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On the Fourier Transformability of Strongly Almost Periodic Measures

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Abstract. In this paper we characterize the Fourier transformability of strongly almost periodic measures in terms of an integrability condition for their Fourier–Bohr series. We also provide a necessary and sufficient condition for a strongly almost periodic measure to be the Fourier transform of a measure. We discuss the Fourier transformability of a measure on $\mathbb{R}^d$ in terms of its Fourier transform as a tempered distribution. We conclude by looking at a large class of such measures coming from the cut and project formalism.

1 Introduction

The Fourier transform of a function plays a fundamental role in many areas of mathematics. In the first half of the 20$^{\text{th}}$ century, Laurent Schwartz extended the Fourier transform to a larger class of objects, namely tempered distributions. This theory extends the classical Fourier transform of functions, and includes all finite measures, all continuous and bounded functions, as well as a large class of unbounded measures. Some of the notions have been extended to arbitrary locally compact Abelian groups (LCAG's) G [12], but so far these extensions have not been as useful for the study of measures as in the case $G = \mathbb{R}^d$.

Motivated by Bochner's Theorem, Argabright and deLamadrid introduced the notion of Fourier transform for unbounded measures over arbitrary locally compact Abelian groups (LCAG's), and proved that positive definite measures are Fourier transformable [1] (see also [10, 29]). Their theory of Fourier transforms of measures generalizes the classical theory of Fourier transforms of functions, as well as the Fourier–Stieltjes transform. The Fourier transforms of measures play a fundamental role for mathematical diffraction and aperiodic order (see, for example, [3, 7, 18, 20, 29, 34, 35, 43]).

There is a hidden strong connection between the Fourier transforms of measures and the class of (weakly) almost periodic functions and measures. Eberlein proved that there exists a canonical decomposition of a weakly almost periodic function into a strongly almost periodic function and a null weakly almost periodic function [15]. We will refer to this decomposition as the *Eberlein decomposition*. Positive definite continuous functions, and hence the Fourier transforms of finite measures, are weakly almost periodic [13]. Given a finite measure, μ, the Eberlein decomposition of the weakly almost periodic function $\widehat{\mu}$ is exactly the Fourier transform [13, 29] of the

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Lebesgue decomposition $\mu = \mu_{\mathsf{pp}} + \mu_{\mathsf{c}}$. deLamadrid and Argabright extended the concept of almost periodicity to translation-bounded measures, via convolution with compactly supported continuous functions [19] (see also [29] for a self contained exposition of these topics). They showed that weakly almost periodic measures also have a canonical Eberlein decomposition. Moreover, the Fourier transform $\widehat{\mu}$ of each transformable measure μ is weakly almost periodic, and the Eberlein decomposition of the Fourier transform is exactly the dual of the Lebesgue decomposition of μ [19]. Recently, the Fourier dual of this result was proved by Moody and me [29]: if any translation-bounded Fourier transformable measure μ is weakly almost periodic, the strong almost periodic component μ_{s} and the null weakly almost periodic component μ_0 are Fourier transformable, and their Fourier transforms are exactly the pure point component $\widehat{\mu}_{\mathsf{pp}}$ and the continuous component $\widehat{\mu}_{\mathsf{c}}$ of $\widehat{\mu}$. This last version of the result is important for mathematical diffraction, since we would like to study the pure point spectrum $\widehat{\gamma}_{\mathsf{pp}}$ and the continuous spectrum $\widehat{\gamma}_{\mathsf{c}}$ of a structure ω, without going to the Fourier dual space. These results allow us to study the pure point and continuous spectra, respectively, by studying the components γ_{s} and γ_0, respectively, of the autocorrelation γ of ω, an idea that has been used effectively in many places (such as [2, 3, 21, 38, 39, 41, 42]). The particular connection between strong almost periodicity and pure point Fourier transform was also exploited in articles such as [3, 5–9, 16, 17, 21–23, 28, 33–35, 40, 43].

It follows from the results in [19] that if a measure μ is Fourier transformable, its Fourier transform $\widehat{\mu}$ is strongly almost periodic exactly when μ is a pure point measure. In this case, the strongly almost periodic measure $\widehat{\mu}$ has a Fourier–Bohr series (see Definition 6.12) $\mathcal{F}_{\mathsf{d}}(\widehat{\mu})$, which is exactly the reflection $\mu^{\dagger}$ of μ. In the same way, if μ is Fourier transformable, its Fourier transform $\widehat{\mu}$ is pure point exactly when μ is strongly almost periodic, and $\widehat{\mu}$ is exactly the Fourier–Bohr series $\mathcal{F}_{\mathsf{d}}(\mu)$.

Every strongly almost periodic measure μ comes with a Fourier–Bohr series $\mathcal{F}_{\mathsf{d}}(\mu)$, which is exactly $\widehat{\mu}$ (resp., $\widecheck{\mu}^{\dagger}$) whenever μ is Fourier transformable (or a Fourier transform). It is natural to ask what extra condition $\mathcal{F}_{\mathsf{d}}(\mu)$ should satisfy in order for μ to be Fourier transformable (resp., a Fourier transform). The main goal of this paper is to answer these two questions.

We show in Theorem 7.1 that a necessary and sufficient condition for a strongly almost periodicity measure μ to be Fourier transformable is a certain integrability condition, which we call *weak admissibility* (see Definitions 3.1 and 6.7), being satisfied by the Fourier–Bohr series. The second question is answered in Theorem 8.1. We show that a strongly almost periodic measure μ is a Fourier transform if and only if μ is weakly admissible and its Fourier–Bohr series is a measure.

In the particular case $G = \mathbb{R}^d$, which is the case in most practical applications, we use a result of Lin [24] to show that the weak admissibility condition can be replaced by the much more concrete notion of translation boundedness. As a consequence, we get that a strongly almost periodic measure $\mu \in \mathcal{SAP}(\mathbb{R}^d)$ is Fourier transformable if and only if its Fourier–Bohr series is a translation-bounded measure. In the same way, a strongly almost periodic measure $\mu \in \mathcal{SAP}(\mathbb{R}^d)$ is a Fourier transform if and only if its Fourier–Bohr series is a measure.

We also study the connection between the Fourier transformability of a measure in $\mathbb{R}^d$ and its Fourier transformability as a tempered distribution. In [1], the authors introduced a measure μ on $\mathbb{R}$, which is positive definite, tempered as a distribution, but for which the variation measure $|\mu|$ is not tempered. In particular, μ is not translation bounded as a measure. Since μ is positive definite, it is Fourier transformable and its Fourier transform $\widehat{\mu} =: \nu$ is translation bounded [1]. It follows that ν is a tempered distribution whose Fourier transform is the measure $\mu^{\dagger}$, but is not Fourier transformable as a measure (see [44] for more details). This raises an interesting question: what is the connection between the Fourier transform of measures on $\mathbb{R}^d$ and their Fourier transform as distributions (compare [44]). We answer this question in Theorem 5.2: we show that a translation-bounded measure μ on $\mathbb{R}^d$ is Fourier transformable as a measure if and only if its Fourier transform in the tempered distribution sense is a translation-bounded measure. Moreover, in this case, the two Fourier transforms coincide.

2 Definitions and Notations

Throughout this paper, G will denote a Locally Compact Abelian Group (LCAG). We will denote by $C_{\mathsf{u}}(G)$ the space of uniformly continuous and bounded functions on G. Then $C_0(G)$ and $C_{\mathsf{c}}(G)$ will denote the subspaces of $C_{\mathsf{u}}(G)$ consisting of functions vanishing at infinity and functions with compact support, respectively.

Recall that each LCAG G comes with a dual group $\widehat{G}$, defined as the set of *continuous characters*

$$\chi\colon G \to U(1) := \{z \in \mathbb{C} : |z| = 1\},$$

where χ is a continuous group homomorphism. The set $\widehat{G}$ becomes a LCAG with the group operation being pointwise multiplication of characters and the topology being that of that of uniform convergence on compact sets (see [29, Sect. 4.2.1] for more details).

In the spirit of Bourbacki [11], by a *Radon measure* we mean a linear function $\mu\colon C_{\mathsf{c}}(G) \to \mathbb{C}$ such that for each compact set $K \subset G$, there exists a constant C_K such that for all $f \in C_{\mathsf{c}}(G)$ with $\operatorname{supp}(f) \subset K$, we have

$$|\mu(f)| \leq C_K \|f\|_\infty.$$

The equivalence between this definition and the measure theory definition of regular Radon measures is provided by the Riesz-Representation Theorem [31, 32] (see also [35, Appendix] for a discussion of this). We will refer to a Radon measure simply as a measure. We will often write $\langle \mu, f\rangle$ or $\int_G f(t)\,d\mu(t)$ instead of $\mu(f)$.

A measure μ is called *positive* if, for all $f \in C_{\mathsf{c}}(G)$ with $f \geq 0$ we have

$$\mu(f) \geq 0.$$

Any regular Radon measure μ can be written as a linear combination

$$\mu := \mu_1 - \mu_2 + i(\mu_3 - \mu_4),$$

of four positive regular Borel measures (see [35, Thm. C4] for example). If all except maybe one of the measures $\mu_1, \mu_2, \mu_3, \mu_4$ can be chosen finite, then μ is a *regular Borel*

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measure. In particular, positive regular Borel and Radon measures coincide (see [35, 37] for more detail details).

Finally, let us emphasize that a regular Borel measure μ assigns a value $\mu(B)$ in the extended complex plane $\mathbb{C} \cup \{\infty\}$ to each Borel set $B \subseteq G$, but the set of regular Borel measures is not closed under addition. In contrast, the set of regular Radon measures is a vector space, but regular Radon measures assign a (finite) value only to pre-compact Borel sets $B \subseteq G$.

