon C_n \to M_N(B)$ for some large integer N, such that

$$[L_1]|_G = \alpha|_G + [h]|_G,$$

where $h\colon C_n \to M_N(B)$ is a one-point evaluation.

Take a small $\sigma > 0$ and $\eta > 0$. Since $M_N(B)$ is TAF, there exists a projection $p \in M_N(B)$ and a finite dimensional C^*-subalgebra $F \subset M_N(B)$ with $1_F = p$ such that

(1) $\|[p, L(x)]\| < \eta$ for all $x \in \mathcal{G}$,
(2) $pL(\mathcal{G})p \subset_\eta F$ and
(3) $\tau(1 - p) < \sigma$ for all tracial states τ on $M_N(B)$.

Thus $L_2(a) = (1-p)L_1(a)(1-p)$ (for $a \in C$) is $\mathcal{G}$-ε-multiplicative, if η is sufficiently small. Note that $K_*(C_n)$ is finitely generated. We may assume that $\mathcal{G}$ is sufficiently large and ε is sufficiently small so that any $\mathcal{G}$-ε-multiplicative contractive completely positive linear morphism L from C_n induces a well defined $[L]$ on $K_*(C_n)$ and on G. By 3.2, that there exists a completely positive linear map $L_3\colon C_n \to F$ such that

$$\|pL_1(x)p - L_3(x)\| < \varepsilon/2$$

for all $x \in \mathcal{G}$. We have

$$[L_1]|_G = [L_2]|_G + [L_3]|_G$$

provided that ε is small enough and $\mathcal{G}$ is large enough. Since the image of L_3 is in F, $[L_3]|_{G\cap K_1(C_n)} = 0$ and $[L_3]|_{G\cap \mathrm{tor}(K_0(C_n))} = 0$. By 4.2, there are a projection $p' \in pM_N(B)p$ with $\tau(p') < \sigma$, a finite dimensional C^*-subalgebra $F_1 \subset p'M_N(B)p'$ with $1_{F_1} = p'$ and $\mathcal{G}$-ε-multiplicative morphism $L_3'\colon C_n \to F_1$ such that

$$[L_3']|_{G\cap K_0(C_n,\mathbf{Z}/k\mathbf{Z})} = [L_3]|_{G\cap K_0(C_n,\mathbf{Z}/k\mathbf{Z})}.$$

Since both L_3 and L_3' factor through a finite dimensional C^*-subalgebra, $[L_3]|_{G\cap K_1(C_n,\mathbf{Z}/k\mathbf{Z})} = 0 = [L_3']_{G\cap K_1(C_n,\mathbf{Z}/k\mathbf{Z})}$. Without loss of generality, we may assume that $L_2(1_{C_n})$ and $L_3'(1_{C_n})$ are projections. With small σ, we may assume that $[(1-p)+p'] \le [1_B]$. Therefore, since $K_0(C_n) = \mathbf{Z} \oplus \mathrm{tor}\bigl(K_0(C_n)\bigr)$, there is a point-evaluation $h_1\colon C_n \to (p-p')M_N(B)(p-p')$ such that $[h_1]|_G = \alpha|_G - [L_2]|_G - [L_3']|_G$. Define $L_4\colon C_n \to M_N(B)$ by defining $L_4(a) = L_2(a) + L_3'(a) + h_1(a)$. We have that $[L_4(1_{C_n})] = \alpha([1_{C_n}])$. Since B is TAF, there is a unitary $U \in M_N(B)$ such that $U^*L_4(1_{C_n})U \le 1_B$. Define $L = \mathrm{ad}(U) \circ L_4$. Then L maps $P_nC_nP_n$ to B. This proves the claim.

Now suppose that H is the unitization of $C_0(X) \otimes S$, where X is a connected finite CW complex with one base point excluded and S is a separable nuclear stably finite C^*-algebra satisfying the UCT and admits tracial states. Let $\alpha \in KK(H,B)_+$. Since $[j]$ is an invertible element in $KK(C,B)_+$, we set $\gamma' = \alpha \times [j]^{-1}$. Without loss of generality, we may assume that $\alpha([1_H]) = [1_B]$.

Let $D' = C_0(Y)$ such that $K_i(D') = K_*\bigl(C_0(X) \otimes S\bigr)$ (see for example 23.10.5 of [Bl]), where Y is a locally compact metric space. Let $D'' = S^2D' = C_0(\mathbf{R}^2) \otimes D' \cong C_0(\mathbf{R}^2 \times Y)$. Let D be the unitization of D''. Then $D = C(Z)$, where Z is the one-point compactification of $\mathbf{R}^2 \times Y$. Z is connected. Thus $K_0(D) = K_0(H)$, $\ker \rho_H = \ker \rho_D$, $K_0(D)_+, K_0(H)_+ \subset \mathbf{N} \oplus \ker \rho_H \cup \{0\}$ (see 4.1) and $K_1(D_1) = K_1(H)$. Let $z \in KK(H,D)$ such that z induces the identity map on $K_i(H)$. The existence of such z follows from the Universal Coefficient Theorem.

Then, it follows from the unsuspended E-theory of [DL1] that for any finite subsets $\mathcal{G} \subset H$ and $\mathcal{P} \subset \mathbf{P}(H)$ and any $\varepsilon > 0$, there is a unital $\mathcal{G}$-$\varepsilon/4$-multiplicative contractive completely positive linear morphism $\psi\colon H \to M_N(D_1)$ for some large N such that

$$[\psi]|_{\mathcal{P}} = z|_{\mathcal{P}} + [\phi]|_{\mathcal{P}},$$

where $\phi\colon H \to H/C_0(X,S) \cong \mathbf{C} \to M_N(D)$ is a unital homomorphism such that $\phi|_{C_0(X)\otimes S} = 0$. Note that for any $\eta \in \ker \rho_D$ there exists $l_0 \in \mathbf{N}$ such that

$(l,\eta) \in K_0(D)_+$ for all $l \geq l_0$. Thus, by adding several copies of ϕ if necessary, we may assume that $[\psi]$ is positive on $G(\mathcal{P})$, the subgroup generated by $\mathcal{P}$. Suppose that $[\phi(1_H)] = (m,\xi)$, where $m \in \mathbf{Z}$ and $\xi \in \ker \rho_D$. There is a homomorphism $\phi'\colon H \to H/C_0(X,S) \cong \mathbf{C} \to D$ such that $[\phi'](1_H) = (l,-\xi)$ for some $l > 0$. So by replacing ϕ by $\phi \oplus \phi'$, we may assume that $[\phi](1_H) = (m+l,0)$. We identify $[\phi](1_H)$ with the integer $M = m+l \geq 0$.

We now show that D satisfies condition (K). Write $D = \lim_{n\to\infty} D_n$, where each $D_n = C(Y_n)$, where each Y_n is a finite connected CW complex. Therefore, by 3.3, it suffices to show each D_n satisfies condition (K). From the claim, we may assume that target algebras are in $\mathcal{C}_0$. Therefore it follows from [Li] (see also Remark 4.5 below) that D_n satisfies condition (K). Denote $\gamma = [\imath] \times z^{-1} \times \gamma'$, where $\imath\colon D_n \to D$ is the embedding.

To show that A is pre-classifiable, by 3.3, it suffices to show that H satisfies condition (K). Without loss of generality, we may assume that the image of ψ are contained in D_n. Thus (since D_n satisfies condition (K)) the proof ends if we show that there is $\beta \in KL(D_n,C)_+$ such that

$$\beta(1+M)[1_{D_n}] \leq [1_C], \quad \beta|_{\ker \rho_{D_n}} = \gamma|_{\ker \rho_{D_n}},$$
$$\beta'_{K_i(D_n,\mathbf{Z}/k\mathbf{Z})} = \gamma|_{K_i(D_n,\mathbf{Z}/k\mathbf{Z})}$$

$(k = 1,2,\dots)$. In fact, let $q \in C$ be a projection such that $[q] = [1_C] - \beta(1_{D_n})$. Then, there is a homomorphism $h_1\colon H \to qCq$ by $h_1 = h_1' \circ \pi$, where $\pi\colon H \to H/C_0(X) \otimes S \cong \mathbf{C}$ and $h_1'\colon \mathbf{C} \to \mathbf{C}q$. Therefore $\beta' \circ \psi \oplus [h_1] = \alpha$. Since we have shown that D_n satisfies condition (K), it remains to show that such β exists.

Suppose that K is the largest torsion order of $K_i(D_n)$ $(i = 0,1)$, by 2.10 of [DL2], we only need to prove the above for $k \leq K!$.

Since C is TAF, it is easy (see the proof of 4.2) to find nonzero mutually orthogonal projection $q_1, q_2 \leq 1_C$ such that

$$[q_1] + (K!)!\,[q_2] = (1+M)[1_C] \quad \text{and} \quad (1+M)[q_1] \leq [1_C].$$

Define a map β as follows: $\beta(m) = m[q_1]$ $(m \in \mathbf{Z})$ on $\mathbf{Z} \subset K_0(D_n) = \mathbf{Z} \oplus \ker \rho_{D_n}$, $\beta|_{\ker \rho_{D_n}} = \gamma|_{\ker \rho_{D_n}}$ and

$$\beta|_{K_i(D_n,\mathbf{Z}/k\mathbf{Z})} = \gamma|_{K_i(D_n,\mathbf{Z}/k\mathbf{Z})}.$$

Since $\gamma \in \mathrm{Hom}_\Lambda\big(\underline{K}(D_n), \underline{K}(C)\big)$, the following diagram commutes for $0 < k \leq (K!)!$.

