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A SET-VALUED GENERALIZATION OF FAN'S BEST APPROXIMATION THEOREM

XIE PING DING AND KOK-KEONG TAN

ABSTRACT. Let (E, T) be a Hausdorff topological vector space whose topological dual separates points of E, X be a non-empty weakly compact convex subset of E and W be the relative weak topology on X. If $F: (X, W) \longrightarrow 2^{(E,T)}$ is continuous (respectively, upper semi-continuous if E is locally convex), approximation and fixed point theorems are obtained which generalize the corresponding results of Fan, Park, Reich and Sehgal-Singh-Smithson (respectively, Ha, Reich, Park, Browder and Fan) in several aspects.

1. **Introduction.** Let X be a non-empty set; we shall denote by 2^X the family of all *non-empty* subsets of X and by $\mathcal{F}(X)$ the family of all non-empty finite subsets of X. If X is a topological space with topology T, we shall use (X, T) and $2^{(X,T)}$ to denote the sets X and 2^X respectively with emphasis on the fact that X is equipped with the topology T. If (X, T) is a topological space and A is a subset of X, we shall denote by $\operatorname{int}(A)$ and ∂A the interior of A in (X, T) and the boundary of A in (X, T) respectively and we shall use the terms "A is T-open (respectively, T-closed, T-compact)" and "A is open (respectively, closed, compact) in (X, T)" interchangeably. Let (X, T) and (Y, S) be topological spaces; a set-valued map $f: (X, T) \longrightarrow 2^{(Y,S)}$ is said to be (i) *upper semi-continuous* (respectively, *lower semi-continuous*) *at* $x_0 \in X$ if for each S-open set G in Y with $f(x_0) \subset G$ (respectively, $f(x_0) \cap G \neq \emptyset$), there exists a T-open neighborhood U of x_0 in X such that $f(x) \subset G$ (respectively, $f(x) \cap G \neq \emptyset$) for all $x \in U$; (ii) *upper semi-continuous* (respectively, *lower semi-continuous*) if f is upper semi-continuous (respectively, lower semi-continuous) at each point of X; (iii) *continuous* if f is both upper semi-continuous and lower semi-continuous. Let E be a topological vector space with topology T. We shall denote by $E^* = (E, T)^*$ the topological dual of (E, T). E^* is said to separate points of E if for each $x \in E$ with $x \neq 0$, there exists an $f \in E^*$ such that $f(x) \neq 0$. We shall denote by $W = W(E, E^*)$ the weak topology of E and by $\mathcal{P} = \mathcal{P}(E, T)$ the family of all continuous semi-norms on (E, T). If X is a non-empty subset of E, we shall denote by $\operatorname{co}(X)$ the convex hull of X and by (X, T) and (X, W) the set X equipped with the relative topology of T to X and the relative topology of W to X respectively. We shall denote by $\mathbb{R}$ the set of all real numbers and if z is a complex number, we shall denote by $\operatorname{Re} z$ the real part of z.

The following is a well-known result of Fan [5, Theorem 1]:

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THEOREM A. *Let X be a non-empty compact convex set in a locally convex Hausdorff topological vector space E. Let $f: X \longrightarrow E$ be a continuous mapping. Then either f has a fixed point in X, or there exist a point $y_0 \in X$ and a continuous semi-norm p on E such that*

$$0 < p\big(y_0 - f(y_0)\big) = \min\big\{p\big(x - f(y_0)\big) : x \in X\big\}.$$

Since then many authors have generalized the above result to set-valued maps or have weakened the compactness condition therein, *e.g.* see [6, 7, 8, 9, 10, 14, 15, 16, 19, 20, 21].

Recently, Sehgal, Singh and Smithson [22] generalized Fan's result to a continuous map $f: (X, W) \longrightarrow (E, T)$ with a weak compactness condition where (E, T) is a locally convex Hausdorff topological vector space and X is a non-empty convex subset of E. Also Park [12] generalized Fan's result to a continuous set-valued map $f: X \longrightarrow 2^E$ where X is a compact convex subset of a Hausdorff topological vector space E whose topological dual E^* separates points of E.