Let us also recall that, given a measure μ, there exists a positive measure $|\mu|$ such that, for all $f \in C_c(G)$ with $f \geq 0$, we have [30, 35]

$$|\mu|(f) = \sup\{|\mu(g)| : g \in C_c(G) \text{ with } |g| \leq f\}.$$

The measure $|\mu|$ is called the *total variation of* μ.

Next, let us recall the definition of Fourier transformability for measures.

Definition 2.1 A measure μ is called *Fourier transformable* if there exists some measure $\widehat{\mu}$ on $\widehat{G}$ such that, for all $f \in C_c(G)$, we have

$$\left|\widecheck{f}\right|^2 \in L^1(\left|\widehat{\mu}\right|) \quad \text{and} \quad \langle \mu, f * \widetilde{f} \rangle = \langle \widehat{\mu}, \left|\widecheck{f}\right|^2 \rangle.$$

Here, for a function $f\colon G \to \mathbb{C}$ we denote by $\widetilde{f}\colon G \to \mathbb{C}$ the function

$$\widetilde{f}(x) := \overline{f(-x)}.$$

In the spirit of [19], we define

$$K_2(G) := \operatorname{Span}\{f * g \mid f, g \in C_c(G)\}.$$

Given a subspace $V \subset L^1(G)$ we will denote by

$$\widehat{V} := \{\widehat{f} \mid f \in V\} \subset C_0(\widehat{G}).$$

Remark 2.2

(i) By the depolarisation identity, a measure is Fourier transformable if and only if there exists some measure $\widehat{\mu}$ on $\widehat{G}$ such that $\widecheck{K_2(G)} \subset L^1(|\widehat{\mu}|)$, and for all $f \in K_2(G)$, we have $\langle \mu, f \rangle = \langle \widehat{\mu}, \widecheck{f} \rangle$.

(ii) Any positive definite measure is Fourier transformable and its transform is positive [1, 10].

Let us now recall the definition of translation boundedness.

Definition 2.3 A measure μ is called *translation bounded* if for all compact sets $K \subset G$, we have

$$\|\mu\|_K := \sup_{x \in G} |\mu|(x + K) < \infty.$$

We denote the space of translation-bounded measures by $\mathcal{M}^\infty(G)$.

Remark 2.4

(i) A measure μ is translation bounded if and only if

$$\|\mu\|_K < \infty,$$

for one compact set K with non-empty interior [9].

(ii) If K is a fixed compact set with non-empty interior, then $\| \cdot \|_K$ is a norm on $\mathcal{M}^\infty(G)$.

An alternate characterisation of translation boundedness is given by the following result.

Theorem 2.5 ([1, Thm. 1.1]) *A measure μ is translation bounded if and only if for all $f \in C_{\mathsf{c}}(G)$, we have $\mu * f \in C_{\mathsf{u}}(G)$.*

2.1 Almost Periodic Measures

In this subsection, we briefly review the basic properties of almost periodic functions and measures. For a more detailed review of this, we refer the reader to [29].

Definition 2.6 A function $f \in C_{\mathsf{u}}(G)$ is called *strong almost periodic* or *Bohr almost periodic* if the closure of $\{T_t f \mid t \in G\}$ with respect to $\| \cdot \|_\infty$ is compact in $(C_{\mathsf{u}}(G), \| \cdot \|_\infty)$.

A function $f \in C_{\mathsf{u}}(G)$ is called *weakly almost periodic* if the closure of $\{T_t f \mid t \in G\}$ with respect to the weak topology of the Banach space $(C_{\mathsf{u}}(G), \| \cdot \|_\infty)$ is weakly-compact.

We denote the space of strong (resp., weakly) almost periodic functions by $SAP(G)$ (resp., $WAP(G)$).

Remark 2.7

(i) $WAP(G)$ and $SAP(G)$ are closed subspaces of $(C_{\mathsf{u}}(G), \| \cdot \|_\infty)$ [13] (see also [29]).

(ii) $WAP(G)$ and $SAP(G)$ are closed under multiplication, complex conjugation, reflection, and taking the absolute value [13].

Next, we review the notion of null weak almost periodicity for functions. We first need to recall the definition of the mean of a weakly almost periodic function, and for this we need to talk first about averaging sequences.

Definition 2.8 A sequence $\{A_n\}$ of compact subsets of G is called a *Fölner sequence* if, for all $x \in G$, we have

$$\lim_{x} \frac{\operatorname{vol}((x + A_n)\Delta A_n)}{\operatorname{vol}(A_n)} = 0.$$

A sequence $\{A_n\}$ of compact subsets of G is called a *van Hove sequence* if, for all compact sets $K \subseteq G$, we have

$$\lim_{x} \frac{\operatorname{vol} \partial^K(A_n)}{\operatorname{vol}(A_n)} = 0,$$

where, the *K-boundary of A_n* is defined as

$$\partial^K(A_n) = \overline{((A_n + K)\backslash A_n)} \cup ((\overline{G\backslash A_n} - K) \cap A_n).$$

It is easy to see that each van Hove sequence is a Fölner sequence.

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Lemma 2.9 ([13, 29]) *Let $f \in WAP(G)$ and let $\{A_n\}$ be a Fölner sequence in G. Then the limit*

$$\lim_n \frac{1}{\mathrm{vol}(A_n)} \int_{x+A_n} f(t)dt$$

exists uniformly in $x \in G$ and is independent of x and of the choice of the Fölner sequence $\{A_n\}$.

Definition 2.10 Let $f \in WAP(G)$ and let $\{A_n\}$ be a Fölner sequence in G. The number

$$M(f) := \lim_n \frac{1}{\mathrm{vol}(A_n)} \int_{A_n} f(t)dt,$$

is called the *mean of f*.

A function $f \in WAP(G)$ is called *null weakly almost periodic* if $M(|f|) = 0$. We denote the space of null weakly almost periodic functions by $WAP_0(G)$.

In the spirit of [19], we extend the notions of almost periodicity to measures (see also [29]).

Definition 2.11 A measure $\mu \in \mathcal{M}^\infty(G)$ is called *strong almost periodic*, *weakly almost periodic*, and *null weakly almost periodic* if for all $f \in C_c(G)$ the function $f * \mu$ is strong almost periodic, weakly almost periodic, null weakly almost periodic, respectively.

We will denote the spaces of almost periodic measures by $\mathcal{SAP}(G), \mathcal{WAP}(G)$, respectively $\mathcal{WAP}_0(G)$.

Similar to functions, weakly almost periodic measures have a well defined mean.

Lemma 2.12 ([19, 29]) *Let $\mu \in \mathcal{WAP}(G)$. Then, there exists a number $M(\mu)$ such that, for all $f \in C_c(G)$, we have*

$$M(\mu * f) = M(\mu) \int_G f(t)dt.$$

Moreover, if $\{A_n\}$ is any van Hove sequence in G, we have

$$M(\mu) = \lim_n \frac{\mu(x + A_n)}{\mathrm{vol}(A_n)},$$

uniformly in x.

As proved by Eberlein for functions [15], and Argabright and deLamadrid for measures [19], the space $\mathcal{SAP}(G)$ is a direct summand in $\mathcal{WAP}(G)$, and $\mathcal{WAP}_0(G)$ is its complement.

Theorem 2.13 ([19])

$$\mathcal{WAP}(G) = \mathcal{SAP}(G) \bigoplus \mathcal{WAP}_0(G).$$

In particular, every measure $\mu \in \mathcal{WAP}(G)$ can be written uniquely

$$\mu = \mu_s + \mu_0,$$

with $\mu_s \in \mathcal{SAP}(G)$, $\mu_0 \in \mathcal{WAP}_0(G)$. *We will refer to this as the* Eberlein decomposition of μ.

For Fourier transformable measures the Eberlein decomposition is the Fourier dual of the Lebesgue decomposition into pure point and continuous components [19, 29].

We complete the section by reviewing the Eberlein convolution.

Theorem 2.14 ([13, 14]) *If* $f, g \in WAP(G)$, *then*

$$f \circledast g(t) = M_x\big(f(t-x)g(x)\big),$$

is well defined and belongs to $SAP(G)$.

We will call $f \circledast g$ the *Eberlein convolution of f and g*.

Theorem 2.15 ([19]) *If* $f \in SAP(G)$ *and* $\mu \in \mathcal{WAP}(G)$, *then*

$$f \circledast \mu(t) = M\big(f(t-\cdot)\mu\big),$$

is well defined and belongs to $\mathcal{SAP}(G)$.

We will call $f \circledast \mu$ the *Eberlein convolution of f and* μ.

Recently, the notion of Eberlein convolution was extended to two weakly almost periodic measures in [23].

Finally, we review the notion of approximate identity for the Eberlein convolution.

Definition 2.16 A net $\{f_\alpha\}$ with $f_\alpha \in \mathcal{SAP}(G)$ is *an approximate identity for* $(SAP(G), \circledast)$ if for all $f \in SAP(G)$ we have

$$f = \lim_\alpha f \circledast f_\alpha$$

in $(SAP(G), \|\cdot\|_\infty)$.

Remark 2.17

(i) Consider the natural embedding $G \hookrightarrow G_b$ of G into its Bohr compactification G_b (see [29, Sect. 4.2.2] for definition and properties).

Then f_α is an approximate identity for $(SAP(G), \circledast)$ if and only if there exists an approximate identity g_α for $(C(G_b), *)$ such that f_α is the restriction to G of g_α [13, 19, 29]. Moreover,

$$M(f_\alpha) = \int_{G_b} g_b(s) d\theta_{G_b}(s).$$

In particular, approximate identities for $(SAP(G), \circledast)$ exist and can be chosen such that $f_\alpha \geq 0$, $f_\alpha(-x) = f_\alpha(x)$ and $M(f_\alpha) = 1$.