Since $\beta|_{K_i(D_n,\mathbf{Z}/m\mathbf{Z})} = \gamma|_{K_i(D_n,\mathbf{Z}/m\mathbf{Z})}$, $i = 0,1$ for all $0 < m \leq (K!)!$ and if $0 < m$, $k \leq K!$, $km \leq (K!)!$, therefore β also preserves the short exact sequences

$$K_{i+1}(-,\mathbf{Z}/k\mathbf{Z}) \to K_i(-,\mathbf{Z}/m\mathbf{Z}) \to K_i(-,\mathbf{Z}/km\mathbf{Z}) \to K_i(-,\mathbf{Z}/k\mathbf{Z}), \quad i \in \mathbf{Z}/2\mathbf{Z}$$

for all $0 < k$, $m \leq K!$. Therefore, by 2.10 of [DL2], $\beta \in KL(D_n,C)$. This ends the proof as shown earlier. ■

Corollary 4.4 *Every simple locally AH-algebra of which satisfies the UCT is pre-classifiable.*

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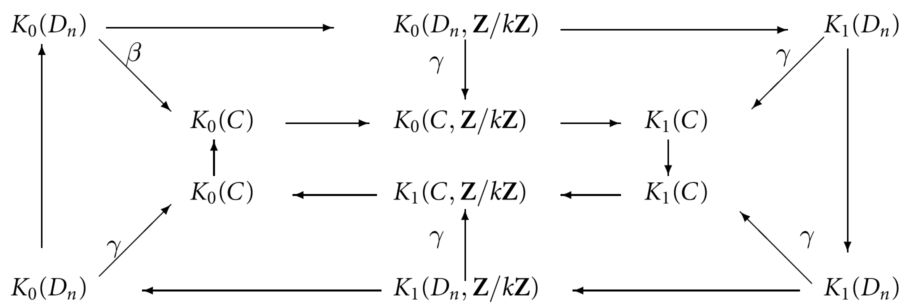

Remark 4.5 By applying [DL1], one can prove 4.4 without using [Li], since here one needs approximate multiplicative maps not homomorphisms. One should note that, in the corollary, we do not assume the condition of slow dimension growth. We do not even assume that the algebras in question are direct limits of so-called homogeneous C^*-algebras. In [DE], examples are given that locally AH-algebras that are not AH, in general. The corollary shows that, if a locally AH-algebra is simple and TAF, then it is an AH-algebra (with no dimension growth). To show all simple C^*-algebras in $\mathcal{C}_0$ are TAF does not use the full strength of [EG]. So the proof here also provides an alternative proof of the main result in [EG].

Remark 4.6 Let A be a unital, separable, nuclear, simple TAF C^*-algebra satisfying the UCT. We know from [Ln6] that A has real rank zero, stable rank one and weakly unperforated $K_0(A)$. Suppose that $B \in \mathcal{C}_0$ with the same ordered and scaled K-theory. From the proof of 3.7, we see that, if we can have a sequence of asymptotically multiplicative morphisms from A to B which induces an identity map on every finite subset of $\mathbf{P}(A)$, then the same proof shows that $A \cong B$. In the next section we attempt to construct such a map.

5 An Unsuspended E-Theory

5.1

The main purpose of this section is to prove 5.9. Let A be a direct limit of residually finite dimensional C^*-algebras. Roughly speaking, 5.9 says that for any B, and any $\alpha \in \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big)_+$, there are asymptotic multiplicative morphisms $\{\phi_n\}$ which induce α modulo some morphisms with images contained in finite dimensional C^*-subalgebras. This section is inspired by Section 3 of [Ph] and [D5].

5.2

If $L\colon A \to B$ is a homomorphism then $[L]|_{G_0}$ is always well-defined and positive (G_0 is as in 1.8). Let $C \subset A$ be a C^*-subalgebra. Suppose that $\mathcal{P} \subset \mathbf{P}(C)$ and the subgroup G_0' of $K_0(C)$ generated by $\mathcal{P}_0$ is isomorphic to G_0 (one always has a homomorphism

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from G_0' onto G_0) and $h\colon C \to B$ is a $\mathcal{F}$-$\varepsilon/2$-multiplicative contractive completely positive linear morphism so that $[h]_{G_0}$ is well defined and positive. Suppose that the finite subset $\mathcal{F}$ required by 1.6 is also in C. Suppose further that either A or B is nuclear. Then there is sequence of completely positive linear contractions $L_n\colon A \to C$ such that $\lim_n L_n(a) = a$ for all $a \in \mathcal{F}$ (3.2). One sees easily that, for large n, $L = h \circ L_n$ gives an $\mathcal{F}$-ε-multiplicative completely positive linear contractions such that $[L]|_{G_0}$ is well defined and positive. This will be used later.

Definition 5.3 Let A be a residually finite dimensional C^*-algebra (RFD C^*-algebra). Recall that A has a separating family of finite dimensional irreducible representations. Let B be a C^*-algebra. We use the notation B^+ for the C^*-algebra obtained by adding a unit to B (even if B is unital). An asymptotic sequential morphism $\phi = \{\phi_n\}$ from A to B is a sequence of completely positive linear contractions $\{\phi_n\}\colon A \to B^+ \otimes \mathcal{K}$ such that

(1) $\|\phi_n(ab) - \phi_n(a)\phi(b)\| \to 0$ as $n \to \infty$, for $a, b \in A$,
(2) there exist two sequences of homomorphisms $h_n, h_n'\colon A \to B^+ \otimes \mathcal{K}$ with finite dimensional range satisfying the following: for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, there is $n > 0$ such that, $[\phi_k]|_{\mathcal{P}}$ is well defined for all $k \geq n$,

$$[\phi_k \oplus h_k]|_{\mathcal{P}} - [h_k']|_{\mathcal{P}} = [\phi_n \oplus h_n]|_{\mathcal{P}} - [h_n']|_{\mathcal{P}}$$

and $\{[\phi_n \oplus h_n] - [h_n']\}$ defines an element in $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B^+ \otimes \mathcal{K})\big)$, *i.e.*, there exists $\alpha \in \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B^+ \otimes \mathcal{K})\big)$ such that, for each finite subset $\mathcal{P} \subset \mathbf{P}(A)$, $([\phi_n \oplus h_n] - [h_n'])|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for all large n.

From the definition, it is easy to check that the composition of two asymptotic sequential morphisms is an asymptotic sequential morphism.

Let ϕ and ψ be two asymptotic sequential morphisms from A to B. We say ϕ is equivalent to ψ and write $\phi \sim \psi$ if there are two sequences of $h_n, h_n'\colon A \to B^+ \otimes \mathcal{K}$ with finite dimensional range and unitaries $u_n \in (B^+ \otimes \mathcal{K})^{\sim}$ such that

$$\big\| u_n^* \operatorname{diag}\big(\phi_n(a), h_n(a)\big) u_n - \operatorname{diag}\big(\psi_n(a), h_n'(a)\big) \big\| \to 0 \quad \text{as} \quad n \to \infty.$$

We denote by $\langle\phi\rangle$ the equivalence class of asymptotic sequential morphisms represented by ϕ and denote by $\mathcal{E}(A, B)$ the set of the equivalence classes of asymptotic sequential morphisms from A to B.

One should note that if $\{h_n\}\colon A \to B^+ \otimes \mathcal{K}$ is a sequence of homomorphisms with finite dimensional range, then $h = \{h_n\}$ is an asymptotic morphism from A to B. Moreover, any two such asymptotic morphisms are equivalent.

Let ϕ and ψ be two asymptotic sequential morphisms from A to B. We define $\phi + \psi$ by $(\phi + \psi)(a) = \operatorname{diag}\big(\phi(a), \psi(a)\big)$. This clearly gives an addition on $\mathcal{E}(A, B)$.

In the rest of this section A is a separable RFD C^*-algebra unless otherwise stated.

Proposition 5.4

(1) $\mathcal{E}(A, B)$ is an abelian group;

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(2) *For fixed A, $\mathcal{E}(A,-) = \mathcal{E}_A(-)$ is a covariant functor from separable C^*-algebras to abelian groups which is homotopy invariant, stable and split exact.*

Proof To obtain (1) we apply Proposition 2 in [D3] which asserts that there is a sequence of completely positive contractions $\tau_n\colon A \to M_{k(n)}(A)$ and a sequence $h_n\colon A \to M_{k(n)+1}(A)$ of homomorphisms with finite dimensional range such that

$$\lim_{n\to\infty} \big\| \operatorname{diag}\big(a, \tau_n(a)\big) - h_n(a) \big\| = 0$$

for all $a \in A$. We claim that $\{\tau_n\}$ forms an asymptotic morphism. Fix a homomorphism with finite dimensional range h'. Then we have

$$([\tau_n] + [h'])|_{\mathcal{P}} - [h_n]|_{\mathcal{P}} = [h'] - [id_A]|_{\mathcal{P}}$$

for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$ and for all large n. Therefore $\{\tau_n\}$ is an asymptotic morphism.

Let F_n be the range of h_n. Suppose that $\{\phi_n\}$ is a asymptotic sequential morphism from A to $B^+ \otimes \mathcal{K}$. Let $\{\mathcal{F}_n\}$ be an increasing sequence of finite subsets of A such that the union is dense in the unit ball of A. Since $\dim F_k < \infty$, there exists $n(k)\big(n(k) > n(k-1)\big)$ such that

$$\|\phi_n(a) - g_n^{(k)}(a)\| < 1/(n+k)$$

for all $a \in h_k(\mathcal{F}_k)$ and $n \geq n(k)$, where $g_n^{(k)}\colon F_k \to B^+ \otimes \mathcal{K}$ is a homomorphism. Therefore, we see, by letting $\tau_n' = \tau_k$, $h_n' = h_k$ and $\psi_n = g_n^{(k)}$ for $n(k) \leq n < n(k+1)$) (and noting that ϕ_n are contractive), that

$$\lim_{n\to\infty} \big\| \operatorname{diag}\big(\phi_n(a), \phi_n \circ \tau_n'(a)\big) - \psi_n \circ h_n'(a) \big\| = 0$$

for all $a \in A$. Note that $\psi_n \circ h_n$ is a homomorphism with finite dimensional range. Since $\{\phi_n\}$ is an asymptotic sequential morphism, $\phi_n \circ \tau_n'$ is also an asymptotic sequential morphism. This implies that $\mathcal{E}_A(B)$ is a group. This proves (1).

