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CONSTRUCTING HERMAN RINGS BY TWISTING ANNULUS HOMEOMORPHISMS

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Abstract

Let $F(z)$ be a rational map with degree at least three. Suppose that there exists an annulus $H \subset \widehat{\mathbb{C}}$ such that (1) H separates two critical points of F, and (2) $F : H \to F(H)$ is a homeomorphism. Our goal in this paper is to show how to construct a rational map G by twisting F on H such that G has the same degree as F and, moreover, G has a Herman ring with any given Diophantine type rotation number.

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1. Introduction

Let f be a rational map with degree not less than two. Let H be an invariant Fatou component of f. We say that H is a Herman ring if $f : H \to H$ is holomorphically conjugate to an irrational rotation of some annulus. There are two known methods to construct a Herman ring. The original method, which is due to Herman, is based on certain Blaschke products and Arnold's linearization theorem on real-analytic circle homeomorphisms. A more general construction was proposed by Shishikura. The idea of Shishikura can be sketched as follows. One starts with two rational maps with Siegel disks of rotation number θ and $-\theta$. In order to fabricate a Herman ring, one needs to cut and paste together the two Siegel disks to get a topological model map. Then by using the Morrey–Ahlfors–Bers Measurable Riemann Mapping theorem, one can conjugate the resulted topological picture to an actual rational map. For more details about these two constructions, the reader may refer to [1, Ch. VI].

In this paper, we extend Herman's idea and provide an another way to construct Herman rings by twisting a rational map on some annulus. Here the word twisting means, for any given Diophantine type irrational number $0 < \theta < 1$, we

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can postcompose the rational map by a certain homeomorphism such that the resulting model map, when restricted to some topological annulus, is quasiconformally conjugate to the rigid rotation $z \to e^{2\pi i\theta} z$. This idea comes from [4] where a Siegel disk is constructed by twisting a linearizable domain centered at an attracting fixed point. Compared with the constructions given by Herman and Shishikura, the new feature of our construction is that the starting point is just a rational map which satisfies a fairly general topological condition. Before we state the Main Theorem, let us first introduce a definition.

DEFINITION 1.1. Let F and G be two rational maps. We say that F and G are topologically equivalent to each other if there exist homeomorphisms $\phi, \psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $F \circ \phi = \psi \circ G$.

Recall that an irrational number $0 < \theta < 1$ is called a Diophantine number if there exist $\beta \geq 2$ and $C > 0$ such that for all positive integers p and q, $|\theta - p/q| > C/q^{\beta}$. We prove the following result.

MAIN THEOREM. *Let $0 < \theta < 1$ be a Diophantine number and $F(z)$ be a rational map with degree at least three. Suppose that there exists an annulus $H \subset \widehat{\mathbb{C}}$ such that (1) H separates two critical points of F, and (2) $F : H \to F(H)$ is a homeomorphism. Then there exists a rational map G which has a Herman ring with rotation number θ such that F and G are topologically equivalent to each other.*

REMARK 1.2. The Main Theorem applies for a slightly larger class of rotation numbers which are called Herman numbers. This is because the arithmetic condition of θ is only needed to apply the following Herman–Yoccoz Theorem and this theorem is actually true for all such rotation numbers (the reader may refer to [3, p. 131] for the definition of Herman numbers).

Let R_θ denote the rigid rotation given by $z \to e^{2\pi i\theta} z$.

THEOREM (HERMAN–YOCCOZ) [3]. *Let $0 < \theta < 1$ be a Diophantine number. Let $f : S^1 \to S^1$ be a real-analytic circle diffeomorphism of rotation number θ. Then f is real-analytically conjugate to the rigid rotation $R_\theta : S^1 \to S^1$.*

REMARK 1.3. The condition that the degree of F is not less than three is forced by the topological restriction: for a quadratic rational map F, there is no annulus which separates the two critical points, and on which F is a homeomorphism.

