BULL.AUSTRAL.MATH.SOC. 20D10
VOL. 32 (1985), 293-297.

NILPOTENT INJECTORS IN FINITE GROUPS

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Nilpotent injectors exist in all finite groups.

For every Fitting class F of finite groups (see [2]), $Inj_F(G)$ denotes the set of all $H \le G$ such that for each $N \trianglelefteq\trianglelefteq G$, $H \cap N$ is an F-maximal subgroup of N (that is, belongs to F and is maximal among the subgroups of N with this property). Let N and N^* denote the Fitting class of all nilpotent and quasi-nilpotent groups, respectively. (For the basic properties of quasi-nilpotent groups, and of the N^*-radical $F^*(G)$ of a finite group G, the reader is referred to [5], X. §13; we shall use these properties without further reference.) Blessenohl and H. Laue have shown in [1] that for every finite group G, $Inj_{N^*}(G) = \{H \le G \mid H \ge F^*(G)\ N^*\text{-maximal in } G\}$ is a non-empty conjugacy class of subgroups of G. More recently, Iranzo and Pérez-Monasor have verified $Inj_N(G) \ne \emptyset$ for all finite groups G satisfying $G = C_G(E(G))E(G)$ (see [6]), and have extended this result to a somewhat larger class M of finite groups G (see [7]). One checks, however, that M does not contain all finite groups; for example, $S_5 \notin M$. Here we shall apply a result of Glauberman [4] together with the Feit-Thompson Theorem to derive from the Blessenohl-Laue result the following

Received 27 February 1985

THEOREM. *For every finite group* G *and every* $T/Z(E(G)) \in Syl_2(E(G)/Z(E(G)))$,

$$\emptyset \neq Inj_{N*}(N_G(T)) \subseteq Inj_N(G).$$

Proof. To begin with, we shall prove:

$(*)$ $H \in Inj_{N*}(N_G(T)) \Rightarrow H \in N$; in particular, $H \in Inj_N(N_G(T))$.

Since $H = F^*(H) = E(H)F(H)$ (a central product), it suffices to show that $E(H) = 1$. The nilpotent (and thus quasi-nilpotent) normal subgroup T of $N_G(T)$ is contained in $F(H)$. Similarly, $F(G) \le C_G(E(G)) \le C_G(T) \le N_G(T)$ gives $F(G) \le F(H)$. Hence

$$E(H) \le C_H(F(H)) \le C_H(TF(G)).$$

Put $Z = Z(E(G))$, $C = C_G(E(G)/Z)$, $D = C_G(T/Z) \ge C$. We have just shown that $D \ge E(H)$. As $E(G)/Z$ is a direct product of non-abelian simple groups, we have that (to within isomorphism) D/C is a subgroup of $C_{Aut(E(G)/Z)}(T/Z)$, which (in view of $T/Z \in Syl_2(E(G)/Z))$ by a result of Glauberman [4], is a 2-nilpotent-group; note that $O_{2'}(E(G)/Z) = 1$ follows from the Feit-Thompson Theorem. Hence $E(H)C/C$ is 2-nilpotent and, by the Feit-Thompson Theorem again, is soluble. We conclude that the perfect group $E(H)$ is contained in C. As $Z(E(G)) = \Phi(E(G))$, our Proposition below therefore shows that $E(H)/C_{E(H)}(E(G))$ is nilpotent. Now perfectness of $E(H)$ yields that $E(H) \le C_G(E(G))$. Together with the above observation this gives

$$E(H) \le C_G(E(G)) \cap C_G(F(G)) = C_G(F^*(G)) = Z(F^*(G)).$$

From this together with perfectness of $E(G)$ again, the desired conclusion that $E(H) = 1$ is immediate.

Using induction on $|G|$, we can now prove that $H \in Inj_N(G)$.

(1) $H \le X \le G,\ X \in N \Rightarrow H = X$:

Clearly, $T/Z \in Syl_2(X/Z \cap E(G)/Z)$, whence $X/Z \cap E(G)/Z \trianglelefteq X/Z \in N$ implies that $T/Z = O_2(X/Z \cap E(G)/Z) \trianglelefteq X/Z$; that is, $X \le N_G(T)$.

Thus (1) follows from $(*)$.