It follows that $\mathcal{E}_A(-)$ is a covariant functor from separable C^*-algebras to abelian groups. It is obvious that it is stable. The homotopy invariance follows from the proof of [D4] (see also 1.4 in [D2]).

To see that $\mathcal{E}_A$ is splitting exact, let

$$0 \to J \xrightarrow{j} D \overset{\pi}{\underset{s}{\rightleftharpoons}} C \to 0$$

be a splitting exact sequence of separable C^*-algebras. It is clear that the embedding induces an injective map from $\mathcal{E}_A(J)$ to $\mathcal{E}_A(D)$. It follows from the fact that the short exact sequence of C^*-algebras splits that the map from $\mathcal{E}_A(D)$ to $\mathcal{E}_A(D/J)$ is surjective. Let $\phi = \{\phi_n\}$ be an asymptotic sequential morphism from A to D. Assume that $\pi_* \circ \langle\phi\rangle = 0$ in $\mathcal{E}(A, C)$. This implies that $\{\pi \circ \phi_n\}$ is equivalent to a sequence $h = \{h_n\}$ of homomorphisms $h_n\colon A \to C^+ \otimes \mathcal{K}$ with finite dimensional range. Without loss of generality, we may assume that $\{\pi \circ \phi_n\} = \{h_n\}$ for some of those $\{h_n\}$. For each n, let F_n be a finite dimensional C^*-subalgebra of $D^+ \otimes \mathcal{K}$ such

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that im $h_n \subset F_n$. Write $F_n = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$ with p_i the central projection corresponding to the summand M_{n_i}. Let $e_i \le p_i$ be a minimal projection in M_{n_i}. There is a C^*-subalgebra $C \subset \mathbf{C} \cdot 1_{D^+} \otimes \mathcal{K}$ with $C \cong M_{n_i} \oplus \cdots \oplus M_{n_k}$ with central projections P_i and minimal projections $q_i \le P_i$ such that $e_i \preceq q_i$ and $p_i \preceq P_i$. Clearly without loss of generality, we may assume that $e_i \le q_i$ and $p_i \le P_i$. Furthermore, we may assume that $d_i = q_i - e_i \ne 0$. Thus we obtain an isomorphism ψ_n from F_n to $(\sum_i P_i)(D^+ \otimes \mathcal{K})(\sum P_i)$ such that

$$\{a \oplus h(a) : a \in F_n\} \subset \Big(\sum P_i\Big)(1_{B^+} \otimes \mathcal{K})\Big(\sum P_i\Big).$$

Let $g_n = \psi_n \circ \pi \circ \phi_n$. Then $\{\pi \circ (\phi_n \oplus g_n)\}$ has the range in $\mathbf{C}1_{C^+} \otimes \mathcal{K}$. This implies that $\phi_n \oplus s \circ g_n$ has the range in $J^+ \otimes \mathcal{K}$. This proves that

$$0 \to \mathcal{E}_A(J) \to \mathcal{E}_A(D) \to \mathcal{E}(D/J) \to 0$$

is exact. The fact that it splits follows immediately, since the corresponding sequence of C^*-algebras splits. ■

5.5

There is a map Γ from $KK(A,B)$ to $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big)$ (see [DL2]). It is known that $KK(A,-)$ is a covariant functor from separable C^*-algebras to abelian groups which is homotopy invariant, stable and split exact (see Section 2 in [H]). $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(-)\big)$ is also a covariant functor from separable C^*-algebras to abelian groups. From the definition, we see immediately that it is homotopy invariant and stable. Suppose that

$$0 \to I \xrightarrow{j} B \underset{s}{\overset{\pi}{\rightleftarrows}} C \to 0$$

is a splitting exact sequence, where $j\colon I \to B$ is an embedding, $\pi\colon A \to C$ is surjective and $\pi \circ s = \mathrm{id}_C$. We obtain the splitting exact sequences

$$0 \to \mathrm{Hom}\big(K_*(A, \mathbf{Z}/n\mathbf{Z}), K_*(I, \mathbf{Z}/n\mathbf{Z})\big) \xrightarrow{j_*} \mathrm{Hom}\big(K_*(A, \mathbf{Z}/n\mathbf{Z}), K_*(B, \mathbf{Z}/n\mathbf{Z})\big)$$
$$\underset{s_*}{\overset{\pi_*}{\rightleftarrows}} \mathrm{Hom}\big(K_*(A, \mathbf{Z}/n\mathbf{Z}), K_*(C, \mathbf{Z}/n\mathbf{Z})\big) \to 0$$

$(n = 0, 1, \dots)$. Since j, π and s are homomorphisms, j_*, π_* and s_* respect the two exact sequences

$$K_i(-) \xrightarrow{\times n} K_i(-) \xrightarrow{\rho} K_i(-, \mathbf{Z}/n\mathbf{Z}) \xrightarrow{\beta_n^i} K_{i+1}(-)$$

and

$$K_{i+1}(-, \mathbf{Z}/n\mathbf{Z}) \xrightarrow{\beta_{m,n}^{i+1}} K_i(-, \mathbf{Z}/m\mathbf{Z}) \xrightarrow{\kappa_{m,n,m}^i} K_i(-, \mathbf{Z}/mn\mathbf{Z}) \xrightarrow{\kappa_{n,mn}^i} K_i(-, \mathbf{Z}/n\mathbf{Z})$$

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for all $m, n \in \mathbf{N}$, $i \in \mathbf{Z}/2\mathbf{Z}$. Therefore we have the splitting exact sequence

$$0 \to \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(I)\big) \xrightarrow{[j]} \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big) \overset{[\pi]}{\underset{[s]}{\rightleftharpoons}} \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(C)\big) \to 0.$$

We define a subset F of $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B^+ \otimes \mathcal{K})\big)$ as follows. We say $\alpha \in F$, if there exists a sequence $\{h_n\}$ of homomorphisms from A to $B^+ \otimes \mathcal{K}$ with finite dimensional range such that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, there exists $n > 0$ such that

$$\alpha|_{\mathcal{P}} = [h_n]|_{\mathcal{P}}.$$

We write $x \sim y$ for $x, y \in \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B^+ \otimes \mathcal{K})\big)$ if there exists a sequence $[h_n'], [h_n''] \in F$ such that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, there exists $n > 0$ satisfying

$$x + [h_m]|_{\mathcal{P}} = y + [h_m']|_{\mathcal{P}}$$

for all $m \geq n$. Clearly if $x \sim y$ then $x + z \sim y + z$. Since $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(-)\big)$ is split exact, we may view $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big)$ as a subgroup of $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B^+ \otimes \mathcal{K})\big)$. Denote by $KL_F(A, B)$ the quotient of $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big) + F$ by the equivalence relation "$\sim$" defined above. So $KL_F(A, B)$ is a quotient of $\mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big)$. We use q for the quotient map. In particular, if $x \in F$, then $q(x) = 0$.

Proposition 5.6 *Let A be as above. Then $KL_F(A, -)$ is a covariant functor from separable C^*-algebras to abelian groups which is stable, homotopy invariant and splitting exact.*

Proof It is clear that $KL_F(A, -)$ is a covariant functor. It is clearly stable. Since homotopy images of finite dimensional C^*-algebras are finite dimensional C^*-algebras, one checks easily that $KL_F(A, -)$ is also homotopy invariant. To see it is split exact, let

$$0 \to J \xrightarrow{j} D \overset{\pi}{\underset{s}{\rightleftharpoons}} C \to 0$$

be a split exact sequence of separable C^*-algebras.

We use the similar argument used in the proof of 5.4. From the splitting exact sequence

$$0 \to \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(J)\big) \xrightarrow{[j]} \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(D)\big) \overset{[\pi]}{\underset{[s]}{\rightleftharpoons}} \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(C)\big) \to 0$$

one easily checks that the induced maps from $KL_F(A, J)$ to $KL_F(A, D)$ is injective and from $KL_F(A, D)$ to $KL_F(A, C)$ is surjective, respectively. Let $x \in \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(D)\big)$ such that $[\pi]\big(q(x)\big) = 0$. Then there is a sequence homomorphism $\{\psi_n\}$ from A to $C^+ \otimes \mathcal{K}$ with finite dimensional range such that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, there exists $n > 0$ satisfying:

$$[\psi_n]|_{\mathcal{P}} = [\pi](x)|_{\mathcal{P}}.$$

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Then $\{s\circ\psi_n\}$ is a sequence of homomorphisms from A to $D^+\otimes\mathcal{K}$ with finite dimensional range. Let $y = x - [s]\big([\pi](x)\big)$ in $\mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(D)\big)$. Then $[\pi](y) = 0$ in $\mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(C)\big)$ which implies that $y\in\mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(J)\big)$. It follows that $q(y)\in KL_F(A,J)$. Since

$$[s]\big([\pi](x)\big)|_{\mathcal{P}} = [s\circ\psi_n]|_{\mathcal{P}},$$

we conclude that $q\Big([s]\big([\pi](x)\big)\Big) = 0$, whence $q(x) = q(y)$. This implies that the splitting exact sequence of C^*-algebras gives a short exact sequence

$$0\to KL_F(A,J)\to KL_F(A,D)\to KL_F(A,C)\to 0.$$

It follows easily that it in fact splits. ∎

5.7

There is a homomorphism β_A^B from $\mathcal{E}_A(B)$ to $KL_F(A,B)$ defined as follows. Let $\phi = \langle\{\phi_n\}\rangle\in\mathcal{E}_A(B)$. By the definition, there are homomorphisms $h_n, h_n'\colon A\to B^+\otimes\mathcal{K}$ with finite dimensional range such that $x = \{[\phi_n\oplus h_n]\} - \{[h_n']\}$ defines an element in $\mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(B^+\otimes\mathcal{K})\big)$. Note that we have a splitting exact sequence

$$0\to\mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(B)\big)$$
$$\to\mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(B^+\otimes\mathcal{K})\big)\ \underset{[s]}{\overset{[\pi]}{\rightleftarrows}}\ \mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(\mathbf{C}\cdot 1_{B^+}\otimes\mathcal{K})\big)\to 0.$$