In this paper, we shall first improve, generalize and unify those results of Park in [12] and Sehgal, Singh and Smithson in [22] to continuous set-valued maps $f: (X, W) \longrightarrow 2^{(E,T)}$ where (E, T) is a Hausdorff topological vector space whose topological dual E^* separates points of E and X is a non-empty convex subset of E and thus generalize Fan's result in many aspects. Next, we improve and generalize those results of Ha in [7] and Reich in [15, 16] to upper semi-continuous set-valued maps $f: (X, W) \longrightarrow 2^{(E,T)}$ where (E, T) is a locally convex Hausdorff topological vector space and X is a nonempty W-compact convex subset of E.

2. **Preliminaries.** Let (E, T) be a topological vector space; for each non-empty subset A of E and for each $p \in \mathcal{P}$, let

$$d_p(x, A) = \inf\{p(x - a) : a \in A\}$$

for each $x \in E$. Then we have the following simple fact which can be easily proved.

LEMMA 1. *Let (E, T) be a topological vector space, A be a non-empty compact convex subset of E and $p \in \mathcal{P}$. Then the function $x \longrightarrow d_p(x, A)$ is lower semi-continuous and convex.*

We shall need the following result of Aubin [1, Theorem 2.5.1, p. 67]:

LEMMA 2. *Let X and Y be topological spaces. Suppose that $h: X \times Y \longrightarrow \mathbb{R}$ is lower semi-continuous and $f: X \longrightarrow 2^Y$ is upper semi-continuous at $x_0 \in X$ such that $f(x_0)$ is compact. Then the function $x \longrightarrow \inf\{h(x, y) : y \in f(x)\}$ is lower semi-continuous at x_0.*

We shall also need the following result of Aubin [1, Theorem 2.5.2, p. 69]:

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LEMMA 3. *Let X and Y be topological spaces. Suppose that $h: X \times Y \longrightarrow \mathbb{R}$ is upper semi-continuous and $f: X \longrightarrow 2^Y$ is lower semi-continuous at $x_0 \in X$. Then the function $x \longrightarrow \inf\{h(x,y) : y \in f(x)\}$ is upper semi-continuous at x_0.*

LEMMA 4. *Let (E,T) be a Hausdorff topological vector space whose topological dual E^* separates points of E. Let X be a non-empty subset of E, $p \in \mathcal{P}$ and $f: (X,W) \longrightarrow 2^{(E,T)}$ be upper semi-continuous such that $f(x)$ is T-compact for each $x \in X$. Then the function $V: (X,W) \longrightarrow \mathbb{R}$ defined by*

$$V(x) = d_p\big(x, f(x)\big) \text{ for } x \in X$$

is lower semi-continuous, i.e. $V: X \longrightarrow \mathbb{R}$ is weakly lower semi-continuous.

PROOF. Define $h: (X,W) \times (E,T) \longrightarrow \mathbb{R}$ by

$$h(x,y) = p(x-y) \text{ for } (x,y) \in X \times E.$$

For each $r \in \mathbb{R}$, let $A(r) = \{(x,y) \in X \times E : h(x,y) \leq r\}$. Let $\{(x_\alpha, y_\alpha)\}_{\alpha \in \Delta}$ be a net in $A(r)$ and $(x,y) \in X \times E$ such that $x_\alpha \longrightarrow x$ in W-topology and $y_\alpha \longrightarrow y$ in T-topology. By the Corollary of Hahn-Banach Theorem (*e.g.* see [17, Corollary 2, p. 29]), there exists $x^* \in E^*$ such that $x^*(x-y) = p(x-y)$ and $|x^*(z)| \leq p(z)$ for all $z \in E$. Since $x_\alpha - y_\alpha \longrightarrow x - y$ in W-topology,

$$\begin{aligned} h(x,y) &= p(x-y) = x^*(x-y) \\ &= \operatorname{Re} x^*(x-y) \\ &= \lim_\alpha \operatorname{Re} x^*(x_\alpha - y_\alpha) \\ &\leq \liminf_\alpha |x^*(x_\alpha - y_\alpha)| \\ &\leq \liminf_\alpha p(x_\alpha - y_\alpha) \leq r \end{aligned}$$

so that $(x,y) \in A(r)$. Thus $A(r)$ is closed in $(X,W) \times (E,T)$. Thus h is lower semi-continuous on $(X,W) \times (E,T)$. By Lemma 2, the function $x \longrightarrow \inf\{h(x,y) : y \in f(x)\} = d_p\big(x, f(x)\big)$ is lower semi-continuous on (X,W). ∎