2. Proof of the Main Theorem

Let F and H be the rational map and the annulus which satisfy the conditions in the Main Theorem. Without loss of generality, we may assume that the two critical points separated by H are 0 and ∞.

2.1. The Riemann isomorphisms g_ξ and h_λ By considering a sub-annulus of H, we may assume that both boundaries of H are real-analytic curves which do not pass any critical point of F. Let γ and η denote the outer and the inner boundary component of H, respectively. It follows that the curves $F(\gamma)$ and $F(\eta)$, which are the two boundary components of $F(H)$, are also real-analytic curves.

Let U and V denote the two components of $\widehat{\mathbb{C}} - \overline{H}$ such that $\partial U = \gamma$ and $\partial V = \eta$. By the assumption, we obtain $\infty \in U$ and $0 \in V$. Let X and Y denote the two components of $\widehat{\mathbb{C}} - \overline{F(H)}$ such that $\partial X = F(\gamma)$ and $\partial Y = F(\eta)$. Take $x \in X$, $y \in Y$, $a \in \partial X$, and $b \in \partial Y$.

For each $\xi \in \gamma$, there is a unique holomorphic isomorphism $g_\xi : X \to U$ such that $g_\xi(a) = \xi$ and $g_\xi(x) = \infty$. The maps $g_\xi \circ F|\gamma : \gamma \to \gamma$, $\xi \in \gamma$, consist of a monotone family of topological circle homeomorphisms. By [2, Proposition 11.1.9], it follows that there is a unique $\xi' \in \gamma$ such that the rotation number of $g_{\xi'} \circ F|\gamma : \gamma \to \gamma$ is θ.

Similarly, for each $\lambda \in \eta$, there is a unique holomorphic isomorphism $h_\lambda : Y \to V$ such that $h_\lambda(b) = \lambda$ and $h_\lambda(y) = 0$. The maps $h_\lambda \circ F : \eta \to \eta$, where $\lambda \in \eta$, consist of a monotone family of topological circle homeomorphisms. Again by [2, Proposition 11.1.9], it follows that there is a unique $\lambda' \in \eta$ such that the rotation number of $h_{\lambda'} \circ F|\eta : \eta \to \eta$ is θ.

2.2. Passing to an Euclidean annulus Take $0 < r < R$. Let Δ_R and Δ_r denote the two Euclidean disks which are centered at the origin, and which have radius R and r, respectively. Let $\mathbb{T}_R = \partial \Delta_R$ and $\mathbb{T}_r = \partial \Delta_r$. Let $\phi : \widehat{\mathbb{C}} - \overline{\Delta_R} \to U$ be the Riemann isomorphism such that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. Similarly, let $\psi : \Delta_r \to V$ be the Riemann isomorphism such that $\psi(0) = 0$ and $\psi'(0) > 0$. Since ξ and η are both real-analytic curves, it follows that ϕ and ψ can be homeomorphically extended to $\mathbb{T}_R$ and $\mathbb{T}_r$, respectively. By the Schwarz Reflection Lemma, one easily obtains the following result.

LEMMA 2.1. *Both the maps $\phi^{-1} \circ g_{\xi'} \circ F \circ \phi : \mathbb{T}_R \to \mathbb{T}_R$ and $\psi^{-1} \circ h_{\lambda'} \circ F \circ \psi : \mathbb{T}_r \to \mathbb{T}_r$ are real-analytic circle diffeomorphisms of rotation number θ.*

Now applying the Herman–Yoccoz Theorem (see Section 1), we obtain the following result.