Then $x-[s\circ\pi](x)$ defines an element z in $\mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(B)\big)$. We define $\beta_A^B(\phi) = q(z)$. To see it is well-defined, suppose that $f_n', f_n''\colon A\to B^+\otimes\mathcal{K}$ are two sequences of homomorphisms with finite dimensional range such that $y = \{[\phi_n\oplus f_n'] - [f_n'']\}$ defines an element in $\mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(B^+\otimes\mathcal{K})\big)$. Then, let $\{\mathcal{P}_n\}$ be an increasing sequence of finite subsets of $\mathcal{P}(A)$ with $\bigcup_{n=1}\mathcal{P}_n = \mathbf{P}(A)$, we have (by passing to a subsequence if necessary)

$$(x-y+[h_n']+[f_n''])|_{\mathcal{P}_n} = ([h_n]+[f_n'])|_{\mathcal{P}_n}.$$

So $q(x) = q(y)$. Suppose that $\phi' = \{\phi_n'\}$ is an asymptotic morphism which is equivalent to ϕ. Then there are two sequences of homomorphisms $g_n, g_n'\colon A\to B^+\otimes\mathcal{K}$ with finite dimensional range and a sequence of unitaries $u_n\in(B^+\otimes\mathcal{K})^\sim$ such that

$$\big\| u_n^*\,\mathrm{diag}\big(\phi_n(a)\oplus g_n(a)\big)\,u_n - \mathrm{diag}\big(\phi_n'(a)\oplus g_n'(a)\big)\big\|\to 0,$$

as $n\to\infty$ for all $a\in A$. Therefore, we may assume that

$$[\phi_n\oplus g_n]|_{\mathcal{P}_n} = [\phi_n'\oplus g_n']|_{\mathcal{P}_n}.$$

A similar argument above shows that ϕ and ϕ' define the same element in $KL_F(A,B)$. This implies that $q(y) = q(x)$. So β_A^B is well defined. It is clear that $\beta\colon\mathcal{E}_A(-)\to KL_F(A,-)$ is a natural transformation.

We denote by $\Gamma L_F(A,B)$ the image of $\Gamma\big(KK(A,B)\big)$ in $KL_F(A,B)$ (see 5.5. for Γ).

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Lemma 5.8 *The transformation β_A maps $\mathcal{E}_A(B)$ onto $\Gamma L_F(A,B)$ for each separable C^*-algebra B.*

Proof By 3.7 in [H], there is a unique natural transformation $\alpha\colon KK(A,-) \to \mathcal{E}_A(-)$ such that $\alpha_A(1_A) = \langle 1_A\rangle$. Let $\gamma\colon KK(A,-) \to KL_F(A,-)$ be the natural transformation induced by the map $\Gamma\colon KK(A,B) \to \mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(B)\big)$ and the quotient map from $\mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(B)\big)$ to $KL_F(A,B)$. We have

$$\beta_A \circ \alpha_A(1_A) = q([1_A]),$$

where $[1_A]$ is the image of the identity map in $KL(A,A)$. Since $\gamma(1_A) = q([1_A])$, by the uniqueness (3.7 in [H]),

$$\beta \circ \alpha = \gamma.$$

Since γ maps $KK(A,B)$ onto $\Gamma L_F(A,B)$, $\beta_A\colon \mathcal{E}_A(B) \to \Gamma L_F(A,B)$ is surjective for each B. ■

Theorem 5.9 *Let A be a separable C^*-algebra satisfying UCT such that A is the closure of an increasing sequence $\{A_n\}$ of RFD C^*-algebras and B be a unital nuclear separable C^*-algebra. Then, for any $\alpha \in \mathrm{Hom}_\Lambda\big(\underline{K}(A),\underline{K}(B)\big)_+$, there exist two sequences of completely positive contractions $\phi_n^{(i)}\colon A \to B\otimes\mathcal{K}$ ($i = 1,2$) satisfying the following:*

(1) $\|\phi_n^{(i)}(ab) - \phi_n^{(i)}(a)\phi_n^{(i)}(b)\| \to 0$ as $n \to \infty$,
(2) for each n, the images of $\phi_n^{(2)}$ are contained in a finite dimensional C^*-subalgebra of $B\otimes\mathcal{K}$ and for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, $[\phi_n^{(2)}]|_{\mathcal{P}}$ and $[\phi_n^{(2)}]|_{G_0}$ are well defined for all large n, where G_0 is the subgroup generated by $\mathcal{P}_0$ (see 1.8),
(3) for each finite subset of $\mathcal{P} \subset \mathbf{P}(A)$, there exists $m > 0$ such that

$$[\phi_n^{(1)}]|_{\mathcal{P}} = \alpha + [\phi_n^{(2)}]|_{\mathcal{P}}$$

for all $n \geq m$.
(4) For each n, we may assume that $\phi_n^{(2)}$ is a homomorphism on A_n.
(5) If $G_0 \cap K_0(A)_+$ is finitely generated, then we may also assume that $[\phi_n^{(2)}]|_{G_0}$ is positive.

(The condition that B is nuclear can be replaced by the condition that each A_n is nuclear).

Proof We first prove this for all RFD C^*-algebras and $\alpha \in \Gamma\big(KK(A,B)\big)$. So we assume that A is a RFD C^*-algebra. It follows from 5.8 that all conclusions remain valid if we do not demand that images of maps are contained in $B\otimes\mathcal{K}$ (they are in $B^+\otimes\mathcal{K}$). Furthermore, $\phi_n^{(2)}$ can be taken to be homomorphisms. Therefore, with the same notation, we may assume that we have obtained $\phi_n^{(i)}$ satisfying (1)–(4) but with images in $B^+\otimes\mathcal{K}$. Since B is unital, we have $B^+\otimes\mathcal{K} = (B\otimes\mathcal{K})\oplus\mathcal{K}$. Therefore, we may write $\phi_n^i = \phi_n^{(i)'} \oplus \phi_n^{(i)''}$, where $\phi_n^{(i)'}\colon A \to B\otimes\mathcal{K}$ and $\phi_n^{(i)''}\colon A \to \mathcal{K}$. It immediately follows that

$$[\phi_n^{(1)'}]|_{\mathcal{P}} = (\alpha + [\phi_n^{(2)'}])_{\mathcal{P}}.$$

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Now we consider the general case. We write $A = \lim_n A_n$, where each A_n is a RFD C^*-algebra. So for a finite subset $\mathcal{P} \subset \mathbf{P}(A)$, we may assume that there is a RFD C^*-algebra $A_m \subset A$ such that $\mathcal{P} \subset \mathbf{P}(A_m)$. We may also assume that $G_0 \subset (j)_*\big(K_0(A_m)\big)$ and $((G_0)_+ \subset (j)_*\big(K_0(A_m)_+\big)$, if $G_0 \cap K_0(A)_+$ is finitely generated), where $j\colon A_m \to A$ is the embedding.

By Lemma 2.2 in [DL2], we have the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \lim^1 KK(A_n, B) \longrightarrow & KK(A,B) & \longrightarrow & \lim KK(A_n,B) & \longrightarrow 0 \\ & \Gamma\downarrow & & \downarrow \lim\Gamma_n & \\ & \mathrm{Hom}_\Lambda\big(\underline{K}(A), \underline{K}(B)\big) & \xrightarrow{\cong} & \lim \mathrm{Hom}_\Lambda\big(\underline{K}(A_n), \underline{K}(B)\big) & \end{array}$$

where the upper row is the Milnor $\lim^1$ exact sequence ([Br] and [Sc1, 7.1]). When A has the UCT, by [DL2], Γ is surjective. So, from the commutative diagram, the map $\lim_n \Gamma_n$ is also surjective. We may assume that there is $\beta \in \Gamma\big(KK(A_m, B)\big) \subset \mathrm{Hom}_\Lambda\big(\underline{K}(A_m), \underline{K}(B)\big)$ such that

$$\alpha \circ [j_m]_{\mathcal{P}} = \beta|_{\mathcal{P}}.$$

Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset A$. By choosing an even larger m, we may assume that $\mathcal{F} \subset A_m$. By applying the first part of the proof to β and A_m, we obtain an $\mathcal{F}$-$\varepsilon/2$-multiplicative completely positive contraction $\phi\colon A_m \to B \otimes \mathcal{K}$ and a homomorphism $h\colon A_m \to B \otimes \mathcal{K}$ such that

$$[\phi]|_{\mathcal{P}} = (\beta + [h])|_{\mathcal{P}}.$$

By 3.2, we may extend ϕ to A. To see that $[\phi]|_{G_0}$ is positive, if $(G_0)_+$ is finitely generated, we note that $(G_0)_+ \subset (j)_*\big(K_0(A_m)_+\big)$ (for some large m) and $h|_{A_m}$ is a homomorphism. ■

Corollary 5.10 *In 5.9, let B be a unital simple C^*-algebra of real rank zero and stable rank one, $\alpha|_{K_0(A)}$ be an isomorphism from $K_0(A)$ onto $K_0(B)$. Suppose that $K_0(B)$ is divisible and torsion free. Then we can assume that, for each finitely generated subgroup of $K_0(A)$, there is n so that $\alpha + [\phi_n^{(2)}]$ is injective on that subgroup. Furthermore, if we assume that A has unique normalized quasitrace τ, and $\tau\big(K_0(A)\big)$ is a divisible dense subgroup of $\mathbf{R}$ containing 1 and $K_0(A)$ is weakly unperforated, then we may assume that $\tau\big(\alpha + [\phi_n]^{(2)}\big](1_A)$ is a rational number.*

Proof Let G_0 be a finitely generated subgroup of $K_0(A)$ and $\gamma = [\phi_n^{(2)}]$ be well defined on it. Since $K_0(A) \cong K_0(B)$, to save notation we may assume that $K_0(A) = K_0(B)$. We view $K_0(A)$ as a linear space over $\mathbf{Q}$. Let G_1 be the subgroup generated by G_0 and $\gamma(G_0)$ and let V be the finite dimensional linear subspace generated by G_1. Since now V is a divisible subgroup of $K_0(A)$, we extend γ so it is defined on V. We also assume $\alpha(x) = x$ on V. Therefore there exists a positive integer $m > 0$ such that $\frac{-1}{1+1/m}$ is not an eigenvalue of γ. In particular,

$$\gamma(x) \neq \frac{-1}{1+1/m}(x)$$

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for any nonzero $x \in V$. This implies that

$$x + \gamma(x) + 1/m\gamma(x) \neq 0$$

for any nonzero $x \in G_0$.