(ii) If f_α is an approximate identity for $(SAP(G), \circledast)$, then

$$\lim_\alpha M(f_\alpha) = 1.$$

3 Weakly Admissible Measures

In this section we introduce a new concept for a measure, which we will call weakly admissible (compare the definition of admissible measures [24]), and study the basic properties of weakly admissible measures.

The definition of weak admissibility is simply the integrability condition from the definition of Fourier transformability, and its importance to the Fourier theory for measures is emphasized by [35, Thm. 3.10] (Theorem 5.1).

Definition 3.1 A measure $\mu \in \mathcal{M}(G)$ is called *weakly admissible* if we have

$$\widecheck{K_2(\widehat{G})} \subset L^1(|\mu|).$$

Note that $\mu \in \mathcal{M}(G)$ is weakly admissible if and only if

$$\widecheck{C_{\mathsf{c}}(\widehat{G})} \subset L^2(|\mu|).$$

We start by stating a simple lemma that contains a few straightforward properties of weakly admissible measures.

Lemma 3.2

(i) *The measure μ is weakly admissible if and only if $|\mu|$ is weakly admissible.*
(ii) *If μ is weakly admissible and $|\nu| \leq |\mu|$, then ν is weakly admissible.*
(iii) *The measure μ is weakly admissible if and only if $\mu_{\mathsf{pp}}, \mu_{ac}$, and μ_{sc} are weakly admissible.*
(iv) *If μ is Fourier transformable, then $\widehat{\mu}$ is weakly admissible.*
(v) *If μ is weakly admissible then $\overline{\mu}, \widetilde{\mu}, \mu^{\dagger}$ and $T_t\mu$ are weakly admissible.*
(vi) *If μ is weakly admissible and $f \in C_{\mathsf{u}}(G)$, then $f\mu$ is weakly admissible.*

Proof (i) This is obvious by the definition of weak admissibility.

(ii) If $f \in C_{\mathsf{c}}(\widehat{G})$, then $|\widecheck{f}|^2$ is continuous, hence measurable. Moreover,

$$\int_G |\widecheck{f}|^2 d|\nu| \leq \int_G |\widecheck{f}|^2 d|\mu| < \infty.$$

This shows that $\widecheck{C_{\mathsf{c}}(\widehat{G})} \subset L^2(|\nu|)$.

(iii) This follows immediately from

$$|\mu| = |\mu_{\mathsf{pp}}| + |\mu_{ac}| + |\mu_{sc}|$$

and (ii).

(iv) This is a consequence of the definition of the Fourier transformability.

(v), (vi) These are obvious. ∎

Next, we show that if μ is a weakly admissible measure and $f \in C_{\mathsf{c}}(\widehat{G})$, then $|\widecheck{f}|^2 * \mu$ defines an uniformly continuous and bounded function. This result is essential for the proof of Theorem 8.1. The proof of the Theorem 3.3 follows the idea of [1, Thm. 2.5] (see also [29, Thm. 9.18], [36, Lemma]).

Theorem 3.3 *Let μ be a weakly admissible measure.*

(i) *For each $K \subset \widehat{G}$, there exists a constant C_K so that for all $f \in C_c(\widehat{G})$ we have*

$$\sqrt{\int_G |\widehat{f}|^2 d|\mu|} \leq C_K \|f\|_\infty .$$

(ii) *For each $f \in C_c(\widehat{G})$, the function*

$$t \longrightarrow \int_G |\widehat{f}(x+t)|^2 d|\mu|(x) =: g(t)$$

belongs to $C_u(G)$.

(iii) *For each $f \in C_c(\widehat{G})$ the function*

$$t \longrightarrow \int_G |\widehat{f}(x+t)|^2 d\mu(x) =: h(t)$$

belongs to $C_u(G)$.

(iv) *The measure μ is translation bounded.*

Proof (i) Let us start by recalling that $L^2(|\mu|)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int_G f(x)\overline{g(x)} d|\mu|(x).$$

The norm induced by this inner product is

$$\|h\|_2 := \sqrt{\int_G |h|^2 d|\mu|},$$

and it depends on the measure μ.

The definition of weak admissibility tells us that we can define a mapping

$$T: C_c(\widehat{G}) \longrightarrow L^2(|\mu|); T(f) = \widehat{f}.$$

It is obvious that T is linear.

Now fix some compact set $K \subset \widehat{G}$ and define, as usual,

$$C(\widehat{G} : K) := \left\{ f \in C_c(\widehat{G}) \mid \operatorname{supp}(f) \subset K \right\}.$$

We claim that the restriction $T: C(\widehat{G} : K) \to L^2(|\mu|)$ has a closed graph and is hence continuous.

Indeed, let $f_\alpha \to f$ in $(C(\widehat{G} : K), \|\cdot\|_\infty)$ be such that $\widehat{f_\alpha} \to g$ in $L^2(|\mu|)$. We need to show that $\widehat{f} = g$ in $L^2(|\mu|)$.

Let $\epsilon > 0$ and let $J \subset G$ be any compact set.

Since $\widehat{f_\alpha} \to g$ in $L^2(|\mu|)$, there exists some β so that for all $\alpha > \beta$, we have

$$\left(\int_G |\widehat{f_\alpha} - g|^2 d|\mu| \right)^{\frac{1}{2}} < \frac{\epsilon}{2}.$$

Moreover, since $f_\alpha \to f$ in $(C(\widehat{G} : K), \|\cdot\|_\infty)$, there exists some $\gamma > \beta$ such that, for all $\alpha > \gamma$ we have

$$\|f_\alpha - f\|_\infty \leq \frac{\epsilon}{2\theta_{\widehat{G}}(K)\sqrt{|\mu|(J)} + 1}.$$

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Then, for some $\alpha > \gamma$, we have by the triangle inequality for $\|\cdot\|_2$

$$\begin{aligned}\Big(\int_J |\widehat{f}-g|^2 d|\mu|\Big)^{\frac{1}{2}} &\le \Big(\int_J |\widehat{f}-\widehat{f_\alpha}|^2 d|\mu|\Big)^{\frac{1}{2}} + \Big(\int_J |g-\widehat{f_\alpha}|^2 d|\mu|\Big)^{\frac{1}{2}} \\ &\le \|\widehat{f}-\widehat{f_\alpha}\|_\infty \sqrt{|\mu|(J)} + \frac{\epsilon}{2} \\ &= \|f-f_\alpha\|_\infty \theta_{\widehat{G}}(K)\sqrt{|\mu|(J)} + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

This shows that $(\int_J |\widehat{f}-g|^2 d|\mu|)^{\frac{1}{2}} < \epsilon$ for all compact sets $J \subset G$. Therefore, by the regularity of the measure $|\widehat{f}-g|^2|\mu|$, we get

$$\Big(\int_G |\widehat{f}-g|^2 d|\mu|\Big)^{\frac{1}{2}} < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we get

$$\Big(\int_G |\widehat{f}-g|^2 d|\mu|\Big)^{\frac{1}{2}} = 0,$$

which proves that $\widehat{f} = g$ in $L^2(|\mu|)$. Therefore, the graph of T is closed, and hence T is continuous.

The continuity of T implies the existence of C_K.

(ii) Fix some $K \subset \widehat{G}$ compact set so that $\operatorname{supp}(f) \subset K$. For the remainder of (ii), f and K are fixed.

For each $s \in G$, we will denote by ϕ_s the character on $\widehat{G}$ defined by s, that is

$$\phi_s(\chi) := \chi(s).$$

Then for all $t \in G$, we have

$$\begin{aligned}\int_G |\widehat{f}(x+t)|^2 d|\mu|(x) &= \int_G |T_{-t}\widehat{f}(x)|^2 d|\mu|(x) = \int_G |\widehat{\phi_{-t}f}(x)|^2 d|\mu|(x) \\ &\le C_K \|\phi_{-t}f\|_\infty = C_K\|f\|_\infty.\end{aligned}$$

This shows that g is bounded.

Next, let $\epsilon > 0$. By Pontryagin duality, the set

$$N\Big(K, \frac{\epsilon}{C_K\|f\|_\infty + 1}\Big) := \Big\{ s \in G \,\Big|\, |\psi_s(\chi) - 1| < \frac{\epsilon}{C_K\|f\|_\infty + 1} \text{ for all } \chi \in K\Big\}$$

is an open neighbourhood of 0 in G.

If $s - t \in N(K, \frac{\epsilon}{2})$, by the triangle inequality for $\|h\|_2$, we have

$$\begin{aligned}|\sqrt{g(s)} - \sqrt{g(t)}| &= |\|T_{-t}\widehat{f}\|_2 - \|T_{-s}\widehat{f}\|_2| \\ &\le \|T_{-t}\widehat{f} - T_{-s}\widehat{f}\|_2 = \|\widehat{\phi_{-t}f} - \widehat{\phi_{-s}f}\|_2 \\ &= \|\widehat{\phi_{-t}f - \phi_{-s}f}\|_2 \le C_K\|\phi_{-t}f - \phi_{-s}f\|_\infty \\ &= C_K\|\phi_{-t}(1-\phi_{t-s})f\|_\infty = C_K\|(1-\phi_{t-s})f\|_\infty \\ &\le C_K \frac{\epsilon}{C_K\|f\|_\infty + 1}\|f\|_\infty < \epsilon.\end{aligned}$$

This proves that $\sqrt{g(t)}$ is uniformly continuous. Therefore, as $0 \leq g(t) \leq C_K \|f\|_\infty$, and as x^2 is uniformly continuous on the compact set $[0, \sqrt{C_K \|f\|_\infty}]$, it follows that g is uniformly continuous.