(2) If N is a maximal normal subgroup of G, then $H \cap N \in Inj_N(N)$:

By the inductive hypothesis, it suffices to show that $H \cap N \in Inj_{N^*}(N_N(S))$ for some $S \geq Z(E(N))$ such that $S/Z(E(N)) \in Syl_2(E(N)/Z(E(N)))$. Now note that $H \cap N \in Inj_{N^*}(N_N(T))$. Therefore, if $E(G) \leq N$ (in which case $E(N) = E(G)$), we may choose $S = T$, proving our claim. Suppose, then, that $E(G) \nleq N$. Then $G = NE(G)$, and $E(G)$ is a central product of $E(N)Z = E(G) \cap N$ and EZ for some component E of $E(G)$. Let $S = T \cap E(N)$. Then from $Z \cap E(N) = Z(E(N))$ we infer that $S/Z(E(N)) \in Syl_2(E(N)/Z(E(N)))$. Moreover,

$$N_N(T) \leq N_N(T \cap E(N)) = N_N(S) \leq N_N(SR) = N_N(T),$$

where $R = T \cap EZ$; observe that $T/Z = (T \cap E(N)Z)/Z \times (T \cap EZ)/Z$ and $[N,R] \leq [N,EZ] \leq N \cap EZ = Z \leq R$. Hence $N_N(T) = N_N(S)$, and the proof of (2) is complete.

Finally, $H \in Inj_N(G)$ follows from (1 + 2). □

The second half of the above proof yields the following

COROLLARY. *Let* $T/Z(E(G)) \in Syl_p(E(G)/Z(E(G)))$. *Then*

$$Inj_N(N_G(T)) \subseteq Inj_N(G)).$$

Note that for $p \nmid |E(G)/Z(E(G))|$, $N_G(T) = G$. Furthermore, it is not in general true that $Inj_N(N_G(T))$ is a single conjugacy class of subgroups of G, as is shown by taking $G = A_{12} (= E(G))$ and $T \in Syl_7(G)$; also, in this example, $Inj_N(N_G(T)) \cap Inj_{N^*}(N_G(T)) = \emptyset$, for $A_5 \times T \cong C_G(T) \in Inj_{N^*}(N_G(T))$.

For any finite group G, let $F'(G) = S(G \bmod \Phi(G))$, where $S(X)$ denotes the socle of X. Clearly, $F^*(G) \leq F'(G))$ and $F'(F^*(G))=F^*(G)$. In our proof of the above Theorem we have applied the following 'global' version of [3], *1.2b*:

PROPOSITION. *Suppose that for some finite group* G *and some* $A \leq Aut(G)$, *we have* $[A,F_i] \leq F_{i-1} (i = 1,\dots,n)$, *where the* $F_i \trianglelefteq G$ *are such that*

$$\Phi(G) = F_0 \leq F_1 \leq \dots \leq F_n = F'(G).$$

Then A *is a nilpotent* $\pi(F'(G))$*-group.*

https://doi.org/10.1017/S0004972700009965 Published online by Cambridge University Press

Proof. Consider the semidirect product $H = A\bar{G}$, where $\bar{G} = G/\Phi(G)$. By [3], *1.1*, $C_{\bar{G}}(F'(\bar{G})) \leq F'(\bar{G})$, and so we obtain from Dedekind's modular law

$$AF'(\bar{G}) = AC_{\bar{G}}(F'(\bar{G}))F'(\bar{G}) = C_H(F'(\bar{G}))F'(\bar{G}) \trianglelefteq H,$$

$$[A,G] \leq AF'(G) \cap G = F'(G).$$

This together with $[A,F_i] \leq F_{i-1}$ shows that A centralises each chief factor of $G/\Phi(G)$. Since the latter condition is inherited by $A/C_A(G/O_{p'}(G))$, $(G/O_{p'}(G)$ from A, G, we may apply [3], *1.2b* to conclude that for every prime p

$$A/C_A(G/O_{p'}(G)) \text{ is a } p\text{-group}.$$

If π denotes $\pi(F'(G))$, then from $S(G) \leq F'(G)$ we get

$$\cap_{p\in\pi} O_{p'}(G) = 1.$$

Consequently, $A \cong A/\cap_{p\in\pi} C_A(G/O_{p'}(G))$ is a nilpotent π-group. □

Notice that a somewhat more careful argument in the above proof would have shown that $[A,G] \leq F(G)$, yielding that A is a $\pi(F(G))$-group.

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https://doi.org/10.1017/S0004972700009965 Published online by Cambridge University Press