Let C be a finite dimensional C^*-subalgebra of $B \otimes \mathcal{K}$ which contains the image of $\phi_n^{(2)}$. Since B has real rank zero, stable rank one and divisible $K_0(B)$, each minimal projection of C can be written as a direct sum of m equivalent projections. From this it is easy to see that there is a monomorphism $\Psi\colon C \to C'$ ($\subset B \otimes \mathcal{K}$) such that $[\Psi]([q]) = 1/m[q]$ on every projection $q \in C$. Now define $\phi_n^{(2)'} = \phi_n^{(2)} \oplus \Psi \circ \phi_n^{(2)}$. If we replace $\phi_n^{(2)}$ by $\phi_n^{(2)'}$, we see that the first part of the corollary follows.

Suppose that A has unique normalized quasitrace and $\tau\big(K_0(A)\big)$ is divisible dense subgroup of $\mathbf{R}$ containing 1. Suppose that $[\phi_n^{(2)}](1_A) = \theta$. If θ is rational, then the above proof works. Otherwise, assume that $k - \theta > 0$ for some positive integer k. Let C be as in the first part of the proof. By considering a corner of C, we may assume that $\phi_n^{(2)}$ is unital. It is easy to obtain a monomorphism $h_1\colon C \to B$ such that $\tau\big(h_1(1_C)\big) = k - \theta$ (see the last paragraph). Let $\psi_n^{(2)} = \phi_n^{(2)} + h_1 \circ \phi_n^{(2)}$. Then $s = [\psi_n^{(2)}](1_A)$ is a rational number. Note also $s + s/m$ is also a rational number. Thus, in the first part of the proof replacing $\phi_n^{(2)}$ by $\psi_n^{(2)}$, we see that we may assume that $\tau\big((\alpha + [\psi_n^{(2)}])(1_A)\big)$ is a rational number. ■

6 Classification of Simple TAF C^*-algebras with Unique Normalized Traces

Every simple separable TAF C^*-algebra is quasidiagonal (see [Ln6]). So a nuclear separable simple TAF algebra is strong NF (see [BK2]). It follows from [BK1] that A is the closure of the union of an increasing sequence of RFD C^*-algebras. So we can apply 5.9. The main result of this paper is that under certain restrictions on K-theory, two unital separable simple nuclear TAF C^*-algebras A and B satisfying the UCT are isomorphic if they have the same K-theory. The typical cases are:

(i) $K_0(A) = \mathbf{Z}[1/p]$ and $K_1(A)$ is torsion free,
(ii) $K_0(A) = \mathbf{Q}$ and $K_1(A)$ is any countable abelian group,
(iii) $K_0(A) = \mathbf{Q} \oplus \mathrm{Tor}\big(K_0(A)\big)$ and $K_1(A)$ is torsion free,
(iv) $K_0(A) = \mathbf{Q} \oplus \mathbf{Z}/p\mathbf{Z}$ (p is prime) and $K_1(A)$ is any countable abelian group without element of order p.

Precise conditions are stated in 6.6, 6.7, 6.8, 6.10, 6.11 and 6.13.

6.1

Let A be a stably finite C^*-algebra. We say $K_0(A)_+$ is *locally finitely generated*, if for any finitely generated subgroup $G \subset K_0(A)$, there exists a finitely generated subgroup $F \subset K_0(A)$ such that $G \subset F$ and $F \cap K_0(A)_+$ is finitely generated, *i.e.*,

$$F \cap K_0(A)_+ = \Big\{ \sum_{i=1}^{n} m_i g_i : m_i \in \mathbf{Z}_+ \text{ and } g_1, \dots, g_n \in F \cap K_0(A)_+ \Big\}.$$

Let $G \subset \mathbf{R}$ be a countable (additive) subgroup containing 1. If G is a subgroup of $\mathbf{Q}$, then G_+ is locally finitely generated. Suppose that G_0 containing 1 is generated by $r_1, r_2, \dots, r_n \in G_0$. We may assume that $r_i = p_i/q_i$, where $p_i, q_i \in \mathbf{Z}_+$, $q_i \neq 0$ and $(p_i, q_i) = 1$. There are $n_1, n_2 \in \mathbf{Z}$ such that $n_1 q_i + n_2 p_i = 1$. Thus $n_1 + n_2(r_i) = 1/q_i$ $(i = 1, 2, \dots, n)$. Therefore $1/q_i \in G_0$. Let $q_1 = k_1^{n(1)} k_2^{n(2)} \cdots k_m^{n(m)}$ be the prime decomposition. Then $1/k_j^{n(j)} \in G_0$. From here, one concludes that $1/m \in G_0$ where m is the least multiple of $q_1, q_2, \dots, q_n$. This implies that G_0 is generated by $1/m$. It is clear that $G_0 \cap G_+$ is generated by $1/|m|$. This also shows that any positive homomorphism $h\colon G_0 \to G_0$ has the form $h(g) = rg$ for some $r \in G_0$.

If G is dense in $\mathbf{Q}$, from the above, it shows that there is a sequence of (increasing) integers $\{p_n\} \in G$ such that $1/p_n \in G$.

On the other hand, if G_+ is locally finitely generated, then G does not contain any irrational number. In fact, if G contains an irrational number $\theta > 0$, $G_0 = \mathbf{Z} + \mathbf{Z}\theta$ is finitely generated. But $(G_0)_+$ contains arbitrary small positive numbers. This shows that $(G_0)_+$ can not be finitely generated.

6.2

Let A be a separable simple C^*-algebra of real rank zero, stable rank one and with weakly unperforated $K_0(A)$. Let S be the normalized quasitrace space of A. Let $d\colon K_0(A) \to \operatorname{Aff}(S)$ be defined by $d(x) = t(x)$ $(t \in S)$. It is known (see [BH]) that, if A is not elementary, then im d is dense in $\operatorname{Aff}(S)$. Furthermore

$$K_0(A)_+ = \{x \in K_0(A) : x = 0 \text{ or } d(x) > 0\}.$$

Let

$$0 \to J \to G \xrightarrow{d} D \to 0$$

be a short exact sequence of countable abelian groups and D be an ordered dense subgroup of $\mathbf{R}$. It is well known that a positive homomorphism $h\colon D \to \mathbf{R}$ has the form $h(r) = h(1)r$, $r \in D$, if $1 \in D$. Let $G_+ = \{g \in G : d(g) > 0, \text{ or } g = 0\}$. Let G_0 be a finitely generated subgroup of G. Since D is torsion free, we may write $G_0 = J' \oplus D_0$, where $J' \subset J$ and d is injective on D_0. Suppose that $1 \in D_0$. It is clear that if $D_0 \in \mathbf{Q}$, then $h(r) = h(1)r$. If D_0 contains an irrational number, then D_0 is dense in $\mathbf{R}$. So $h(r) = h(1)r$ for all $r \in D$.

Lemma 6.3 *Let G, J, D, J' and G_0 be as above. Suppose that $\gamma\colon G_0 \to G$ is a positive homomorphism on $(G_0, G_0 \cap G_+)$. Suppose that $\gamma(x) \neq 0$. Then $\gamma(x) > 0$ if and only if $x > 0$. Therefore, γ maps J' to J. In particular, if γ is injective then its inverse is also positive.*

Proof We need to show that if $x \in G_0$ with $\gamma(x) > 0$ then $x > 0$. Write $x = x_0 \oplus x_1$ according to the decomposition $G_0 = J' \oplus D_0$. Suppose that $x_1 \neq 0$. If x is not positive and nonzero, then $x_1 < 0$. Therefore $-x = -x_0 \oplus (-x_1) > 0$. In particular, $\gamma(-x) > 0$. This would imply that $\gamma(x) \leq 0$. This is impossible. Therefore $x_1 = 0$, *i.e.*, $x \in J'$. Let $g \in (G_0)_+$ and $g \neq 0$. Since $\gamma(x) > 0$, we have a positive integer $n > 0$

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such that $nd\big(\gamma(x)\big) > d\big(\gamma(g)\big)$. This implies that $d\big(\gamma(-nx+g)\big) < 0$. However, $-nx+g \geq 0$. This contradicts the assumption that γ is positive. So far we have shown that $\gamma(x) > 0$ implies that $x > 0$. The proof also implies that γ maps J' into J. The last part of the lemma also follows. ■

Lemma 6.4 *Let G, J, D, J' and G_0 be as above. Suppose that $\gamma\colon G_0 \to G$ is a positive homomorphism on $(G_0, G_0 \cap G_+)$ such that $d \circ \gamma(x) = rd(x)$ for some $r > 0$ in D and $rs \in D$ for all $s \in D$. Suppose further that J is divisible. Then there exists a positive homomorphism $\bar{\gamma}\colon G \to G$ such that $\bar{\gamma}|_{G_0} = \gamma$. In particular, this holds when $J = 0$.*