(iii) Consider the decomposition

$$\mu = \mathrm{Re}(\mu) + i\mathrm{Im}(\mu).$$

of μ.

Since

$$\mathrm{Re}(\mu) = \frac{1}{2}(\mu + \bar{\mu}),$$

we have

$$|\mathrm{Re}(\mu)| \leq \frac{1}{2}(|\mu| + |\bar{\mu}|) = |\mu|.$$

In the same way, we get

$$|\mathrm{Im}(\mu)| \leq \frac{1}{2}(|\mu| + |\bar{\mu}|) = |\mu|.$$

Therefore, by Lemma 3.2 (ii), the measures $\mathrm{Re}(\mu)$ and $\mathrm{Im}(\mu)$ are weakly admissible.

Next, consider the Jordan decomposition

$$\mathrm{Re}(\mu) = \mathrm{Re}(\mu)_+ - \mathrm{Re}(\mu)_- .$$

It follows from the properties of Jordan decomposition that

$$|\mathrm{Re}(\mu)_\pm| \leq |\mathrm{Re}(\mu)|.$$

This shows that $\mathrm{Re}(\mu)_\pm$ are weakly admissible measures, and hence by (ii) the functions

$$t \longrightarrow \int_G |\widehat{f}(x+t)|^2 d|\mathrm{Re}(\mu)_\pm|(x)$$

belong to $C_{\mathrm{u}}(G)$. As $\mathrm{Re}(\mu)_\pm \geq 0$, we get that

$$t \longrightarrow \int_G |\widehat{f}(x+t)|^2 d\mathrm{Re}(\mu)_\pm(x)$$

belong to $C_{\mathrm{u}}(G)$, and hence, so does their difference

$$t \to \int_G |\widehat{f}(x+t)|^2 d\mathrm{Re}(\mu)(x).$$

In exactly the same way, the function

$$t \to \int_G |\widehat{f}(x+t)|^2 d\mathrm{Im}(\mu)(x)$$

belongs to $C_{\mathrm{u}}(G)$.

Now, the equality

$$\int_G |\widehat{f}(x+t)|^2 d\mu(x) = \int_G |\widehat{f}(x+t)|^2 d\mathrm{Re}(\mu)(x) + i \int_G |\widehat{f}(x+t)|^2 d\mathrm{Im}(\mu)(x)$$

proves the claim.

(iv) Let $K \subset G$ be compact. Then there exists some $h \in C_{\mathrm{c}}(\widehat{G})$ so that ([10, 29])

$$\widehat{h} \geq 1_K.$$

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Then, for all $x \in G$, we have

$$|\mu|(-x+K) = \int_G 1_K(x+t)d|\mu|(t) \leq \int_G |\widehat{h}(x+t)|^2 d|\mu|(x).$$

Therefore, by (ii),

$$\|\mu\|_K = \sup_{x\in G}\{|\mu|(-x+K)\} < \infty. \qquad \blacksquare$$

A natural question to ask now is whether translation boundedness implies weakly admissible. We will show in the next section that for $G = \mathbb{R}^d$ the answer is yes, but in general, to our knowledge, the question is still open.

Next, we show that for weakly almost periodic measures, weak admissibility is compatible with Eberlein decomposition.

Theorem 3.4 *Let* $\mu \in \mathcal{WAP}$*. Then* μ *is weakly admissible if and only if* μ_{s} *and* μ_0 *are weakly admissible.*

Proof ($\Leftarrow$) This is obvious.

($\Rightarrow$) Let $f \in C_{\mathsf{c}}(\widehat{G})$. Fix some compact K and pick some $g \in C_{\mathsf{c}}(G)$ such that $g \geq 1_K$.

Let $h := g|\widehat{f}|^2$. Then $h \in C_{\mathsf{c}}(G)$, $h \geq 0$ and $h = |\widehat{f}|^2$ on K.

Finally, let $f_\alpha \in SAP(G)$ be an approximate identity for the Eberlein convolution, such that $f_\alpha \geq 0$ and $f_\alpha(-x) = f_\alpha(x)$. Then

$$\mu_{\mathsf{s}} = \lim_\alpha \mu \circledast f_\alpha$$

in the product topology on $\mathcal{M}^\infty(G)$ [19, Cor. 7.2]. In particular, $\mu \circledast f_\alpha$ converges in the vague topology to μ_{s}.

Next, we have

$$\int_G h(t)d|\mu_{\mathsf{s}}|(t) = \sup\Big\{\Big|\int_G \phi(t)d\mu_{\mathsf{s}}(t)\Big| \mid \phi \in C_{\mathsf{c}}(G), |\phi| \leq h\Big\}. \tag{3.1}$$

Let $\phi \in C_{\mathsf{c}}(G)$ be so that $|\phi| \leq h$. Then

$$\begin{aligned} \Big|\int_G \phi(t)d\mu_{\mathsf{s}}(t)\Big| &= \lim_\alpha \Big|\int_G \phi(t)d\mu \circledast f_\alpha(t)\Big| \\ &= \lim_\alpha |\phi^\dagger * (\mu \circledast f_\alpha)(0)| \\ &= \lim_\alpha |(\phi^\dagger * \mu) \circledast f_\alpha(0)|, \end{aligned} \tag{3.2}$$

with the last equality following from [19, Thm. 6.4].

Now, since μ is weakly admissible, by Theorem 3.3 (ii), there exists a constant C_f that depends only on f such that for all $t \in G$, we have

$$\int_G |\widehat{f}|^2(t+s)d|\mu|(s) \leq C_f.$$

This implies that for all $t \in G$, we also have

$$\int_G h(t+s)d|\mu|(s) \leq C_f,$$

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and hence,

$$\int_G |\phi|(t+s)d|\mu|(s) \le C_f.$$

Therefore, if A_n is any Fölner sequence, by the definition of Eberlein convolution, we have, for all α,

$$\begin{aligned}|(\phi^\dagger * \mu) \circledast f_\alpha(0)| &= \Big|\lim_n \frac{1}{\mathrm{vol}(A_n)} \int_{A_n} (\phi^\dagger * \mu)(t) f_\alpha(-t)dt\Big| \\ &\le \limsup_n \frac{1}{\mathrm{vol}(A_n)} \int_{A_n} |\phi^\dagger * \mu|(t) f_\alpha(t)dt \\ &\le \limsup_n \frac{1}{\mathrm{vol}(A_n)} \int_{A_n} \Big|\int_G \phi(-t+s)d\mu(s)\Big|(t) f_\alpha(t)dt \\ &\le \limsup_n \frac{1}{\mathrm{vol}(A_n)} \int_{A_n} \Big(\int_G |\phi|(-t+s)d|\mu|(s)\Big) f_\alpha(t)dt \\ &\le \limsup_n \frac{1}{\mathrm{vol}(A_n)} C_f f_\alpha(t)dt = C_f M(f_\alpha).\end{aligned}$$

Hence, by (3.2), we have

$$\Big|\int_G \phi(t)d\mu_{\mathsf{s}}(t)\Big| \le \limsup_\alpha C_f M(f_\alpha) = C_f.$$

By (3.1), we get

$$\int_G h(t)d|\mu_{\mathsf{s}}|(t) \le C_f.$$

This shows that

$$\int_K \Big|\widehat{f}\Big|^2(t)d|\mu_{\mathsf{s}}|(t) \le C_f.$$

As the constant is independent of the compact set K, and $K \subset G$ was an arbitrary compact set, by the regularity of the measure $\Big|\widehat{f}\Big|^2(t)|\mu_{\mathsf{s}}|$, we get

$$\int_G \Big|\widehat{f}\Big|^2(t)d|\mu_{\mathsf{s}}|(t) \le C_f.$$

This proves that μ_{s} is weakly admissible.

Finally, $\mu_0 = \mu - \mu_{\mathsf{s}}$ is weakly admissible as a difference of two weakly admissible measures. ∎

4 Weakly Admissible Measures on $\mathbb{R}^d$

In this section we connect our concept of weak admissibility with the concepts of admissibility and uniform boundedness that appeared in the work of Lin [24], Thornett [45], and Robertson and Thornett [36].

Let us first recall some of their definitions:

Definition 4.1 A positive Borel measure μ on $\mathbb{R}^d$ is called *r-admissible* if for all $f \in L^2(\mathbb{R}^d)$ with $\mathrm{supp}(f) \subset [-r,r]^d$, we have

$$\int_{\mathbb{R}^d} |\widehat{f}(y)|^2 d\mu(y) < \infty.$$

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Note that since $f \in L^2(\mathbb{R}^d)$ and $\operatorname{supp}(f) \subset [-r,r]^d$, we have $f \in L^1(\mathbb{R}^d)$, and hence $\widehat{f}(y)$ is defined pointwise for all $y \in \mathbb{R}^d$.

The following theorem is of importance to us; see [24, Thm. 1], [45, Thm. 4.2, Thm. 4.3].

Theorem 4.2 *A positive Borel measure μ on $\mathbb{R}^d$ is r-admissible for some r if and only if it is translation bounded. In particular, a measure is r-admissible for some $r > 0$ if and only if it is r-admissible for all $r > 0$.*

Because of this, we will simply call a measure admissible instead of r-admissible.

As a consequence, we get the following simple characterisation of weak admissibility on $\mathbb{R}^d$.