We shall need the following useful result:

LEMMA 5. *Let E be a Hausdorff topological vector space whose topological dual E^* separates the points of E. Let A be a non-empty compact convex subset of E and $x \in E$. If for each $f \in E^*$, $\inf_{a \in A} |\operatorname{Re} f(x-a)| = 0$, then $x \in A$. In particular, if $d_p(x,A) = 0$ for each continuous semi-norm p on E, then $x \in A$.*

PROOF. Suppose $x \notin A$. Then for each $a \in A$, as E^* separates the points of E, there exists $f_a \in E^*$ such that $f_a(x) \neq f_a(a)$. Let O_a and U_a be disjoint open convex sets containing $f_a(a)$ and $f_a(x)$ respectively. Then $f_a^{-1}(O_a)$ and $f_a^{-1}(U_a)$ are disjoint open convex sets in E containing a and x respectively. Since A is compact, there exist $a_1, \ldots, a_n \in A$ such that $A \subset \bigcup_{i=1}^n f_{a_i}^{-1}(O_{a_i})$. Let $U = \bigcap_{i=1}^n f_{a_i}^{-1}(U_{a_i})$, then U is an open convex set containing

x such that $U \cap A = \emptyset$. By Theorem 3.4 of Rudin [18, p. 58], there exist $f \in E^*$ and $\gamma \in \mathbb{R}$ such that $\mathrm{Re} f(x) < \gamma \leq \mathrm{Re} f(a)$ for all $a \in A$. It follows that

$$\inf_{a \in A} |\,\mathrm{Re} f(x-a)| \geq \gamma - \mathrm{Re} f(x) > 0,$$

which is a contradiction. Thus we must have $x \in A$. The last assertion follows from the fact that for each $f \in E^*$, the function $p \colon E \longrightarrow \mathbb{R}$ defined by $p(y) = |\,\mathrm{Re} f(y)|$ for all $y \in E$ is a continuous semi-norm on E. ∎

We emphasize here that in the above lemma, E is *not* assumed to be locally convex. We also remark here that even when E is Hausdorff, the conclusion of Lemma 5 is false if E^* does not separate points of E; *e.g.*, the completely metrizable topological vetor space $\mathcal{L}^p$ (where $0 < p < 1$) contains no open convex sets other than $\emptyset$ and $\mathcal{L}^p$ (see, *e.g.* [18, p. 35]) so that $\mathcal{L}^p$ has no non-zero continuous linear functionals and no non-zero continuous semi-norms.

LEMMA 6. *Let (E,T) be a Hausdorff topological vector space whose topological dual E^* separates points of E, X be a non-empty W-compact subset of E and $f \colon (X,W) \longrightarrow 2^{(E,T)}$ be upper semi-continuous such that for each $x \in X$, $f(x)$ is T-compact and convex. If for each $p \in \mathcal{P}$, there exists $x_p \in X$ such that $d_p\big(x_p, f(x_p)\big) = 0$, then f has a fixed point in X.*