LEMMA 2.2. *There exist two analytic circle homeomorphisms $h_1 : \mathbb{T}_R \to \mathbb{T}_R$ and $h_2 : \mathbb{T}_r \to \mathbb{T}_r$ such that $\phi^{-1} \circ g_{\xi'} \circ F \circ \phi(z) = h_1^{-1} \circ R_\theta \circ h_1(z)$ for all $z \in \mathbb{T}_R$, and $\psi^{-1} \circ h_{\lambda'} \circ F \circ \psi(z) = h_2^{-1} \circ R_\theta \circ h_2(z)$ for all $z \in \mathbb{T}_r$.*

Let A denote the annulus $\{z \,\big|\, r < |z| < R\}$. Since $h_1 : \mathbb{T}_R \to \mathbb{T}_R$ and $h_2 : \mathbb{T}_r \to \mathbb{T}_r$ are both real-analytic homeomorphisms, it is not difficult to obtain the following result.

LEMMA 2.3. *There exists a quasiconformal homeomorphism $\sigma : A \to A$ such that $\sigma|\mathbb{T}_R = h_1$ and $\sigma|\mathbb{T}_r = h_2$.*

Since γ and η are real-analytic curves, one can easily obtain the following result.

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LEMMA 2.4. *There is a quasiconformal homeomorphism $\tau : A \to H$ such that $\tau|\mathbb{T}_R = \phi$ and $\tau|\mathbb{T}_r = \psi$.*

2.3. The quasiconformal model map Since $F : H \to F(H)$ is a homeomorphism, there is an inverse branch of F defined on $F(H)$, say $\chi : F(H) \to H$ such that $\chi \circ F = \mathrm{id}$ on H. Since both η and ξ are analytic curves and are therefore quasiconformally erasable, we can define a quasi-conformal homeomorphism $L : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$L(z) = \begin{cases} g_{\xi'}(z) & \text{for all } z \in X, \\ \tau \circ \sigma^{-1} \circ R_\theta \circ \sigma \circ \tau^{-1} \circ \chi(z) & \text{for all } z \in F(\overline{H}), \\ h_{\chi'}(z) & \text{for all } z \in Y. \end{cases} \tag{2.1}$$

Now define the quasiconformal model map $\widetilde{F} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by $\widetilde{F}(z) = L \circ F(z)$. Note that, by construction, $\widetilde{F}(z)$ is holomorphic on the outside of $\widetilde{F}^{-1}(\overline{H})$.

2.4. Realize the quasiconformal map $\widetilde{F}$ by a rational map By the construction of $\widetilde{F}$, it follows that when restricted on H, $\widetilde{F}$ is quasiconformally conjugate to the irrational rotation $R_\theta : A \to A$. Therefore, $\widetilde{F}$ has an invariant complex structure μ defined in H. Since $\widetilde{F}$ is holomorphic on the outside of $\widetilde{F}^{-1}(\overline{H})$, one can pull back μ by the iteration of $\widetilde{F}$ and finally obtain a $\widetilde{F}$-invariant complex structure ν on the whole Riemann sphere. Now applying the Morrey–Ahlfors–Bers Measurable Riemann Mapping Theorem, one has a quasiconformal homeomorphism ω of the sphere which fixes 0, 1, and ∞, and which solves the Beltrami equation given by ν. Note that when restricted to H, $\nu = \mu$. This implies that the map $G(z) = \omega \circ \widetilde{F} \circ \omega^{-1}(z)$ is a rational map which, when restricted to $\omega(H)$, is holomorphically conjugated to the irrational rotation $R_\theta : A \to A$. Let W be the Fatou component of G which contains $\omega(H)$. It follows that W is either a Siegel disk or a Herman ring. Since $\omega(H)$ separates 0 and ∞ which are critical points of G, by construction, it follows that W cannot be simply connected. This implies that W is a Herman ring of G which has rotation number θ. By the construction, it is clear that G is topologically equivalent to F. The proof of the Main Theorem is complete.

REMARK 2.5. There are many ways to construct a rational map F which satisfies the conditions in the Main Theorem. For instance, let $F(z) = z + \epsilon(z^m + 1/z^n)$ where $m, n \geq 2$ are integers. It is not difficult to see that when $|\epsilon| > 0$ is small, there is an annulus H which separates the two critical points 0 and ∞ and on which F is a homeomorphism.

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