Theorem 4.3 *Let μ be a Radon measure on $\mathbb{R}^d$. Then the following are equivalent:*

(i) *The measure μ is translation bounded.*
(ii) *The measure $|\mu|$ is translation bounded.*
(iii) *The measure $|\mu|$ is admissible.*
(iv) *The measure μ is weakly admissible.*
(v) *For all bounded Borel sets $A \subset \mathbb{R}^d$, we have $\widehat{1_A} \in L^2(\mu)$.*

Proof The implication (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) This follows from Theorem 4.2.

(iii) $\Rightarrow$ (iv) This is immediate, as $|\mu|$ is admissible implies that $|\mu|$ is weakly admissible, which in turn implies that μ is weakly admissible.

(iv) $\Rightarrow$ (i) This follows from Theorem 3.3.

The equivalence (ii) $\Leftrightarrow$ (v) is [36, Theorem] applied to $|\mu|$. ■

5 Weak Admissibility and the Fourier Transform

In this section, we take a closer look at weak admissibility and Fourier transformability. We start by reviewing a criterion for twice Fourier transformability of a transformable measure. Next, we give a criterion for Fourier transformability of a measure μ on $\mathbb{R}^d$ in terms of its Fourier transform as a tempered distribution.

Theorem 5.1 ([35, Thm. 3.10]) *Let μ be a Fourier transformable measure. Then μ is twice Fourier transformable if and only if μ is a weakly admissible measure.*

In this case, we have $\widehat{\widehat{\mu}} = \mu^\dagger$.

Next, consider a measure μ on $\mathbb{R}^d$. If μ is tempered as a distribution, then μ has a Fourier transform ψ that is a tempered distribution. If ψ is not a measure, it is easy to see that μ cannot be Fourier transformable in the measure sense. An interesting question is: what happens when ψ is a measure?

As shown in [1], it does not necessary follow that μ is Fourier transformable as a measure.

In the following theorem we prove that in this situation, the Fourier transformability of μ in the measure sense is equivalent to the weak admissibility, and hence translation boundedness, of ψ.

Theorem 5.2 *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. Then μ is Fourier transformable as a measure if and only if the following hold:*

(i) *The measure μ is tempered as a distribution.*
(ii) *The Fourier transform ν of μ as a tempered distribution is a translation-bounded measure.*

Moreover, in this case, we have $\widehat{\mu} = \nu$.

Proof ($\Rightarrow$) Since μ is Fourier transformable, the measure $\widehat{\mu}$ is translation bounded and

$$\langle \mu, g \rangle = \langle \widehat{\mu}, \widetilde{g} \rangle$$

for all $g \in C_{\mathsf{c}}(\mathbb{R}^d)$. Moreover, since ν is translation bounded, it is tempered as a distribution.

As $\widehat{\mu}$ is a translation-bounded measure, it is a tempered distribution. Therefore, it is the Fourier transform of a tempered distribution ν.

Then, for all $f \in \mathcal{S}^{\infty}(\mathbb{R}^d)$, we have $\langle \nu, g \rangle = \langle \widehat{\mu}, \widetilde{g} \rangle$. This shows that for all $g \in C_{\mathsf{c}}^{\infty}(\mathbb{R}^d)$, we have $\langle \nu, g \rangle = \langle \widehat{\mu}, \widetilde{g} \rangle = \langle \mu, g \rangle$. Therefore, $\nu = \mu$.

As ν is a tempered distribution, it follows that μ is tempered as a measure, and that $\widehat{\mu}$ is the Fourier transform of μ as a tempered distribution.

As $\widehat{\mu}$ is a translation-bounded measure, the claim follows.

($\Leftarrow$) We have $\langle \mu, g \rangle = \langle \nu, \widetilde{g} \rangle$ for all $g \in \mathcal{S}(\mathbb{R}^d)$. Therefore, $\langle \mu, g \rangle = \langle \nu, \widetilde{g} \rangle$ for all $g \in C_{\mathsf{c}}^{\infty}(\mathbb{R}^d)$.

Next, fix some $h_n \in \mathcal{S}(\mathbb{R}^d)$ such that $\|h_n\|_\infty = 1$, $h_n = 1$ on $B_n(0)$ and $\operatorname{supp}(h_n) \subset B_{n+1}(0)$. Let $g_n := \widetilde{h_n}$.

Now, pick some $f \in C_{\mathsf{c}}(G)$. Then, as $f \in L^2(\mathbb{R}^d)$, we have $\widehat{f} \in L^2(\mathbb{R}^d)$. Therefore, by the Lebesgue Dominated Convergence Theorem,

$$\|(\widehat{f})h_n - \widehat{f}\|_2 \to 0.$$

This shows that $f * g_n \to f$ in $L^2(\mathbb{R}^d)$. Then

$$(f * g_n) * (\widetilde{f * g_n}) \to f * \widetilde{f} \text{ in } (C_{\mathsf{c}}(\mathbb{R}^d), \| \cdot \|_\infty).$$

This gives

$$\langle \mu, f * \widetilde{f} \rangle = \lim_n \left\langle \mu, (f * g_n) * (\widetilde{f * g_n}) \right\rangle = \lim_n \langle \nu, \left| \widetilde{f} \right|^2 h_n^2 \rangle.$$

Finally, since $\left| \widetilde{f} \right|^2 \in L^1(|\nu|)$ and $\left| \widetilde{f} \right|^2 h_n^2$ is increasing and converges pointwise to $\left| \widetilde{f} \right|^2$, we get by the monotone convergence theorem,

$$\lim_n \langle \nu, \left| \widetilde{f} \right|^2 h_n \rangle = \langle \nu, \left| \widetilde{f} \right|^2 \rangle.$$

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Therefore, for all $f \in C_c(\mathbb{R}^d)$, we have $\left|\widetilde{f}\right|^2 \in L^1(|\nu|)$ and

$$\langle \mu, f * \widetilde{f} \rangle = \langle \nu, |\widetilde{f}|^2 \rangle.$$

This shows that μ is Fourier transformable and $\widehat{\mu} = \nu$. ∎

6 Fourier–Bohr Series and Formal Sums

Given a weakly almost periodic measure μ, we can introduce its Fourier–Bohr series (see Def. 6.12). If μ is Fourier transformable, then its Fourier–Bohr series is a measure, but there is no guarantee that this happens in general. For this reason, when we deal with the Fourier–Bohr series of a weakly almost periodic measure, we need to treat it as a formal sum (see also [19]).

In this section, we review the basic properties of formal sums and the Fourier–Bohr series of weakly almost periodic measures.

6.1 Formal Sums

We start by defining the notion of formal sums.

Definition 6.1 By a *formal sum* we mean an expression of the form

$$\omega = \sum_{x \in G} \omega_x \delta_x,$$

where $\omega_x \in \mathbb{C}$.

For such an expression we define the *support* of ω as

$$\operatorname{supp}(\omega) := \{x \in G | \omega_x \neq 0\}.$$

Remark 6.2 Any formal sum is a measure on G_{d}. Our interest will be in formal sums that are measures on G, so we will simply treat them as formal sums.

We will often speak of integrals of functions against formal sums. Note that we can multiply a formal sum by a function in an obvious way, and we obtain a new formal sum. We will say that the function f is integrable against the formal sum ω if the product $f\omega$ is an absolutely summable series.

Definition 6.3 Let ω be a formal sum and let $f\colon G \to \mathbb{C}$ be a function. We say that f is *integrable* with respect to ω if

$$\sum_{x \in G} |f(x)\omega(x)| < \infty.$$

In this case, we define the *integral*

$$\int_G f d\omega = \langle f, \omega \rangle := \sum_{x \in G} f(x)\omega(x).$$

We also let

$$\mathsf{L}^1(\omega) := \{f\colon G \to \mathbb{C} \mid f \text{ is integrable with respect to } \omega\}.$$

and

$$\mathsf{L}^2(\omega) := \{f\colon G \to \mathbb{C} \mid f^2 \text{ is integrable with respect to } \omega\}.$$

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Note that for pure point measures ω, we have

$$\mathsf{L}^1(\omega) = L^1(|\omega|) \qquad \mathsf{L}^2(\omega) = L^2(|\omega|).$$

Remark 6.4

(i) If f is integrable with respect to ω then $f(x)\omega_x = 0$ for all but at most countably many $x \in G$.

(ii) A function f is integrable with respect to ω exactly when $\sum_{x\in G} f(x)\omega(x)$ is absolutely convergent.

(iii) If we treat ω as a measure on G_{d}, then a function is integrable with respect to ω exactly when it is integrable with respect to ω as a measure and $\mathsf{L}^1(\omega)$ is just the standard $L^1(|\omega|)$ space.

We start by characterizing formal sums which are measures on G. It is clear that every such measure is pure point.

Theorem 6.5 *Let $\omega = \sum_{x\in G} \omega_x \delta_x$ be a formal sum. Then ω is a measure if and only if for all compact sets K we have*

$$\sum_{x\in K} |\omega_x| < \infty.$$

Proof ($\Rightarrow$) Let K be a compact set. Since ω is a measure, so is its variation measure $|\omega|$.

Therefore we have

$$\sum_{x\in K} |\omega_x| = |\omega|(K) < \infty.$$

($\Leftarrow$) We first prove that $C_{\mathsf{c}}(G) \subset \mathsf{L}^1(\omega)$.

Let $f \in C_{\mathsf{c}}(G)$, and let K be any compact set such that $\operatorname{supp}(f) \subset K$. Then

$$\sum_{x\in G} |f(x)\omega(x)| = \sum_{x\in K} |f(x)\omega(x)| \leq \|f\|_\infty \sum_{x\in K} |\omega(x)| < \infty.$$

Next it is trivial to show that ω is linear on $C_{\mathsf{c}}(G)$.