Proof Since J is divisible, we may write $G = J \oplus D$. Suppose that $G_0 = J' \oplus D_0$, where $J' \subset J$ and $D_0 \subset D$. Let $p_1\colon G \to J$ and $p_2\colon G \to D$ be the projections. Define $\gamma_1 = p_1 \circ \gamma$ and $\gamma_2 = p_2 \circ \gamma$. Clearly $\gamma_2(x) = rd(x)$ and $\gamma_2(x) = rp_2(x)$. Define $\bar{\gamma}_2(x) = rp_2(x)$ for all $x \in G$. Since J is divisible, there is $\bar{\gamma}_1\colon G \to J$ such that $(\bar{\gamma}_1)|_{G_0} = \gamma_1$. Now define $\bar{\gamma} = \bar{\gamma}_1 + \bar{\gamma}_2$. To show it is positive, let $0 < x \in G$. So $d(x) > 0$. Then

$$d\big(\bar{\gamma}(x)\big) = d\big(\bar{\gamma}_2(x)\big) = rp_2(x) = rd(x) > 0.$$

Thus $\bar{\gamma}$ is positive. It is clear that $\bar{\gamma}|_{G_0} = \gamma$. ■

Theorem 6.5 *Let A be a unital nuclear separable simple TAF C^*-algebra with the unique normalized trace τ, torsion free $K_*(A)$ and let $K_0(A)$ be a dense subring of $\mathbf{R}$ with $[1_A] = 1$ such that $K_0(A)_+$ is locally finitely generated, and let $B \in \mathcal{C}_0$ be a unital separable simple C^*-algebra with the same (ordered) K-theory and let $\alpha \in KL(A,B)_+$ be such that $\alpha|_{K_i(A)} = \mathrm{id}$, $i = 0, 1$. Then for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, a finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists an $\mathcal{F}$-ε-multiplicative completely positive contraction $\Phi\colon A \to B$ such that*

$$[\Phi]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

Proof It follows from [BK2] that A is a simple strong NF algebra. By [BK1], A is the closure of the union of an increasing sequence of $\{A_n\}$ of (nuclear) RFD C^*-algebras. We will apply 5.9. First we assume that A is not elementary. Otherwise the theorem is known. Let $\mathcal{P}$ be a finite subset of $\mathbf{P}(A)$ which contains 1_A and let $\mathcal{P}_0$ be the subset of $\mathcal{P}$ which represents the element in $K_0(A)$. (Note that here we only consider $K_0(A)$ and $K_1(A)$.) Let G_0 be the subgroup generated by $\mathcal{P}_0$. Let $D = K_0(A)$ which we identify it with a dense subring of $\mathbf{Q}$ (see 6.1).

Let $\phi_n = \phi_n^{(1)}$ and $\phi_n^{(2)}$ be as in 5.9 (corresponding to α here). Without loss of generality, we may assume that $[\phi_n^{(1)}]|_{G(\mathcal{P})}$ is well defined. By the assumption, we may assume that $(G_0)_+ = G_0 \cap K_0(A)_+$ is finitely generated. Therefore, we may assume that both $[\phi_n]|_{G_0}$ and $[\phi_n^{(2)}]|_{G_0}$ are positive (5.9), $s = [\phi_n](1_A) \in D$ and $s = 1 + t$, where $t = [\phi_n^{(2)}](1_A) \in D$. Furthermore, $\phi_n(1_A)$ is a projection. From the discussion of 6.1, there exists a sufficiently large integer $q \in \mathbf{Z}_+$ such that $1/q \in D$ and

$$q = Nt + l,$$

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where $N > 0$ and $l > 0$ are integers such that

$$l/q \leq 1/s.$$

Let $k_1 = (q/t)\big(1 - (l/q)s\big)$. Then

$$k_1 = (1/t)(q - l - lt) = (1/t)(Nt - lt) = N - l$$

is in $\mathbf{Z}_+$.

Define $\beta(r) = (l/q)r$ for all $r \in D$. Thus $\beta\colon K_0(B) \to K_0(B)$ is a positive homomorphism. From the classification of circle algebras (see [Ell1]), there is a homomorphism $h\colon B \to B$ such that $[h]|_{K_0(B)} = \beta$ and $[h]|_{K_1(B)} = \mathrm{id}_{K_1(B)}$ (here we identify $K_0(B)$ with $K_0(A) = D$). If $1/q = 1/s$, we let $\Phi = h \circ \phi_n$. Otherwise, we let $\beta_1(r) = (k_1/q)r$ for $r \in D$ and $h_1\colon B \to B$ such that $[h_1]|_{K_0(B)} = \beta_1$ and $[\beta_1]|_{K_1(B)} = \mathrm{id}_{K_1(B)}$. Set $\Psi = h_1 \circ \phi_n^{(2)}$. Note that

$$(k_1/q)t = (q/t)\big(1 - (l/q)s\big)(t/q) = 1 - (l/q)s.$$

So

$$(l/q)s + (k_1/q)t = 1.$$

Therefore

$$[h \circ \phi_n]([1_A]) + [\Psi]([1_A]) = 1 \quad \text{and} \quad [h \circ \phi_n] + [\Psi]|_{K_1(A)} = \alpha|_{K_1(A)}.$$

Set $\Phi = h \circ \phi_n \oplus \Psi$. Since $[\Phi]$ is positive on $G_0 \subset D$, from the discussion of 6.1,

$$[\Phi]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$$

as desired. ■

Theorem 6.6 *Let A and B be two unital separable simple nuclear TAF C^*-algebras satisfying the UCT, with the unique normalized trace, torsion free K_* such that $K_0(A)$ is a subring of* $\mathbf{Q}$. *Suppose that*

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big) \cong \big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big).$$

Then $A \cong B$.

Proof We may assume that $B \in \mathcal{C}_0$. In particular B is pre-classifiable (so the desired map from B to A exists). Thus the theorem follows immediately from the proof of 3.7 and the proof of 6.5. ■

Let Q be the unital UHF algebra with $K_0(Q) = \mathbf{Q}$.

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Theorem 6.7 **(The Rationalization Theorem)** *Let A and B be two unital separable simple nuclear TAF C^*-algebras satisfying the UCT and with unique traces. Suppose that $K_0(A)_+$ is locally finitely generated and*

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big) \cong \big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big).$$

Then $A \otimes Q \cong B \otimes Q$.

Corollary 6.8 *Let A be a separable unital nuclear simple TAF C^*-algebra with the unique normalized trace and satisfying the UCT. Suppose that $K_1(A) = 0$ and $K_0(A)_+$ is locally finitely generated. Then $A \otimes Q$ is a simple AF-algebra.*

Theorem 6.6 includes those C^*-algebras A with $K_0(A) = \mathbf{Q}$ or $K_0(A) = \mathbf{Z}[1/p]$, where $p \in \mathbf{N}$ is an integer, with usual order.

Finally we expand the classification results to include C^*-algebras with any K_1 and with torsion in their K_0.

Lemma 6.9 *Let A be a unital simple C^*-algebra in $\mathcal{C}_0$ (see 1.4). Then, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, a finite subset $\mathcal{F} \subset A$, $\delta > 0$ and $\varepsilon > 0$, there exists a projection $p \in A$ with $\tau(p) < \delta$ for all $\tau \in T(A)$ such that*

(1) $\|[p, x]\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $\mathrm{id}_A \approx_\varepsilon \Psi \oplus L$ on $\mathcal{F}$, where $\Psi(a) = pap$ and L is an $\mathcal{F}$-ε-multiplicative completely positive contraction from A to a finite dimensional C^*-subalgebra $C \subset (1-p)A(1-p)$ with the property that $[\Psi]|_{\mathcal{P}}$ and $[L]|_{\mathcal{P}}$ are well defined.

Moreover, if A has the unique normalized trace, $K_0(A)/\operatorname{tor}\big(K_0(A)\big)$ is divisible and elements in $\operatorname{tor}\big(K_0(A)\big)$ have bounded order, we may choose p so that $\tau(p)/\tau(1_A)$ is a rational number. Furthermore, if $K_0(A)_+$ is locally finitely generated, $[\Psi]$ and $[L]$ can be assumed to be positive on G_0.

Proof The first part of the lemma follows from 3.27 and 4.18 in [EG]. It is the second part of the lemma requires a proof. By [EG], we write $A = \lim_n(A_n, \psi_n)$, where $A_n = \bigoplus A_{n,i}$, $A_{n,i} = P_{n,i}M_{[n,i]}\big(C(X_{n,i})\big)P_{n,i}$, and each $X_{n,i}$ is a connected CW complex with dimension no more than 3. We write (see 3.27 and 4.18 in [EG]) each partial map by $\phi^{(i,j)}_{n,n+1} \colon A_{n,i} \to A_{n,j}$ and (with $L(n,i,j) \to \infty$ as $n \to \infty$)

$$\phi^{(i,j)}_{n,n+1}(f) = \operatorname{diag}\big(\psi_{n,i,j}(f), h^{(1)}_{n,i,j}(f), \dots, h^{(L(n,i,j))}_{n,i,j}(f), h^{(0)}_{n,i,j}(f)\big),$$

where $h^{(t)}_{n,i,j}$ ($t = 0, 1, \dots, L(n,i,j)$) are one-point-evaluations, $L(n,i,j) > 1/\delta$, $h^{(1)}_{n,i,j}(1_{A_{n,i}}), \dots, h^{(L(n,i,j))}_{n,i,j}(1_{A_{n,i}})$ are unitarily equivalent trivial projections in $A_{n,j}$ and $\psi_{n,i,j}(1_{A_{n,i}})$ and $h^{(1)}_{n,i,j}(1_{A_{n,i}})$ have the same rank. Therefore $\tau\Big(\psi_n\big(\psi_{n,i,j}(1_{A_{n,i}})\big)\Big) = \tau\Big(\psi_n\big(h^{(1)}_{n,i,j}(1_{A_{n,i}})\big)\Big)$. Moreover, $h^{(s)}_{n,i,j}(A_{n,i}) \cong M_{l(n,i)}$ for all j and s. Note we may assume that $\tau(1_A) = 1$. Let $k > 0$ be an integer such that $1/k < \delta$ and b be the largest order of elements in $\operatorname{tor}\big(K_0(A)\big)$. In particular, $b!\,z = 0$ for all $z \in \operatorname{tor}\big(K_0(A)\big)$. Set

$l = kb! + 1$. So $lz = z$ for all $z \in \operatorname{tor}\big(K_0(A)\big)$. By passing to a subsequence, we may assume that $L(n,i,j) \geq l+1$ for all i, j.