PROOF. For each $p \in \mathcal{P}$, the set

$$A(p) = \big\{x \in X : d_p\big(x, f(x)\big) = 0\big\}$$

is non-empty as $x_p \in A(p)$ and also W-closed as $x \longrightarrow d_p\big(x, f(x)\big)$ is W-lower semi-continuous by Lemma 4. If $\{p_1, \ldots, p_n\}$ is an finite subset of $\mathcal{P}$, then $\sum_{i=1}^n p_i \in \mathcal{P}$ and $A\big(\sum_{i=1}^n p_i\big) \subset \bigcap_{i=1}^n A(p_i)$. Thus $\{A(p) : p \in \mathcal{P}\}$ has the finite intersection property. By weak compactness of X, $\bigcap_{p \in \mathcal{P}} A(p) \neq \emptyset$. Take any $\hat{u} \in \bigcap_{p \in \mathcal{P}} A(p)$, then $d_p\big(\hat{u}, f(\hat{u})\big) = 0$ for all $p \in \mathcal{P}$. Since $f(\hat{u})$ is T-compact and convex, by Lemma 5, $\hat{u} \in f(\hat{u})$. ∎

Let (E,T) be a topological vector space and X be a non-empty subset of E. It is clear that if $f \colon (X,W) \longrightarrow 2^{(E,T)}$ is upper semi-continuous (respectively, lower semi-continuous, continuous), then $f \colon (X,T) \longrightarrow 2^{(E,T)}$ is upper semi-continuous (respectively, lower semi-continuous, continuous). The following result shows that the converse also holds under additional conditions on E and on X:

LEMMA 7. *Let (E,T) be a Hausdorff topological vector space whose topological dual E^* separates points of E and X be a non-empty T-compact subset of E. If $f \colon (X,T) \longrightarrow 2^{(E,T)}$ is upper semi-continuous (respectively, lower semi-continuous, continuous), then $f \colon (X,W) \longrightarrow 2^{(E,T)}$ is upper semi-continuous (respectively, lower semi-continuous, continuous).*

PROOF. Suppose $f \colon (X,T) \longrightarrow 2^{(E,T)}$ is upper semi-continuous (respectively, lower semi-continuous). Let U be any T-open set in E. Then the set $A = \{x \in X : f(x) \not\subset U\}$ (respectively, $A = \{x \in X : f(x) \cap U = \emptyset\}$) is T-closed in X and hence A is T-compact since X is T-compact. Thus A is W-compact. Since E^* separates points of E, W

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is Hausdorff so that A is W-closed. Therefore $f\colon (X, W) \longrightarrow 2^{(E,T)}$ is upper semi-continuous (respectively, lower semi-continuous). ■

The following general minimax inequality due to Ding and Tan [3, Theorem 1] will be needed to prove our main result. For completeness we shall provide its proof.

LEMMA 8. *Let X be a non-empty convex subset of a topological vector space and $g\colon X \times X \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ be such that*

(i) for each fixed $x \in X$, $g(x, y)$ is a lower semi-continuous function of y on C for each non-empty compact subset C of X;

(ii) for each $A \in \mathcal{F}(X)$ and for each $y \in \mathrm{co}(A)$, $\min_{x\in A} g(x, y) \leq 0$;

(iii) there exist a non-empty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x \in \mathrm{co}(X_0 \cup \{y\})$ with $g(x, y) > 0$.

Then there exists $\hat{y} \in K$ such that $g(x, \hat{y}) \leq 0$ for all $x \in X$.

PROOF. For each $x \in X$, let

$$K(x) = \{y \in K : g(x, y) \leq 0\}.$$

By (i), $K(x)$ is closed in K for each $x \in X$. We claim that the family $\{K(x) : x \in X\}$ has the finite intersection property. For any fixed $\{x_1, \ldots, x_n\} \in \mathcal{F}(x)$, let

$$D = \mathrm{co}(X_0 \cup \{x_1, \ldots, x_n\}),$$

then D is a compact convex subset of X. Define $G\colon D \longrightarrow 2^D$ by

$$G(x) = \{y \in D : g(x, y) \leq 0\}.$$

By hypotheses, it is easy to check that all hypotheses of Fan's Lemma 1 [4] (note that "Hausdorff" was never needed in its proof) are satisfied. Thus $\bigcap_{x\in X} G(x) \neq \emptyset$; that is there exists $\bar{y} \in D$ such that $g(x, \bar{y}) \leq 0$ for all $x \in D$. By (iii), we must have $\bar{y} \in K$ so that $\bar{y} \in \bigcap_{i=1}^{n} K(x_i)$. This shows that $\{K(x) : x \in X\}$ has the finite intersection property so that by compactness of K, $\bigcap_{x\in X} K(x) \neq \emptyset$. Take any $\hat{y} \in \bigcap_{x\in X} K(x)$, then $\hat{y} \in K$ and $g(x, \hat{y}) \leq 0$ for all $x \in X$. ■

The following result is Theorem 1 of Ha [7].