Finally, if $K \subset G$ is a fixed compact set and $f \in C_{\mathsf{c}}(G)$ is so that $\operatorname{supp}(f) \subset K$, by the above computation we have

$$|\langle \omega, f\rangle| \leq C_K \|f\|_\infty,$$

where

$$C_K = \sum_{x\in K} |\omega_x| < \infty.$$

Therefore, by the Riesz representation theorem, ω is a measure. ∎

We next introduce a simpler criterion which involves a single compact set with non-empty interior.

Corollary 6.6 *Let $\omega = \sum_{x\in G} \omega_x \delta_x$ be a formal sum and let K be a fixed compact set with a non-empty interior. Then ω is a measure if and only if for all $t \in G$ we have*

$$\sum_{x\in (t+K)} |\omega_x| < \infty.$$

Proof Let $K' \subset G$ be any compact set. Then, since K has non-empty interior, there exists $t_1, \ldots, t_k \in G$ such that

$$K' \subset \bigcup_{j=1}^{k} t_j + K.$$

Then

$$\sum_{x \in K} |\omega_x| \leq \sum_{j=1}^{k} \Big(\sum_{y \in (t_j + K)} |\omega_y| \Big) < \infty.$$

Therefore, by Theorem 6.5, ω is a measure. ∎

6.2 Weakly Admissible Formal Sums

We can now extend the definition of weak admissibility to formal sums. We will see in this subsection that all weakly admissible formal sums are in fact measures.

The reason we are interested in extending the definition to formal sums is because we will be interested in weak admissibility of a Fourier–Bohr series, which may or may not be a measure.

Definition 6.7 A formal sum ω is called a *weakly admissible formal sum* if $\widecheck{K_2(\widehat{G})} \subset \mathsf{L}^1(\omega)$.

Remark 6.8

(i) A formal sum ω is weakly admissible if and only if for all $f \in C_c(\widehat{G})$, we have

$$\sum_{x \in G} |\omega_x| \Big| \widecheck{f} \Big|^2 (x) < \infty.$$

(ii) A formal sum ω is weakly admissible if and only if $\widecheck{C_c(\widehat{G})} \subset \mathsf{L}^2(\omega)$.

(iii) Any formal sum that is weakly admissible is a linear function on $\widecheck{K_2(\widehat{G})}$.

We start by proving that weakly admissible formal sums are measures.

Lemma 6.9 *Let ω be a weakly admissible formal sum. Then ω is a translation-bounded measure.*

Proof Let $K \subset G$ be compact. Then there exists a function $f \in C_c(\widehat{G})$ such that $\widecheck{f} \geq 1_K$ [10, 29].

Then

$$\sum_{x \in K} |\omega_x| \leq \sum_{x \in G} |\omega_x| \Big| \widecheck{f} \Big|^2 < \infty.$$

Then by Theorem 6.5, ω is a measure, which is trivially a weakly admissible measure. Hence, by Theorem 3.3, ω is a translation-bounded measure. ∎

Corollary 6.10 *Let ω be a formal sum on $\mathbb{R}^d$. Then ω is weakly admissible if and only if ω is a translation-bounded measure.*

We complete the section by providing a slight generalisation to [35, Thm. 5.5]: we prove that translation-bounded measures with Meyer set support are weakly admissible. For definitions and properties of Meyer sets and cut and project schemes we refer the reader to [25–27, 43]. We would like to emphasize that this stronger version was actually proved but not stated in [35]. Our proof is identical to the one in [35].

Theorem 6.11 *Let $\mu = \sum_{x\in\Lambda} \omega_x \delta_x$. If Λ is a subset of a Meyer set and $\{\omega_x\}$ is bounded, then*

$$\omega := \sum_{x\in\Lambda} \omega_x \delta_x$$

is a weakly admissible formal sum.

Proof Let $(G, H, \mathcal{L})$ be a cut and project scheme and let $W \subset H$ be a compact set such that $\Lambda \subset \curlywedge(W)$.

Let $(\widehat{G}, \widehat{H}, \mathcal{L}^0)$ be the dual cut and project scheme. Then, there exists some $h \in C_{\mathsf{c}}(\widehat{H})$ such that $\widetilde{h} \geq 1_W$ [10, 29].

Then $\omega_{h*\widetilde{h}}$ is Fourier transformable and ([35])

$$\widehat{\omega_{h*\widetilde{h}}} = \omega_{|\widetilde{h}|^2}.$$

Therefore, as the Fourier transform of a measure, $\omega_{|\widetilde{h}|^2}$ is weakly admissible. As $|\omega| \leq \omega_{|\widetilde{h}|^2}$, it follows from Lemma 3.2 that $|\omega|$ is weakly admissible, and hence by Lemma 3.2, ω is weakly admissible. ∎

6.3 Fourier Bohr Series

In the spirit of [19] we have the following definition.

Definition 6.12 Let $\mu \in \mathcal{WAP}(G)$. The *Fourier–Bohr series* of μ is defined as

$$\mathcal{F}_{\mathsf{d}}(\mu) := \sum_{\chi\in\widehat{G}} c_\chi(\mu)\delta_\chi.$$

As shown in [19], the Fourier–Bohr series uniquely identifies the strongly almost periodic component of a weakly almost periodic measure.

Theorem 6.13 ([19])

(i) *For each $\mu \in \mathcal{WAP}(G)$, we have $\mathcal{F}_{\mathsf{d}}(\mu) = \mathcal{F}_{\mathsf{d}}(\mu_{\mathsf{s}})$.*
(ii) *For $\mu, \nu \in \mathcal{SAP}(G)$, we have $\mathcal{F}_{\mathsf{d}}(\mu) = \mathcal{F}_{\mathsf{d}}(\nu)$ if and only if $\mu = \nu$.*

Let us recall that Fourier–Bohr series have the following summability property.

Remark 6.14 ([19, Sect. 8]) If $\mu \in \mathcal{WAP}(G)$, then, for all $g \in C_{\mathsf{c}}(G)$ we have

$$\sum_{\chi\in\widehat{G}} |c_\chi(\mu)|^2 |\widehat{g}(\chi)|^2 < \infty.$$

The importance of the Fourier–Bohr series for the Fourier transform of measures is given by the following result.

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Theorem 6.15 *If $\mu \in \mathcal{WAP}(G)$ is Fourier transformable, then*

$$\widehat{\mu}_{\mathsf{pp}} = \mathcal{F}_{\mathsf{d}}(\mu),$$

and $\mathcal{F}_{\mathsf{d}}(\mu)$ is a weakly admissible formal sum.

Proof Since μ is Fourier transformable, for all $\chi \in \widehat{G}$, we have ([29])

$$\widehat{\mu}(\{\chi\}) = c_\chi(\mu).$$

Therefore,

$$\widehat{\mu}_{\mathsf{pp}} = \sum_{\chi \in \widehat{G}} \widehat{\mu}(\{\chi\})\delta_\chi = \sum_{\chi \in \widehat{G}} c_\chi(\mu)\delta_\chi = \mathcal{F}_{\mathsf{d}}(\mu).$$

Moreover, $\widehat{\mu}$ is weakly admissible, and hence so is $\widehat{\mu}_{\mathsf{pp}}$. ∎

As an immediate consequence of Theorem 6.15, we get the following corollary.

Corollary 6.16 *If $\mu \in \mathcal{SAP}(G)$ is Fourier transformable, then $\widehat{\mu} = \mathcal{F}_{\mathsf{d}}(\mu)$, and $\mathcal{F}_{\mathsf{d}}(\mu)$ is a weakly admissible formal sum.*

The main result in this paper (Theorem 7.1) shows that the converse of this also holds.

7 Fourier Transformability of Strongly Almost Periodic Measures

In this section we proceed to prove the main result of this paper. We then look at few consequences.

Theorem 7.1 *Let $\mu \in \mathcal{SAP}(G)$. Then, μ is Fourier transformable if and only if $\mathcal{F}_{\mathsf{d}}(\mu)$ is a weakly admissible formal sum.*

Moreover, in this case, we have $\widehat{\mu} = \mathcal{F}_{\mathsf{d}}(\mu)$.

Proof ($\Rightarrow$) This follows from Corollary 6.16.

($\Leftarrow$) First, since $\mathcal{F}_{\mathsf{d}}$ is a weakly admissible formal sum, it is a measure by Lemma 6.9. For simplicity, let us denote this measure by

$$\nu := \mathcal{F}_{\mathsf{d}}(\mu) = \sum_{\chi \in \widehat{G}} c_\chi(\mu)\delta_\chi.$$

To complete the proof we show that ν satisfies the definition of the Fourier transform of μ. In order to achieve this conclusion, for each $f \in C_{\mathsf{c}}(G)$ we show that $\mu * f * \widetilde{f}$ and $|\widecheck{f}|^2\nu$ are Bohr almost periodic functions with the same Fourier–Bohr series, and hence equal. Equating them at zero gives the desired conclusion. We proceed along this line.

Let $f \in C_{\mathsf{c}}(G)$ be arbitrary. Then, as ν is a weakly admissible measure, we have $\widecheck{f} \in L^2(\nu)$. Therefore,

$$|\widecheck{f}|^2\nu = \sum_{\chi \in \widehat{G}} |\widecheck{f}(\chi)|^2 c_\chi(\mu)\delta_\chi$$

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is a finite measure. Let $g(x)$ denote the inverse Fourier transform of this measure. Then $g \in C_{\mathsf{u}}(G)$ is a Bohr almost periodic function [13], whose Fourier–Bohr series is $\sum_{\chi\in\widehat{G}} |\widetilde{f}(\chi)|^2 c_\chi(\mu)\delta_\chi$.