Let $e^{(K)}_{n,i,j,t}$, $K = 1, \dots, l(n,i)$ be minimal projections in $h^{(t)}_{n,i,j}(A_{n,i})$.

Fix n, i, j, and s. Since $K_0(A)/\operatorname{tor}\big(K_0(A)\big)$ is divisible, there are $x' \in K_0(A)_+$ such that $lx' - [e^{(K)}_{n,i,j,t}] = z \in \operatorname{tor}\big(K_0(A)\big)$. Define $x = x' - z$ (in $K_0(A)_+$). Then $lx = lx' - lz = lx' - z = [e^{(K)}_{n,i,j,t}]$. Hence there are mutually orthogonal and mutually equivalent projections $q_{i,j,t,s,K}$, $s = 1, 2, \dots, l$, $t = 1, 2, \dots, L(n,i,j)$ in A such that $\sum_s q_{i,j,t,s,K} = e^{(K)}_{n,i,j,t}$. We also have mutually orthogonal and mutually equivalent projections $q_{i,j,0,s,K}$, $s = 1, 2, \dots, l$ in A such that $\sum_s q_{i,j,0,s,K} = e^{(K)}_{n,i,j,0}$, where $e^{(K)}_{n,i,j,0}$ is a minimal projection of $h^{(0)}_{n,i,j}(A_{n,i})$. Set $q_{i,j,t,s} = \sum_K q_{i,j,K,t,s}$, $s = 1, 2, \dots, l$ and $t = 0, 1, \dots, L(n,i,j)$. We have $[q_{i,j,t,s}, c] = 0$ for all $c \in h^{(t)}_{n,i,j}(A_{n,i})$. Defining $h^{ts}_{n,i,j}(f) = h^{(t)}_{n,i,j}(f) q_{i,j,t,s}$. Let $m(i,j) = 1 + L(n,i,j) - l$. Rename $h^{ts}_{n,i,j}$, $s = 1, 2, \dots, l$ and $t = 1, 2, \dots, L(n,i,j)$, by $H^J_{n,i,j}$, $J = 1, 2, \dots, lL(n,i,j)$, and $h^{0s}_{n,i,j}$, $s = 1, 2, \dots, l$, by $F^s_{n,i,j}$, $s = 1, 2, \dots, l$. Set

$$d_{i,j} = \operatorname{diag}\big(\psi_{n,i,j}(1_{A_{n,i}}), H^{(1)}_{n,i,j}(1_{A_{n,i}}), \dots, H^{(m(i,j))}_{n,i,j}(1_{A_{n,i}}), F^{(1)}_{n,i,j}(1_{A_{n,i}})\big).$$

To save notation, in the following computation, we do not distinguish projections in A_{n+1} and their image in A. Note that $\tau\big(\psi_{n,i,j}(1_{A_{n,i}})\big) = \tau\big(h^{(1)}_{n,i,j}(1_{A_{n,i}})\big)$. So

$$\begin{aligned}
\tau(d_{i,j}) &= l\tau\big(H^{(1)}_{n,i,j}(1_{A_{n,i}})\big) + m(i,j)\tau\big(H^{(1)}_{n,i,j}(1_{A_{n,i}})\big) + (1/l)\tau\big(h^{(0)}_{n,i,j}(1_{A_{n,i}})\big) \\
&= \big(1 + L(n,i,j)\big)\tau\big(H^{(1)}_{n,i,j}(1_{A_{n,i}})\big) + (1/l)\tau\big(h^{(0)}_{n,i,j}(1_{A_{n,i}})\big) \\
&= (1/l)\tau\Big[\psi_{n,i,j}(1_{A_{n,i}}) \oplus \sum_t^{L(n,i,j)} h^{(t)}_{n,i,j}(1_{A_{n,i}})\Big] + (1/l)\tau\big(h^{(0)}_{n,i,j}(1_{A_{n,i}})\big) \\
&= (1/l)\tau(1_{A_{n,i}}).
\end{aligned}$$

Set $d_i = \bigoplus_i d_{i,j}$. Then $[\psi_n(d_i)] = (1/l)[\psi_n(1_{A_{n,i}})]$. Define $e = \sum_i d_i$ and $p = \psi_{n+1}(e)$. Then $[p] = (1/l)[1_A]$. From the above, we see that with the choice of p the lemma follows.

Finally, if $K_0(A)_+$ is locally finitely generated, as in the proof of (5) of 5.9, we may assume that both $[\Psi]$ and $[L]$ are positive on G_0. ■

Lemma 6.10 *Let A be a unital nuclear separable simple TAF C^*-algebra with the unique normalized trace τ, $K_0(A) = K_0(A)/\operatorname{tor}\big(K_0(A)\big) \oplus \operatorname{tor}\big(K_0(A)\big)$, $K_0(A)/\operatorname{tor}\big(K_0(A)\big)$ is divisible, $K_0(A)_+$ is locally finitely generated and the order in $\operatorname{tor}\big(K_0(A)\big)$ is bounded. We also assume that $(k,m) = 1$ if k is the (positive) order of an element in $\operatorname{tor}\big(K_0(A)\big)$ and m is the order of an element in $\operatorname{tor}\big(K_1(A)\big)$ (for example, $\operatorname{tor}\big(K_0(A)\big) = 0$). Suppose that $B \in \mathcal{C}_0$ is a unital separable simple C^*-algebra with the same K-theory and $\alpha \in KL(A,B)_+$ such that $\alpha|_{K_i(A)} = \mathrm{id}$, $i = 0, 1$. Then, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, a finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists an $\mathcal{F}$-ε-multiplicative completely positive contraction $\Psi\colon A \to B$ such that*

$$[\Psi]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

Proof It follows from [BK2] that A is a simple strong NF algebra. By [BK1], A is the closure of the union of an increasing sequence of $\{A_n\}$ of RFD C^*-algebras. We will apply 5.9. First we assume that A is not elementary. Otherwise the theorem is known. Let $\mathcal{P}$ be a finite subset of $\mathbf{P}(A)$ which contains 1_A and let $\mathcal{P}_0$ be the subset of $\mathcal{P}$ which represents the element in $K_0(A)$. Let $G_1 = K_0(A)/\operatorname{Tor}\big(K_0(A)\big)$ and G_0 be the subgroup generated by $\mathcal{P}_0$. Since G_1 is assumed to be divisible, we may write $K_0(A) = G_1 \oplus \operatorname{tor}\big(K_0(A)\big)$. Therefore we may write $G_1 = D \oplus G_1 \cap \ker d$, where $D = d\big(K_0(A)\big)$. Consequently, we have $K_0(A) = D \oplus \ker d$. Let $\pi_1\colon K_0(A) \to D$ and $\pi_2\colon K_0(A) \to \ker d$ be the projections.

Let $\phi_n^{(1)}$ and $\phi_n^{(2)}$ be as in 5.9 (corresponding to α here). Denote $\phi_n = \phi_n^{(1)}$. We assume that $[\phi_n]|_G$ is well defined, $[\phi_n]|_{G_0}$ and $[\phi_n^{(2)}]|_{G_0}$ are positive. Furthermore the image of $\phi_n^{(2)}$ is contained in C', a finite dimensional C^*-subalgebra. Since $[\phi_n^{(2)}]|_{G_0}$ is positive, $d\big([\phi_n^{(2)}](g)\big) = d\big(\pi_1(g)\big)$ for all $g \in G_0$.

We may assume that $1_{C'} = \phi_n^{(2)}(1_A)$. Choose an integer $K > 0$ so that $K - d([1_{C'}]) > 0$. Since D is divisible, it is easy to obtain a monomorphism $h_0'\colon C' \to M_S(B)$ for some possibly large integer S such that $d(h_0')_*(1_{C'}) = K - d([1_{C'}])$. By replacing $\phi_n^{(2)}$ by $\phi_n^{(2)} \oplus h_0' \circ \phi_n^{(2)}$, we may assume that $\theta = d\big([\phi_n^{(2)}](1_A)\big) = d[1_{C'}]$ is a rational number. Let $j'\colon C' \to M_S(B)$ be the embedding (for some large $S > 0$). Write $[1_{C'}] = (\theta, x)$ in $D \oplus \ker d$. Define $\lambda(r,z) = (r, -z)$ for $(r,z) \in D \oplus \ker d$. Then λ is positive. So one has a monomorphism $h'\colon C' \to M_S(B)$ such that $(h')_* = \lambda \circ (j')_*$. Then, if $g = (r,z)$, where $r = \pi_1(g)$ and $z = \pi_2(g)$,

$$\big([\phi_n^{(2)}] + (h')_* \circ [\phi_n^{(2)}]\big)(g) = (2\theta \cdot r, 0)$$

for $g \in G_0$. So, by replacing $\phi_n^{(2)}$ by $\phi_n^{(2)} \oplus h' \circ \phi_n^{(2)}$, we may assume that $[\phi_n^{(2)}](r,z) = (2\theta r, 0)$ for $g = (r,z) \in G_0$. Therefore, $[\phi_n](r,z) = (sr, z)$ for all $g = (r,z) \in G_0$, where $s = 1 + 2\theta$ is a rational number.