LEMMA 9. *Let E, F be Hausdorff topological vector spaces, $X \subset E$, $Y \subset F$ be nonempty convex subsets, Y be compact. Let $f\colon X \longrightarrow 2^Y$ be an upper semicontinuous map with nonempty closed and convex values and $g\colon X \times Y \longrightarrow \mathbb{R}$ be such that*

(a) for each $x \in X$, $g(x, y)$ is a lower semi-continuous function of y in Y;

(b) for each $y \in Y$, $g(x, y)$ is a quasi-concave function of x in X.

Then

$$\inf_{y\in Y} \sup_{x\in X} g(x, y) \leq \sup_{\substack{u\in f(x)\\ x\in X}} g(x, u).$$

3. **Approximation of continuous maps.** *In this section, we shall prove several approximation theorems and fixed point theorems for continuous set-valued maps in a Hausdorff topological vector space whose topological dual separates points.*

Let X be a non-empty subset of a topological vector space (E, T). For each $x \in X$, the inward set and outward set of X at x, denoted by $I_X(x)$ and $O_X(x)$ respectively, are defined by

$$I_X(x) = \{x + r(y - x) : y \in X \text{ and } r > 0\},$$
$$O_X(x) = \{x - r(y - x) : y \in X \text{ and } r > 0\}.$$

The closures of $I_X(x)$ and $O_X(x)$ in (E, T), denoted by $\overline{I_X}(x)$ and $\overline{O_X}(x)$ respectively, are called the *weakly inward set* and the *weakly outward set* of X at x respectively. We shall use $Q(x)$ to denote either $\overline{I_X}(x)$ or $\overline{O_X}(x)$.

THEOREM 1. *Let (E, T) be a Hausdorff topological vector space whose topological dual E^* separates points of E, X be a non-empty convex subset of E and $f: (X, W) \longrightarrow 2^{(E,T)}$ be continuous on C for each non-empty W-compact subset C of X such that for each $x \in X$, $f(x)$ is T-compact and convex. Let X_0 be a non-empty W-compact and convex subset of X and K be a non-empty W-compact subset of X. If $p \in \mathcal{P}$ has the following property: "for each $y \in X \setminus K$, there exists $x \in \mathrm{co}(X_0 \cup \{y\})$ such that $d_p\big(x, f(y)\big) < d_p\big(y, f(y)\big)$", then there exists $u \in K$ such that*

$$d_p\big(u, f(u)\big) = \min\big\{d_p\big(x, f(u)\big) : x \in \overline{I_X}(u)\big\}.$$

Moreover, $u \in K \cap \partial X$ whenever $d_p\big(u, f(u)\big) > 0$.

PROOF. Define $g: X \times X \longrightarrow \mathbb{R}$ by

$$g(x, y) = d_p\big(y, f(y)\big) - d_p\big(x, f(y)\big).$$

Then we have

(a) For each fixed $x \in X$, by Lemma 3, $y \longrightarrow d_p\big(x, f(y)\big)$ is W-upper semi-continuous on C for each non-empty W-compact subset C of X (where $h(t, u) = p(x - u)$ for all $(t, u) \in X \times X$ in applying Lemma 3) so that together with Lemma 4, $g(x, y)$ is a W-lower semi-continuous function of y on C for each non-empty W-compact subset C of X.

