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A NEW PROOF OF THE ABSENCE OF LIMIT CYCLES IN A QUADRATIC AUTONOMOUS SYSTEM

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Dedicated to Professor George Szekeres on his 65th birthday

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Abstract

The absence of limit cycles in the system (S): $x' = -y + dx^2 + exy + fy^2$, $y' = x$, $(' \equiv d/dt)$, for $0 \leqq f^2 < 2$, was shown by Yeh et al. (1963). The general case was established by Kukles and Šakhova using various special results and relying on an extraneous auxiliary system. In this paper we give a new proof of the theorem; in it, we draw mainly on the basic properties of the characteristic exponent of a cycle and use an intrinsic system, viz. the case $d = -f(\neq 0)$ of (S) as an auxiliary system.

1. Introduction

We are concerned with the following.

THEOREM. *The system*

$$\left.\begin{aligned} x' &= -y + dx^2 + exy + fy^2, \\ y' &= x, \end{aligned}\right\} \qquad (S)$$

$(' \equiv d/dt)$, *with d, e, f real constants, has no limit cycles.*

The result appears to have been found first by Yeh et al. (1963), and subsequently, using different methods, by Kukles and Šakhova (1967). The proof in Yeh et al. (1963), however, is valid only if $0 \leqq f^2 < 2$; that in Kukles and Šakhova (1967), on the other hand, introduces an auxiliary (topographic) system of a rather involved kind and refers to sources which, for the most part, are inaccessible to the western reader.

In the present note we give a new proof of the Theorem. In it, no restrictions are imposed on the coefficients of (S) and, unlike in Kukles and Šakhova (1967), an intrinsic system, viz. the case, $d = -f \neq 0$ of (S) is used as an auxiliary system.

The proof relies on the general properties of the characteristic exponent of a cycle (see Sansone and Conti (1964); Chapter iv, §2) and on the well-known results of the Poincaré theory of small perturbations of differential systems (see Andronov, Vitt and Khaikin (1966); Chapter vi, §5); for particular results concerning quadratic autonomous systems reference is made to the survey article by Coppel (1966).

Considering the form of (S), it was found convenient to divide the Theorem into two parts: one dealing with the case of the origin being a centre (Theorem A), the other with the contrary case of the origin being a weak focus of (S) (Theorem B).

2. The case of a centre

A criterion by which, in a quadratic autonomous system of the relevant form, a centre can be distinguished from a weak focus is given in Coppel (1966). We shall state it in the following.

PROPOSITION. *The real autonomous system* (C) *given by*

$$\left.\begin{aligned} x' &= \quad - y - bx^2 - (2c + \beta)xy - dy^2, \\ y' &= x + \cdot + ax^2 + (2b + \alpha)xy + cy^2, \end{aligned}\right\} \qquad (C)$$

with $' \equiv d/dt$, *has a centre at the origin if and only if one of the following three conditions is satisfied*:

i. $a + c = b + d = 0$;

ii. $\alpha(a + c) = \beta(b + d)$,
$a\alpha^3 - (3b + \alpha)\alpha^2\beta + (3c + \beta)\alpha\beta^2 - d\beta^3 = 0$;

iii. $\alpha + 5(b + d) = \beta + 5(a + c) = ac + bd + 2(a^2 + d^2) = 0$.

By applying this result, we have

THEOREM A. *If* $e(d + f) = 0$, *then* (S) *has no limit cycles.*

PROOF. The system (S) can be identified with the standard form in the Proposition above, with a and c equal to zero, and $-d$, $-f$, $2d$, $-e$ replacing b, d, α, β respectively. Applying parts I and II of Proposition, we conclude that the origin, 0 is a centre of (S) if and only if $e(d + f) = 0$.

Beside 0, the system has no other finite critical points except for a saddle at $M(0, 1/f)$ when $f \neq 0$. We, therefore, conclude (see Coppel (1966); Theorems 2 and 6) that (S) can have a limit cycle only when the origin is a weak focus. Hence, the result follows.

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3. The case of a weak focus

We shall first prove the following two lemmas.