Now since μ is translation bounded, $\mu * (f * \widetilde{f})^\dagger \in C_{\mathsf{u}}(G)$ and ([19, 29])

$$c_\chi(\mu * (f * \widetilde{f})^\dagger) = \widecheck{f * \widetilde{f}}(\chi) c_\chi(\mu) = |\widecheck{f}(\chi)|^2 c_\chi(\mu).$$

Therefore, $\mu * (f * \widetilde{f})^\dagger$ has Fourier–Bohr series $\sum_{\chi\in\widehat{G}} |\widecheck{f}(\chi)|^2 c_\chi(\mu)\delta_\chi$.

It follows that the functions $\mu * (f * \widetilde{f})^\dagger$ and g are two Bohr almost periodic functions with the same Fourier–Bohr series, and hence they are equal [13, 19, 29].

Making the expressions equal at $x = 0$, we get

$$\langle \mu, f * \widetilde{f} \rangle = \mu * (f * \widetilde{f})^\dagger(0) = g(0) = \widecheck{|\check{f}|^2 \nu}(0) = \langle \nu, |\widecheck{f}|^2 \rangle.$$

This shows that for all $f \in C_{\mathsf{c}}(G)$, we have $\widecheck{f} \in L^2(\nu)$ and

$$\langle \mu, f * \widetilde{f} \rangle = \langle \nu, |\widecheck{f}|^2 \rangle.$$

Therefore, μ is Fourier transformable and $\widehat{\mu} = \nu = \mathcal{F}_{\mathsf{d}}(\mu)$. The last claim now follows from Corollary 6.16. ■

In the particular case $G = \mathbb{R}^d$, we get the following theorem.

Theorem 7.2 *Let $\mu \in \mathcal{SAP}(\mathbb{R}^d)$. Then, μ is Fourier transformable if and only if $\mathcal{F}_{\mathsf{d}}(\mu)$ is a translation-bounded measure.*

Moreover, in this case we have

$$\widehat{\mu} = \mathcal{F}_{\mathsf{d}}(\mu).$$

By combining Theorem 7.1 with Theorem 5.1 we get the following theorem.

Theorem 7.3 *Let $\mu \in \mathcal{SAP}(G)$. Then μ is twice Fourier transformable if and only if μ is a weakly admissible measure and $\mathcal{F}_{\mathsf{d}}(\mu)$ is a weakly admissible formal sum.*

Since strongly almost periodic measures are by definition translation bounded, in the particular case $G = \mathbb{R}^d$, we get the following corollary.

Corollary 7.4 *Let $\mu \in \mathcal{SAP}(\mathbb{R}^d)$. Then μ is twice Fourier transformable if and only if $\mathcal{F}_{\mathsf{d}}(\mu)$ is translation-bounded measure.*

Remark 7.5 Consider the class

$$S := \left\{ \mu \in \mathcal{SAP}(\mathbb{R}^d) \mid \mathcal{F}_{\mathsf{d}}(\mu) \text{ is a translation-bounded measure} \right\}.$$

Let

$$T := \{ \mathcal{F}_{\mathsf{d}}(\mu) \mid \mu \in S \}.$$

Then, by Corollary 7.4, all measures in S and T, respectively, are Fourier transformable, and the Fourier transform gives two bijection $\widehat{}: S \to T; \widehat{}: T \to S$, whose composition is a reflection.

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Another immediate consequence of Theorem 7.1 is the following simple characterisation of Fourier transformable measures with pure point transform.

Corollary 7.6 *Let $\mu \in \mathcal{M}^\infty(G)$. Then the following are equivalent:*

(i) *The measure μ is Fourier transformable, and $\widehat{\mu}$ is pure point.*
(ii) *The measure μ is strongly almost periodic, and its Fourier–Bohr series $\mathcal{F}_{\mathsf{d}}(\mu)$ is a weakly admissible formal sum.*

Moreover, in this case, we have $\widehat{\mu} = \mathcal{F}_{\mathsf{d}}(\mu)$.

Proof We know that for a Fourier transformable measure μ, we have $\widehat{\mu}$ is pure point if and only if $\mu \in \mathcal{SAP}(G)$ [29].

The claim now follows from Theorem 7.1. ∎

Corollary 7.7 *Let $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$. Then the following are equivalent:*

(i) *The measure μ is Fourier transformable, and $\widehat{\mu}$ is pure point.*
(ii) *The measure μ is strongly almost periodic, and its Fourier–Bohr series $\mathcal{F}_{\mathsf{d}}(\mu)$ is a translation-bounded measure.*

Moreover, in this case, we have $\widehat{\mu} = \mathcal{F}_{\mathsf{d}}(\mu)$.

Proof We know that for a Fourier transformable measure μ, we have $\widehat{\mu}$ is pure point if and only if $\mu \in \mathcal{SAP}(G)$ [29].

The claim follows now from Theorem 7.1. ∎

Theorem 7.1 also produces the following criterion for a pure point measure to be the Fourier transform of a measure.

Corollary 7.8 *Let ν be a pure point measure on $\widehat{G}$. Then ν is the Fourier transform of a measure if and only if ν is weakly admissible, and ν is the Fourier–Bohr series of a strongly almost periodic measure.*

Corollary 7.9 *Let ν be a pure point measure on $\mathbb{R}^d$. Then ν is the Fourier transform of a measure if and only if ν is translation bounded, and ν is the Fourier–Bohr series of a strongly almost periodic measure.*

Theorem 7.1 gives an independent proof of the following result, which was proved recently in [29].

Theorem 7.10 *Let $\mu \in \mathcal{M}^\infty(G)$ be a Fourier transformable measure. Then μ_{s} and μ_0 are Fourier transformable and*

$$\widehat{\mu}_{\mathsf{pp}} = \widehat{(\mu_{\mathsf{s}})}; \quad \widehat{\mu}_{\mathsf{c}} = \widehat{(\mu_0)}.$$

Proof Since $\mu \in \mathcal{M}^\infty(G)$ is Fourier transformable, we get that $\mu \in \mathcal{WAP}(G)$ [29]. Also, $\widehat{\mu}$ is a weakly admissible measure.

Therefore, $\widehat{\mu}_{\mathsf{pp}}$ is a weakly admissible formal sum.

Moreover, we have $\widehat{\mu}_{\mathsf{pp}} = \mathcal{F}_{\mathsf{d}}(\mu) = \mathcal{F}_{\mathsf{d}}(\mu_{\mathsf{s}})$.

Therefore, μ_{s} is a strongly almost periodic measure with a weakly admissible Fourier–Bohr series, and hence Fourier transformable. Moreover, its Fourier transform is

$$\widehat{\mu_{\mathsf{s}}} = \mathcal{F}_{\mathsf{d}}(\mu_{\mathsf{s}}) = \widehat{\mu}_{\mathsf{pp}}.$$

Finally, as a difference of two Fourier transformable measures, $\mu_0 = \mu - \mu_{\mathsf{s}}$ is Fourier transformable and

$$\widehat{\mu_0} = \widehat{\mu - \mu_{\mathsf{s}}} = \widehat{\mu} - \widehat{\mu_{\mathsf{s}}} = \widehat{\mu} - (\widehat{\mu})_{\mathsf{pp}} = (\widehat{\mu})_{\mathsf{c}}.$$ ∎

We complete the section by providing a characterisation for the class of positive definite strong almost periodic measures in terms of positivity and weak admissibility of the Fourier–Bohr series.

Theorem 7.11 *Let $\mu \in \mathcal{SAP}(G)$. Then μ is positive definite if and only if $\mathcal{F}_{\mathsf{d}}(\mu)$ is a positive weakly admissible formal sum.*

Proof $\Rightarrow$: Since μ is positive definite, it is Fourier transformable and $\widehat{\mu}$ is positive [1, 10, 29]. The claim now follows from Theorem 7.1.

$\Leftarrow$: By Theorem 7.1, μ is Fourier transformable and $\widehat{\mu} = \mathcal{F}_{\mathsf{d}}(\mu) \geq 0$. Therefore, μ is a Fourier transformable measure with positive Fourier transform, and hence positive definite [1, 29]. ∎

Corollary 7.12 *Let $\mu \in \mathcal{SAP}(\mathbb{R}^d)$. Then μ is positive definite if and only if $\mathcal{F}_{\mathsf{d}}(\mu)$ is a positive translation-bounded measure.*

8 Strongly Almost Periodic Measures as Fourier Transforms

In this section we provide a simple necessary and sufficient condition for a strongly almost periodic measure μ to be a Fourier transform, and list some of its consequences.

The result in Theorem 8.1 complements Theorem 7.1. We would like to point out that if the strongly almost periodic measure μ is twice Fourier transformable, then Theorem 7.1 and Theorem 8.1 become equivalent via Theorem 5.1, but, in general, they are independent of each other.

Theorem 8.1 *Let $\mu \in \mathcal{SAP}(\widehat{G})$. Then the following are equivalent:*

(i) *There exists some measure ν on G with $\widehat{\nu} = \mu$.*
(ii) *The Fourier–Bohr series $\mathcal{F}_{\mathsf{d}}(\mu)$ is a measure, and μ is weakly admissible.*

Moreover, in this case, we have $\nu = \left(\mathcal{F}_{\mathsf{d}}(\mu)\right)^{\dagger}$.

Proof (i) $\Rightarrow$ (ii): Since ν is Fourier transformable, and $\widehat{\mu} \in \mathcal{SAP}(\widehat{G})$, the measure ν is pure point [19].

