NOTES ON NUMBER THEORY I

On the product of the primes not exceeding n.

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One of the most elegant results of the elementary theory of the distribution of primes is that

(1) $$R(n) = \prod_{p \leqslant n} p < 4^n ,$$

where the product runs over primes. A very simple proof of (1) has recently been given by Erdős and Kalmar [1], [2]. A form of the prime number theorem [2] states that

(2) $$\theta(n) = \log R(n) \sim n .$$

This implies that for every $\varepsilon > o$ and $n > n_o(\varepsilon)$

(3) $$R(n) < (e + \varepsilon)^n ,$$

and that in (3) Euler's constant e = 2.718 ... cannot be replaced by any smaller number.

If, however, we are interested in improvements of (1) valid for all n, then the best available result is the following estimate due to Rosser [3]:

(4) $$R(n) < 2.83^n .$$

Rosser's proof of (4) is definitely not elementary and moreover involves much computation. The object of the present note is to give an elementary proof of

(5) $$R(n) < c^n ,$$

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where c is the positive number defined by

(6) $$c^5 = 2^4 3^3 \qquad (c < 3.37) .$$

Our proof depends on an analysis of the number

(7) $$A_m = (6m + 1)! \,/\, m!\,(2m)!\,(3m)! .$$

We note first that A_m is $6m + 1$ times a multinomial coefficient and hence is an integer. Next we prove

LEMMA 1.

$$A_m = (2^4 3^3)^m \prod_{k=1}^{m}(1 - 1/2^2 3^2 k^2) < c^{5m} \qquad (m \geq 1).$$

Proof. The equality follows by a straightforward induction on m and the inequality is then an immediate consequence of (6).

We will further require

LEMMA 2. For prime p, $m < p \leq 6m + 1$, p divides A_m.

Proof. Consider separately cases where p lies in the ranges:

(i) $3m < p \leq 6m + 1$,
(ii) $2m < p \leq 3m$,
(iii) $3m/2 < p \leq 2m$,
(iv) $m < p \leq 3m/2$.

In range (i) p divides the numerator of A_m (see (7)) but not the denominator. In range (ii) p^2 divides the numerator while p, but not p^2, divides the denominator. In range (iii) p^3 divides the numerator while the highest power of p dividing the denominator is p^2. Finally, in range (iv), p^4 divides the numerator while p^3 is the highest power of p which divides the denominator.

We now proceed to the proof of (5) by complete induction over n. The result is trivially true for 2 and 3 and by the induction hypothesis will be assumed true up to n. In proving it at $n + 1$ we may assume that $n + 1$ is a prime for otherwise $R(n) = R(n + 1)$. Further, for $n > 3$ all primes have the form $6m \pm 1$. Hence we need only consider the cases (i) $n = 6m + 1$ and (ii) $n = 6m - 1$.

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In case (i) the lemmas and the induction hypothesis yield

$$(8) \qquad R(6m+1) = R(m) \prod_{m<p\leq 6m+1} p \leq c^m A_m < c^{6m+1}.$$

For case (ii) we first note that for $m < p \leq 6m$, p divides $A_m / (6m+1)$ and the latter is an integer less than c^{5m-1}. Hence

$$(9) \qquad R(6m-1) = R(m) \prod_{m<p\leq 6m-1} p \leq c^m A_m/(6m+1) < c^{6m-1}$$

and the proof is complete.

It would be nice to have an equally elementary proof that $R(n) < 3^n$. In conclusion we remark that it does not seem entirely hopeless to seek by elementary methods the smallest constant k for which $R(n) < k^n$ for all n.

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