LEMMA 1. *If $e(d+f) \neq 0$, then the origin as a weak focus of (S) is stable or unstable according as $e(d+f)$ is negative or positive.*

PROOF. The system (S) can now be identified with a standard form under small perturbations from the weak-focus — or centre-position at the origin (see Andronov, Vitt and Khaikin (1966); Chapter vi, §5, section 4, (6.23) p. 409) with $h(x,y)$ vanishing identically and $\alpha(\lambda)$, $\beta(\lambda)$, $g(x,y,\lambda)$ being represented by λ, 1, $dx^2+exy+fy^2$, respectively. Putting

$$P(\theta)=g(\cos\theta,\sin\theta,0),$$

we obtain

$$R_2(\theta,0)=P(\theta)\cos\theta$$

and

$$R_3(\theta,0)=2P^2(\theta)\sin\theta\cos\theta$$

as the respective auxiliary functions $R_2(\theta,\lambda)$ and $R_3(\theta,\lambda)$ for $\lambda=\lambda_0=0$. Substituting these into the equation

$$\frac{du_2}{d\theta}=R_2(\theta,0), \quad \text{with} \quad u_2(0)=0,$$

we find that

$$u_2(\theta)=-\tfrac{1}{3}e(\cos^3\theta-1)+\tfrac{1}{3}(f-d)\sin^3\theta+d\sin\theta,$$

and hence that

$$\int_0^{2\pi}u_2(\theta)R_2(\theta,0)\,d\theta=0.$$

Substituting for $u_2(\theta)$ in the equation

$$\frac{du_3}{d\theta}=2u_2(\theta)R_2(\theta,0)+R_3(\theta,0), \quad \text{with} \quad u_3(0)=0,$$

and integrating over the interval $[0,2\pi]$, we obtain

$$\begin{aligned}u_3(2\pi)&=2\int_0^{2\pi}R_3(\theta,0)\,d\theta\\&=2\int_0^{2\pi}P^2(\theta)\sin\theta\cos\theta\,d\theta\\&=\tfrac{1}{4}\pi e(d+f).\end{aligned}$$

Since $\alpha'(\lambda) = 1 > 0$, it follows (see Andronov, Vitt and Khaikin (1966); Chapter vi, section 4, cases (a), (b), p. 412) from Theorem A of the section above that the origin is a weak focus of (S) and is stable or unstable according as $e(d+f)$ is negative or positive.

LEMMA 2. *Let $\gamma \equiv (\lambda(t), \mu(t))$ be a cycle (a closed path) of (S) with period $T > 0$, where $\lambda(t)$, $\mu(t)$ are functions of class C^1 on the interval $[0,T]$. If h is the characteristic exponent of γ, then*

$$h = \frac{e}{T}\int_0^T \mu(t)dt = \frac{e}{T}\int_0^T [d\lambda^2(t) + f\mu^2(t)]\,dt.$$

PROOF. Applying the divergence formula for the characteristic exponent of a cycle, we obtain

$$h = \frac{1}{T}\int_0^T [2\,d\lambda(t) + e\mu(t)]\,dt.$$

But

$$\int_0^T \lambda(t)dt = \int_0^T \mu'(t)dt = 0,$$

and

$$\int_0^T \mu(t)\lambda(t)\,dt = \frac{1}{2}\int_0^T \frac{d}{dt}\mu^2(t)dt = 0,$$

from which the first and the second identities, respectively, follow.

We now proceed to the second part of Theorem.

THEOREM B. *If $e(d+f) \neq 0$, then (S) has no limit cycles.*

PROOF. *Case* 1. $df \geqq 0$.

This is obviously so when $df \geqq 0$. For, if $e \neq 0$ and $|d| + |f| \neq 0$, then, by Lemma 2 above, the sign of the characteristic exponent of each cycle of (S) is that of $e(d+f)$. Hence (see Sansone and Conti (1966); Chapter iv, §2, section 5, Theorem 13, p. 170), the system (S) can, therefore, have only one limit cycle which is stable or unstable, according as $e(d+f)$ is negative or positive. This, however, contradicts Lemma 1 above.

Case 2. $df < 0$.

Let us thus assume that $df < 0$ and that the system (S) has a limit cycle γ defined as in Lemma 2 above. For the sake of definiteness, we take d to be negative and f positive. (If the opposite were the case, we could reverse the signs of both x and y.)