Let F be a finite dimensional C^*-subalgebra such that $\phi_n^{(2)}(A) \subset F$. Since $K_1(F) = K_1(F, \mathbf{Z}/k\mathbf{Z}) = 0$, we have $[\phi_n^{(2)}]|_{G \cap K_1(A)} = [\phi_n^{(2)}]|_{G \cap K_1(A, \mathbf{Z}/k\mathbf{Z})} = 0$. From the exact sequence

$$0 \to K_0(A)/kK_0(A) \to K_0(A, \mathbf{Z}/k\mathbf{Z}) \to \ker \mathbf{k} \to 0,$$

where $\ker \mathbf{k}$ is a subgroup of $\operatorname{tor}\big(K_1(A)\big)$ (see 1.6) and the fact that $[\phi_n^{(2)}]|_{G \cap K_0(A, \mathbf{Z}/k\mathbf{Z})}$ factors through $K_0(F, \mathbf{Z}/k\mathbf{Z})$, we conclude that $[\phi_n^{(2)}]$ maps $G \cap K_0(A, \mathbf{Z}/k\mathbf{Z})$ into $K_0(B)/kK_0(B) = \operatorname{tor}\big(K_0(A)\big)/kK_0(A)$ ($K_0(A)/\operatorname{tor}\big(K_0(A)\big)$ is divisible) and $[\phi_n^{(2)}]|_{G \cap K_0(A, \mathbf{Z}/k\mathbf{Z}) \cap \operatorname{tor}(K_0(A))/kK_0(A)} = 0$. Therefore $[\phi_n^{(2)}]|_{G \cap K_0(A, \mathbf{Z}/k\mathbf{Z})}$ is induced by a homomorphism from $\ker \mathbf{k}$ to $\operatorname{tor}\big(K_0(B)\big)/kK_0(B)) = \operatorname{tor}\big(K_0(A)\big)/kK_0(A)$. Since $(m_1, m_2) = 1$, if m_1 is the order of an element in $\operatorname{tor}\big(K_0(A)\big)$ and m_2 is the order of an element in $\operatorname{tor}\big(K_1(A)\big)$, there is no nonzero maps from any subgroup of $\ker \mathbf{k}$ to $\operatorname{tor}\big(K_0(A)\big)/kK_0(A)$. Hence $[\phi_n^{(2)}]|_{G \cap K_0(A, \mathbf{Z}/k\mathbf{Z})} = 0$. Therefore

$$[\phi_n]|_{G \cap K_1(A)} = \alpha|_{G \cap K_1(A)} \quad \text{and} \quad [\phi_n]|_{G \cap K_*(A, \mathbf{Z}/k\mathbf{Z})} = \alpha|_{G \cap K_*(A, \mathbf{Z}/k\mathbf{Z})}.$$

We denote $\bar{G}_0$ the image of G_0 under $[\phi_n]$.

Fix a sufficiently large n. We now apply 6.9. We will use some of notation there. Assume that $P = \phi_n(1_A)$ is a projection in $M_N(B)$. Write

$$\mathrm{id}_{PM_N(B)P} \approx \Psi \oplus L,$$

https://doi.org/10.4153/CJM-2001-007-8 Published online by Cambridge University Press

where both Ψ and L are $\mathcal{G}$-η-multiplicative contractive completely positive linear morphisms with any given $\mathcal{G}$ and η, the image of L is contained in a finite dimensional C^*-subalgebra C_1, $[\Psi]|_{\mathcal{P}}$ and $[L]|_{\mathcal{P}}$ are well defined, $[\Psi]_{\bar{G}_0}$ and $[L]_{\bar{G}_0}$ are positive. Furthermore, $\tau\Big(\Psi\big(\phi_n(1_A)\big)\Big) = t$ so that t/s is a rational number, where $s = \tau\big(\phi_n(1_A)\big)$ is also a rational number and $t < 1/2$. This also implies that t is a rational number. Denote $L_1 = \Psi \circ \phi_n$ and $L_2 = L \circ \phi_n$. Since the image of L is contained in a finite dimensional C^*-subalgebra and $K_*(B) = K_*(A)$, as above, we may assume that $[L_1]|_{G\cap K_1(A)} = \alpha_{G\cap K_1(A)}$ and $[L_1]|_{G\cap K_*(A,\mathbf{Z}/k\mathbf{Z})} = \alpha|_{G\cap K_*(A,\mathbf{Z}/k\mathbf{Z})}$. Denote by $j\colon C_1 \to PM_N(B)P$ the embedding. Let $\lambda_1\colon K_0(B) \to K_0(B)$ be defined by $\lambda_1\big((r,z)\big) = \big((1-t)r/(s-t), z\big)$. Note $(1-t)/(s-t)$ is a rational number. So λ_1 is a positive homomorphism. There is a homomorphism $H\colon C_1 \to \Big(1 - [\Psi]\big(\phi_n(1_A)\big)\Big) M_N(B) \Big(1 - [\Psi]\big(\phi_n(1_A)\big)\Big)$ so that $H_* = \lambda_1 \circ j$. Set $\Phi = \Psi \circ \phi_n \oplus H \circ L \circ \phi_n$. Note that, on G_0, $[\Psi \circ \phi_n] = [\phi_n] - [L \circ \phi_n]$. So (with $g = (r,z) \in G_0$),

$$[L \circ \phi_n](r,z) = \big((s-t)r, \pi_2 \circ [L \circ \phi_n](g)\big)$$

and

$$[\Psi \circ \phi_n](r,z) = \big(\mathrm{tr}, z - \pi_2 \circ [L \circ \phi_n](g)\big).$$

One then computes that

$$\begin{aligned}
[\Phi](r,z) &= \big(\mathrm{tr}, z - \pi_2 \circ [L \circ \phi_n](g)\big) + \lambda_1\big((s-t)r, \pi_2 \circ [L \circ \phi_n](g)\big) \\
&= \big(\mathrm{tr}, z - \pi_2 \circ [L \circ \phi_n](g)\big) + \big((1-t)r, \pi_2 \circ [L \circ \phi_n](g)\big) \\
&= (r,z) = \alpha(r,z)
\end{aligned}$$

for $(r,z) \in G_0$. From the above, we have $[\Phi]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$. ∎

Theorem 6.11 *Let A and B be two unital separable simple nuclear TAF C^*-algebras satisfying the UCT, with unique normalized trace such that $K_0(A) = K_0(A)/\operatorname{tor}\big(K_0(A)\big) \oplus \operatorname{tor}\big(K_0(A)\big)$, $K_0(A)/\operatorname{tor}\big(K_0(A)\big)$ is divisible, $K_0(A)_+$ is locally finitely generated, $\operatorname{tor}\big(K_0(A)\big)$ has bounded order and $(k,m) = 1$ for any positive order k in $\operatorname{tor}\big(K_0(A)\big)$ and any positive order m in $\operatorname{tor}\big(K_1(A)\big)$.*

Suppose that

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big) \cong \big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big).$$

Then $A \cong B$.

(This includes the cases in which $K_0(A) = \mathbf{Q}$ and any $K_1(A)$, and the cases in which $K_0(A) = \mathbf{Q} \oplus G$, where G is any finite abelian group, and $K_1(A)$ is torsion free. Also included are the cases that $K_0(A) = \mathbf{Q} \oplus \mathbf{Z}/2\mathbf{Z}$ and any $K_1(A)$ such that $K_1(A)$ has no finite even order elements.)

Proof The proof is the same as 6.6 but we apply 6.10. ∎

6.12

The condition that $K_0(A)_+$ is locally finitely generated is used (only) to make sure that both $[\phi_n^{(1)}]|_{G_0}$ and $[\phi_n^{(2)}]|_{G_0}$ are positive. From 5.9, it suffices to assume $[\phi_n^{(2)}]|_{G_0}$ is positive. A unital separable simple TAF C^*-algebra is a direct limit of RFD C^*-algebras, *i.e.*, $A = \lim_{n\to\infty}(A_n, \psi_n)$, where each A_n is a unital RFD C^*-algebra. Suppose that each $(\psi_n)_*$ does not map non-positive elements to (non-zero) positive element, then $(\psi_{k,\infty})_*\big(K_0(A_k)\big)_+ \subset (\psi_{k,\infty})_*\big(K_0(A_k)_+\big)$. By (4) in 5.9, we may assume each $\phi_n^{(2)}|_{A_k}$ is a homomorphism. Therefore, we may always have $[\phi_n^{(2)}]|_{G_0}$ is positive. Consequently, we may always assume that $[\phi_n^{(1)}]|_{G_0}$ is positive. Therefore, the condition $K_0(A)_+$ is locally finitely generated can be dropped. So we have the following:

Corollary 6.13 *Let $A = \lim_n(A_n\psi_n)$ be a unital separable nuclear simple TAF C^*-algebra where each A_n is a unital RFD C^*-algebra and homomorphism $(\psi_n)_*$ does not map non-positive elements to (nonzero) positive elements. Then all conclusions of 6.5, 6.6, 6.7, 6.8, 6.10 and 6.11 hold without assuming that $K_0(A)_+$ is locally finitely generated.*

6.14 Final Remark

A theorem for classification of nuclear C^*-algebras usually contains two parts: a uniqueness theorem and an existence theorem. In earlier situations, the harder part always lies in the uniqueness. Theorem A (2.1) is powerful enough to handle all simple TAF C^*-algebras. This is clearly demonstrated in Theorem 3.7. An existence theorem would completely classify simple separable nuclear TAF C^*-algebras. However the existence theorem (5.9), which is unlikely to be improved under present assumptions, is not powerful enough. It is really an existence theorem modulo some morphisms with finite dimensional range. It seems technically difficult to recover the information lost in the quotient. In particular, the order of K-theory seems hard to recover. When C^*-algebras have unique normalized traces, we are able to recover most of them. That is the reason that we are able to prove 6.6–6.11. There is no doubt these results can be improved. However, to get the general classification theorem, one needs to improve what we have in 5.9. Some technical improvement has been made as this paper is revised.

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https://doi.org/10.4153/CJM-2001-007-8 Published online by Cambridge University Press