(b) For each $A \in \mathcal{F}(X)$ and for each $y \in \mathrm{co}(A)$, we must have $\min_{x \in A} g(x, y) \le 0$; if this were not true, then there exist $A = \{x_1, \ldots, x_n\} \in \mathcal{F}(X)$ and $y = \sum_{i=1}^n \lambda_i x_i \in \mathrm{co}(A)$ with $\lambda_1, \ldots, \lambda_n \ge 0$ and $\sum_{i=1}^n \lambda_i = 1$ such that

$$(*) \qquad g(x_i, y) = d_p\big(y, f(y)\big) - d_p\big(x_i, f(y)\big) > 0 \text{ for all } i = 1, \ldots, n.$$

Since $f(y)$ is T-compact, for each $i = 1, \ldots, n$, there exists $u_i \in f(y)$ such that $p(x_i - u_i) =$

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$d_p\big(x_i, f(y)\big)$. Let $u = \sum_{i=1}^n \lambda_i u_i$; then $u \in f(y)$ since $f(y)$ is also convex. It follows that

$$\begin{aligned} d_p\big(y, f(y)\big) &\leq p(y-u) \\ &= p\Big(\sum_{i=1}^n \lambda_i (x_i - u_i)\Big) \\ &\leq \sum_{i=1}^n \lambda_i p(x_i - u_i) \\ &= \sum_{i=1}^n \lambda_i d_p\big(x_i, f(y)\big) \\ &< d_p\big(y, f(y)\big) \text{ by } (*) \end{aligned}$$

which is impossible.

(c) By hypothesis, there exist a non-empty W-compact and convex subset X_0 of X and a non-empty W-compact subset K of X such that for each $y \in X \setminus K$, there exists $x \in \operatorname{co}(X_0 \cup \{y\})$ with $g(x,y) > 0$.

Now equip E with the weak topology W, then all hypotheses of Lemma 8 are satisfied so that there exists $u \in K$ such that $g(x,u) \leq 0$ for all $x \in X$; that is,

$$(1) \qquad d_p\big(u, f(u)\big) \leq d_p\big(x, f(u)\big) \text{ for all } x \in X.$$

Now fix an arbitrary $v \in I_X(u) \setminus X$; as X is convex, there exist $z \in X$ and $r > 1$ such that $v = u + r(z-u)$. Suppose that

$$(2) \qquad d_p\big(v, f(u)\big) < d_p\big(u, f(u)\big).$$

Since $f(u)$ is T-compact, there exist $z_1, z_2 \in f(u)$ such that $p(u - z_1) = d_p\big(u, f(u)\big)$ and $p(v - z_2) = d_p\big(v, f(u)\big)$. Let $\bar{z} = (1 - 1/r)z_1 + (1/r)z_2$, then $\bar{z} \in f(u)$ since $f(u)$ is also convex. Since $z = (1-1/r)u + (1/r)v \in X$, we have

$$\begin{aligned} d_p\big(z, f(u)\big) &\leq p(z - \bar{z}) \\ &= p\big((1-1/r)(u - z_1) + (1/r)(v - z_2)\big) \\ &\leq (1-1/r)p(u - z_1) + (1/r)p(v - z_2) \\ &= (1-1/r)d_p\big(u, f(u)\big) + (1/r)d_p\big(v, f(u)\big) \\ &< d_p\big(u, f(u)\big) \text{ by } (2) \end{aligned}$$

which contradicts (1). Thus we must have

$$d_p\big(u, f(u)\big) \leq d_p\big(x, f(u)\big) \text{ for all } x \in I_X(u).$$

By the continuity of p, we have

$$d_p\big(u, f(u)\big) \leq d_p\big(x, f(u)\big) \text{ for all } x \in \overline{I_X}(u).$$

Hence $d_p\big(u, f(u)\big) = \min\big\{d_p\big(x, f(u)\big) : x \in \overline{I_X}(u)\big\}$.