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In order to determine the sign of the characteristic exponent, h of γ, we consider a system (S^*) of the same form as (S) given by

$$\left.\begin{aligned} x' &= -y - fx^2 + exy + fy^2, \\ y' &= \quad x. \end{aligned}\right\} \qquad (S^*)$$

The system (S^*) has a common saddle with (S) at the point $M(0,1/f)$, and the locus of contacts of the two systems consists of the three open line segments ∞O, OM, $M\infty$ of the y-axis. Moreover, it follows from Theorem A of the section above (see Coppel (1966); section 2, pp. 294–295) that (S^*) has a centre at the origin and that through each point of the open line segment OM of the y-axis passes a unique cycle of the system. (S^*) may, therefore, be regarded as a most convenient auxiliary (topographic) or comparison system for the study of (S).

Concerning the sign of $d+f$, two alternatives now remain to be considered.

Case 2α. $d + f > 0$.

Let us first assume that $d + f > 0$. Since (see Coppel (1966); section 3, Theorems 1 and 2, p. 296, and section 4, Theorem 6, p. 299) each cycle of (S) or (S^*) is a convex curve containing the origin as the only critical point in its interior, it follows that the limit cycle γ crosses the y-axis at two points, $E \equiv (0,y_E)$, say, between O and M, and $I \equiv (0,y_I)$, say, below the origin 0. Similarly, if γ_e^* is the (unique) cycle of (S^*) through E, then γ_e^* will cross the y-axis at another point $E' \equiv (0,y_{E'})$ say, where $y_{E'} < 0$.

Now, since the x'-component of (S) exceeds that of (S^*) by $(d+f)x^2$, whilst the y'-components of both systems are equal to x, it follows that, at the point $P(x,y)$, not on the y-axis, the angle between the field vector of (S) and that of (S^*) is positive or negative according as $x < 0$ or $x > 0$. The cycle γ_e^*, therefore, runs externally to γ, with the point E as the only point of contact. (See Fig. 1.)

Also, since the point I is situated inside γ_e^*, we have a unique cycle, γ_i^*, say, of (S^*) passing through it. From an argument similar to that used above it follows that γ runs externally to γ_i^* and that I is their only point of contact.

Now let $I' \equiv (0,y_{I'})$ be the point at which the cycle γ_i^* crosses the positive y-axis and $\bar{H}H$ be the chord of γ tangent to γ_i^* at I', where the points $H, \bar{H}$ have positive and negative abscissa, respectively. The right- (left-) hand part $IHE(I\bar{H}E)$ of γ is thus the graph of a function $\phi(y)(\bar{\phi}(y))$, say, which is of class C' and satisfies the equation

$$\frac{dx}{dy} = \frac{-y + dx^2 + exy + fy^2}{x}$$

on the open interval (y_I, y_E), and is continuous at the endpoints of the interval. *In an analogous fashion we introduce the functions $\phi_i^*(y)$ and $\bar{\phi}_i^*(y)$ for the* cycle γ_i^* on the interval $(y_I, y_{I'})$.

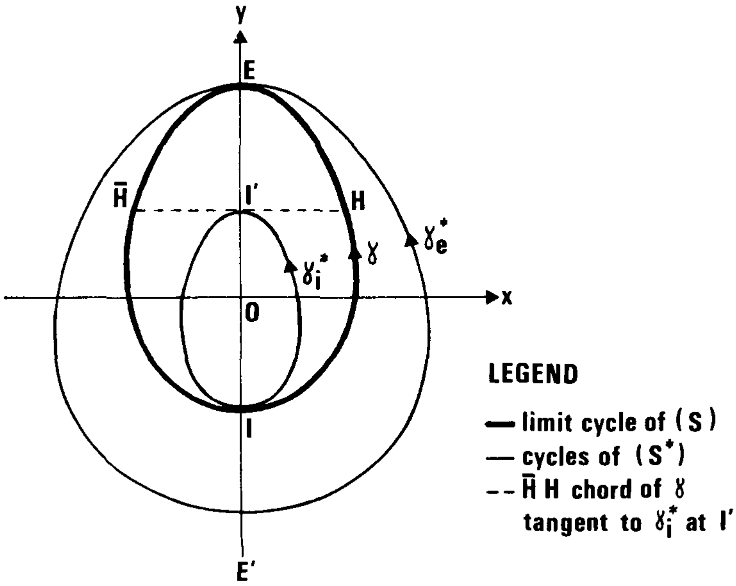

Figure 1.