Moreover, for all $x \in G$, we have ([19])

$$\nu(\{x\}) = M(x\widehat{\nu}) = c_{-x}(\mu).$$

This shows that $\nu = \left(\mathcal{F}_{\mathsf{d}}(\mu)\right)^{\dagger}$.

Therefore, as ν is a measure, $\mathcal{F}_{\mathsf{d}}(\mu)$ is a measure. Finally, as the Fourier transform of ν, μ is weakly admissible.

(ii) ⇒ (i): Define $\nu = (\mathcal{F}_{\mathsf{d}}(\mu))^{\dagger}$. We claim that ν is Fourier transformable, and $\widehat{\nu} = \mu$. Let $f \in C_{\mathsf{c}}(G)$. Then $(f * \widetilde{f})\nu$ is a finite pure point measure, and hence $g = \widehat{(f * \widetilde{f})\nu}$ is a strongly almost periodic function.

Moreover, by Theorem 3.3(iii), $|\widehat{f}|^2$ is convolvable as a function with μ, and the convolution $|\widehat{f}|^2 * \mu$ is continuous.

Finally, by [19, Prop. 7.3], we have $\left|\widehat{f}\right|^2 * \mu \in \mathcal{SAP}(\widehat{G})$, and the Fourier–Bohr coefficients satisfy ([19, Prop. 8.2])

$$c_x(|\widehat{f}|^2 * \mu) = \widehat{|\widehat{f}|^2}(x) c_x(\mu) = f * \widetilde{f}(-x) c_x(\mu) = f * \widetilde{f}(-x)\nu(\{-x\}).$$

As g is the Fourier transform of the finite pure point measure $\widehat{(f * \widetilde{f})\nu}$, it is also strongly almost periodic as measure and ([19, 29])

$$c_x(g) = f * \widetilde{f}(-x)\nu(\{-x\}).$$

This shows that g and $|\widehat{f}|^2 * \mu$ are two strongly almost periodic measures that have the same Fourier–Bohr series; therefore, they are equal. We also know that $g \in C_{\mathsf{u}}(G)$ and, by Theorem 3.3(iii) we have $|\widehat{f}|^2 * \mu \in C_{\mathsf{u}}(G)$. It follows that $g = |\widehat{f}|^2 * \mu$ as functions. In particular,

$$\langle \nu, f * \widetilde{f} \rangle = g(0) = |\widehat{f}|^2 * \mu(0) = \langle \mu, |\widecheck{f}|^2 \rangle.$$

Hence, by the weak admissibility of μ for all $f \in C_{\mathsf{c}}(G)$, we have

$$|\widecheck{f}|^2 \in L^1(|\mu|) \quad \text{and} \quad \langle \nu, f * \widetilde{f} \rangle = \langle \mu, |\widecheck{f}|^2 \rangle.$$

Therefore, by the definition of Fourier transformability, ν is Fourier transformable and $\widehat{\nu} = \mu$. ∎

As above, when $G = \mathbb{R}^d$ we get the following theorem.

Theorem 8.2 *Let $\mu \in \mathcal{SAP}(\widehat{\mathbb{R}^d})$. Then μ is the Fourier transform of a measure if and only if $\mathcal{F}_{\mathsf{d}}(\mu)$ is a measure. Moreover, in this case, we have*

$$\widehat{\left(\mathcal{F}_{\mathsf{d}}(\mu)\right)^{\dagger}} = \mu.$$

As a consequence of Theorem 8.1 we also get a new proof of the following result.

Theorem 8.3 ([19, Thm. 11.2]) *Let μ be a Fourier transformable measure. Then $\mu_{\mathsf{pp}}, \mu_{\mathsf{c}}$ are Fourier transformable and*

$$\widehat{(\mu)_{\mathsf{pp}}} = (\widehat{\mu})_{\mathsf{s}} \quad \textit{and} \quad \widehat{(\mu)_{\mathsf{c}}} = (\widehat{\mu})_0.$$

Proof Since μ is Fourier transformable, $\widehat{\mu} \in \mathcal{WAP}(G)$ [19] is weakly admissible. Then, by Theorem 3.4 $(\widehat{\mu})_{\mathsf{s}}$ is weakly admissible.

Moreover, we have ([19, Thm. 11.3] or [29])

$$c_{\chi}(\widehat{\mu}) = \mu(\{-\chi\}),$$

which shows that $\mathcal{F}_{\mathsf{d}}(\widehat{\mu}) = (\mu_{\mathsf{pp}})^{\dagger}$, and hence $\mathcal{F}_{\mathsf{d}}(\widehat{\mu})$ is a measure.

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Therefore, by Theorem 8.1, the measure $\mathcal{F}_{\mathsf{d}}(\widehat{\mu})^{\dagger} = \mu_{\mathsf{pp}}$ is Fourier transformable and $\widehat{(\mu)_{\mathsf{pp}}} = (\widehat{\mu})_{\mathsf{s}}$. By taking differences, it follows that μ_{c} is also Fourier transformable and

$$\widehat{(\mu)_{\mathsf{c}}} = (\widehat{\mu})_0. \qquad \blacksquare$$

9 On a Special Class of Cut and Project Formal Sums

In this section we review a large class of strongly almost periodic measures and discuss their Fourier transformability.

Consider a cut and project scheme $(G, H, \mathcal{L})$; for $h \in C_0(H)$ we define the formal sum

$$\omega_h := \sum_{(x,x^\star)\in\mathcal{L}} h(x^\star)\delta_x.$$

The following Lemma is trivial; see [9, 43].

Lemma 9.1 *If $h \in C_{\mathsf{c}}(H)$, then ω_h is strongly almost periodic measure.*

We next calculate the Fourier–Bohr series of this measure. Computations like this have been made in many places [21, 33, 35, 43].

Lemma 9.2 *If $h \in C_{\mathsf{c}}(H)$, then* $\mathcal{F}_{\mathsf{d}}(\omega_h) = \operatorname{dens}(\mathcal{L})\omega_{\check{h}}$.

Proof The computation is standard, as follows. Let $\chi \in \widehat{G}$; then by [21, Thm. 9.1], we have

$$c_\chi(\omega_h) = \operatorname{dens}(\mathcal{L}) \int_H \chi^\star(t)h(t)dt = \operatorname{dens}(\mathcal{L})\check{f}(\chi^\star). \qquad \blacksquare$$

Also, let us recall the following result.

Theorem 9.3 ([34]) *If ω_h is a translation-bounded measure, then $h \in L^1(H)$.*

We are now ready to prove the following result, (compare [34]).

Theorem 9.4 *Let $(G, H, \mathcal{L})$ be a cut and project scheme and let $h \in C_{\mathsf{c}}(H)$. Then the following are equivalent:*

(i) *The measure ω_h is Fourier transformable.*
(ii) *The formal sum $\omega_{\check{h}}$ is weakly admissible.*
(iii) *The formal sum $\omega_{\check{h}}$ is a translation-bounded measure.*
(iv) *We have $\check{h} \in L^1(\widehat{H})$.*

Proof The equivalence (i) ⇔ (ii) follows from Theorem 7.1.

(ii) ⇒ (iii) follows from Theorem 3.3, while (iii) ⇒ (iv) follows from Theorem 9.3.

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(iv) ⇒ (ii) Let $g \in K_2(G)$. Then $g \otimes h \in C_c(G \times H)$ and $\widehat{g \otimes h} \in L^1(G \times H)$, and hence [1, 32], $\widehat{g \otimes h} \in L^1(\delta_{\mathcal{L}^0})$. This is equivalent to $|\omega_{\check{h}}|(|\widecheck{g}|) < \infty$, which gives the $\widehat{K_2(G)}$-boundedness. ∎

Remark 9.5 If $h \in C_c(H)$ and $\check{h} \notin L^1(\widehat{H})$, then it follows that $\omega_h \in \mathcal{SAP}(G)$ but ω_h is not Fourier transformable as a measure.

This provides many examples of non Fourier transformable strongly almost periodic measures. In particular, for all these measures, the Fourier–Bohr series is not weakly admissible.

We complete the section by recalling a result of [43]. This result, together with Theorem 9.4, provides a characterisation for Fourier transformability for strongly almost periodic measures supported inside Meyer sets.

Theorem 9.6 *Let ω be a translation-bounded measure with Meyer set support. Then ω is strongly almost periodic if and only if there exists a cut and project scheme $(G, H, \mathcal{L})$ and a function $h \in C_c(H)$ such that $\omega = \omega_h$.*

As a consequence, we get the following theorem.

Theorem 9.7 *Let ω be a strongly almost periodic measure with Meyer set support. Then the following are equivalent:*

(i) *The measure ω is Fourier transformable.*
(ii) *There exists a cut and project scheme $(G, H, \mathcal{L})$ and a function $h \in C_c(H)$ with $\widehat{h} \in L^1(\widehat{H})$ such that $\omega = \omega_h$.*
(iii) *For each cut and project scheme $(G, H, \mathcal{L})$ and function $h \in C_c(H)$ such that $\omega = \omega_h$, we have $\widehat{h} \in L^1(\widehat{H})$.*

Proof (i) ⇒ (ii) Theorem 9.6 gives the existence of the cut and project scheme. Now, since ω is Fourier transformable, by Theorem 9.4 we get $\widehat{h} \in L^1(\widehat{H})$.

(ii) ⇒ (i) Follows from Theorem 9.4.

(i) ⇒ (iii) Follows from Theorem 9.4.

(iii) ⇒ (i) Theorem 9.6 gives that there exists a cut and project scheme and some $h \in C_c(H)$ such that $\omega = \omega_h$. Now, by (iii), we have $\widehat{h} \in L^1(\widehat{H})$. (i) follows now from Theorem 9.4. ∎

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