Now assume $d_p\big(u,f(u)\big) > 0$, then $f(u)\cap X = \emptyset$. Since $f(u)$ is T-compact, there exists $\bar{u} \in f(u)$ such that $p(u-\bar{u}) = d_p\big(u,f(u)\big)$. Note that $\bar{u} \notin X$. If $u \in \operatorname{int} X$, then there exists a real number λ with $0 < \lambda < 1$ such that $z = \lambda u + (1-\lambda)\bar{u} \in X$. It follows that

$$\begin{aligned} 0 < p(u-\bar{u}) &= d_p\big(u,f(u)\big) \\ &\le d_p\big(z,f(u)\big) \le p(z-\bar{u}) \\ &= \lambda p(u-\bar{u}) < p(u-\bar{u}) \end{aligned}$$

which is impossible. Thus $u \notin \operatorname{int} X$ so that $u \in K \cap \partial X$. This completes the proof. ∎

THEOREM 2. *Let (E,T), E^*, X, f, X_0 and K be given as in Theorem 1 such that for each $p \in \mathcal{P}$, the following property holds: for each $y \in X\backslash K$, there exists $x \in \operatorname{co}(X_0 \cup \{y\})$ such that $d_p\big(x,f(y)\big) < d_p\big(y,f(y)\big)$. Then either (a) f has a fixed point in K or (b) there exist $p \in \mathcal{P}$ and $u \in K \cap \partial X$ such that $0 < d_p\big(u,f(u)\big) = \min\{d_p\big(x,f(u)\big) : x \in \overline{I_X}(u)\}$.*

PROOF. By Theorem 1 and Lemma 6, the conclusion follows. ∎

Theorem 1 generalizes Theorem 1 of Sehgal, Singh and Smithson [22] in several ways: (1) f is multi-valued; (2) continuity of f is weakened; (3) the space E need not be locally convex; (4) our non-compactness condition is weaker than that of (1) in [22]; (5) our conclusion is strengthened. In view of Lemma 7, Theorem 2 generalizes Fan's result in many ways.

COROLLARY 1. *Let (E,T) be a Hausdorff topological vector space whose topological dual E^* separates points of E, X be a non-empty weakly compact convex subset of E and $f\colon (X,W) \longrightarrow 2^{(E,T)}$ be continuous such that for each $x \in X$, $f(x)$ is T-compact and convex. Then either (a) f has a fixed point in X or (b) there exist $p \in \mathcal{P}$ and $u \in \partial X$ such that*

$$0 < d_p\big(u,f(u)\big) = \min\{d_p\big(x,f(u)\big) : x \in Q(u)\}.$$

PROOF. By taking $X_0 = K = X$, Theorem 2 proves the case when $Q(u) = \overline{I_X}(u)$.

Next for each $x \in X$, let $g(x) = 2x - f(x)$; then $g\colon (X,W) \longrightarrow 2^{(E,T)}$ is also continuous (*e.g.* see Propositions 1 and 2 in [23]) such that for each $x \in X$, $g(x)$ is T-compact and convex. Again by applying Theorem 2 with $X_0 = K = X$, either (a) g has a fixed point in X and hence f has a fixed point in X as f and g have the same fixed points in X or (b) there exist $p \in \mathcal{P}$ and $v \in \partial X$ such that

$$0 < d_p\big(v,g(v)\big) = \min\{d_p\big(x,g(v)\big) : x \in \overline{I_X}(v)\}.$$

For each $z \in O_X(v)$, let $z' = 2v - z$, then $z' \in I_X(v)$ so that

$$\begin{aligned} d_p\big(v,f(v)\big) &= d_p\big(v,g(v)\big) \\ &\le d_p\big(z',g(v)\big) \\ &= d_p\big(z,f(v)\big) \end{aligned}$$

so that by continuity of p, $d_p\big(v,f(v)\big) \le d_p\big(z,f(v)\big)$ for all $z \in \overline{O_X}(v)$. Thus $0 < d_p\big(v,f(v)\big) = \min\big\{d_p\big(z,f(v)\big) : z \in \overline{O_X}(v)\big\}$ which proves the case when $Q(v) = \overline{O_X}(v)$. This completes the proof. ∎

In view of Lemma 7, Corollary 1 generalizes Theorem 3 of Park [12] as follows: (1) the set X is weakly compact instead of compact; (2) the conclusion is strengthened: $u \in \partial X$ instead of $u \in X$. Corollary 1 also generalizes Lemma 1.6 of Reich [14] in several aspects.