Let T_i^*, h_i^* denote the period and the characteristic exponent of γ_i^*, respectively. Since the corresponding integrals in terms of the variable t exist, we have from Lemma 2 above that

$$\frac{hT}{e} = \int_{y_I}^{y_E} \frac{y}{\phi(y)}\,dy + \int_{y_E}^{y_I} \frac{y}{\bar{\phi}(y)}\,dy,$$

and that

$$\frac{h_i^* T_i^*}{e} = \int_{y_I}^{y_{I'}} \frac{y}{\phi_i^*(y)}\,dy + \int_{y_{I'}}^{y_I} \frac{y}{\bar{\phi}_i^*(y)}\,dy.$$

Clearly, for all y on $(y_I, y_{I'})$,

$$\frac{1}{\phi(y)} - \frac{1}{\phi_i^*(y)} < 0 \quad \text{and} \quad \frac{1}{\bar{\phi}(y)} - \frac{1}{\bar{\phi}_i^*(y)} > 0.$$

Also, since $y_I < 0$, it follows that

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$$\int_{y_I}^{0} y\left[\frac{1}{\phi(y)} - \frac{1}{\bar{\phi}(y)}\right] dy > 0,$$

and that

$$\int_{0}^{y_I} y\left[\frac{1}{\bar{\phi}(y)} - \frac{1}{\bar{\phi}_i^*(y)}\right] dy > 0.$$

By an analogous argument we find that

$$\int_{0}^{y_{I'}} y\left[\frac{1}{\phi(y)} - \frac{1}{\phi_i^*(y)}\right] dy + \int_{y_{I'}}^{y_E} \frac{y}{\phi(y)} dy > 0,$$

and that

$$\int_{y_E}^{y_{I'}} \frac{1}{\bar{\phi}(y)} + \int_{y_{I'}}^{0} y\left[\frac{1}{\bar{\phi}(y)} - \frac{1}{\bar{\phi}_i^*(y)}\right] dy > 0.$$

Now (see Sansone and Conti (1964); Theorem 13, p. 170), we have $h_i^* = 0$; hence, combining the last four inequalities above, we obtain

$$\frac{1}{e}[hT - h_i^* T_i^*] = \frac{hT}{e} > 0.$$

But, since, by hypothesis, both $d + f$ and T are positive, it follows that h and $e(d + f)$ must have the same sign. This, however, as in Case 1. above, leads to a contradiction.

Case 2β. $d + f < 0$

Let us thus assume that $d + f < 0$. By an argument analogous to that used in Case 2α above, we find that the signs of the ordinates of the points of contact of γ with the cycles γ_i^*, γ_e^* of (S^*) that are interior, exterior to it, respectively, are now reversed. This means that in the argument used in Case 2α, the cycles γ and γ_i^* can now be replaced by γ_e^* and γ, respectively. Since $h_e^* = 0$, we have $-hT/e > 0$, and thus $he < 0$. But, with $d + f < 0$, this implies that h and $e(d + f)$ are of the same sign, and so, as in the above case, a contradiction follows.

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References

A. A. Andronov, A. A. Vitt and S. E. Khaikin (1966), *Theory of Oscillations* (translated from the Russian, Pergamon Press, U.S.A., 1966); *MR* 33 # 6888.

W. A. Coppel, 'A survey of quadratic systems (1966)', *J. Differential Equations* **2**, 293–304; *MR* 33 # 4374.

I. S. Kukles and L. V. Šakhova (1967), 'On the absence of limit cycles for a certain differential equation', *Izv. Vysš. Učebn. Zaved., Matematika* 1967, No. 3 (58), 46–49 (Russian); *MR* 35 # 5694.

G. Sansone and R. Conti (1964), *Non-linear differential equations* (translated from the Italian, Pergamon Press, U.S.A., 1964); *MR* 31 # 1417.

Yeh Yen-Ch'ien, Ho Ch'ung-Yu, Wang Ming-Shu, Hsü Ming-Wei, Lo Ting-Chün (1963). 'A qualitative study of the integral curves of the differential equation

$$\frac{dy}{dx} = \frac{q_{00} + q_{10}x + q_{01}y + q_{20}x^2 + q_{11}xy + q_{02}y^2}{p_{00} + p_{10}x + p_{01}y + p_{20}x^2 + p_{11}xy + p_{02}y^2},$$

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