THEOREM 3. *Under the same hypotheses as in Theorem 2, if for each $x \in K \cap \partial X$, there exists a real number λ such that $0 \le \lambda < 1$ and $\big(\lambda x + (1-\lambda)f(x)\big) \cap \overline{I_X}(x) \neq \emptyset$, then f has a fixed point in K.*

PROOF. Suppose f has no fixed point in K. Then by Theorem 2, there exist $p \in \mathcal{P}$ and $u \in K \cap \partial X$ such that

$$0 < d_p\big(u,f(u)\big) = \min\big\{d_p\big(x,f(u)\big) : x \in \overline{I_X}(u)\big\}.$$

Since $u \in K \cap \partial X$, by assumption, there exists λ with $0 \le \lambda < 1$ and $v \in f(u)$ such that $z = \lambda u + (1-\lambda)v \in \overline{I_X}(u)$. Thus

$$\begin{aligned} 0 &< d_p\big(u,f(u)\big) \le d_p\big(z,f(u)\big) \\ &\le \lambda d_p\big(u,f(u)\big) \quad \text{(by Lemma 1)} \\ &< d_p\big(u,f(u)\big) \end{aligned}$$

which is a contradiction. Therefore f must have a fixed point in K. ∎

COROLLARY 2. *Under the same hypotheses as in Corollary 1, if for each $x \in \partial X$, there exists a real number λ such that $0 \le \lambda < 1$ and $\big(\lambda x + (1-\lambda)f(x)\big) \cap \overline{I_X}(x) \neq \emptyset$, then f has a fixed point in X.*

PROOF. By applying Theorem 3 with $X_0 = K = X$, the result follows. ∎

4. **Approximation of upper semi-continuous maps.** In this section, we shall prove some approximation theorems for upper semi-continuous set-valued maps in locally convex topological vector spaces.

THEOREM 4. *Let (E,T) be a locally convex Hausdorff topological vector space and X be a non-empty W-compact convex subset of E. Suppose that $f\colon (X,W) \longrightarrow 2^{(E,T)}$ is an upper semi-continuous map with non-empty T-compact convex values. Then either f has a fixed point in X or there exist $x_0 \in X$, $u_0 \in f(x_0)$ and a weakly continuous semi-norm p on E such that*

$$0 < p(x_0 - u_0) \le p(x - u_0) \text{ for all } x \in \overline{I_X}(x_0)$$

(respectively, $0 < p(x_0 - u_0) \le p(x - u_0)$ for all $x \in \overline{O_X}(x_0)$).

PROOF. Suppose that f has no fixed point in X, then for each $x \in X$, $\theta \notin x - f(x)$ and $x - f(x)$ is a T-compact convex subset of E. By Theorem 3.4 of Rudin [18, p. 58], there exist $\delta_x > 0$ and a linear functional $p_x \in E^*$ such that

$$d_{p_x}\big(x,f(x)\big) = \inf_{u\in f(x)} |p_x(x-u)| > \delta_x.$$

By Lemma 4, there exists a W-neighborhood $N(x)$ of x in X such that

$$d_{p_x}\big(z,f(z)\big) > \delta_x \text{ for all } z \in N(x).$$

Since $X = \bigcup_{x\in X} N(x)$ and X is W-compact, there exists $\{x_1,\ldots,x_n\} \subset X$ such that $X = \bigcup_{i=1}^{n} N(x_i)$. Let $p = \max\{|p_{x_i}| : i = 1,2,\ldots n\}$ and $\delta = \min\{\delta_{x_i} : 1 \le i \le n\}$. Then p is a weakly continuous semi-norm on E. For each $x \in X$, there exists x_{i_0} such that $x \in N(x_{i_0})$; it follows that for each $u \in f(x)$,

$$p(x-u) = \max_{1\le i\le n} |p_{x_i}(x-u)| \ge |p_{x_0}(x-u)| \ge d_{p_{x_0}}\big(x,f(x)\big)$$

so that

$$d_p\big(x,f(x)\big) = \inf\{p(x-u) : u \in f(x)\} \ge d_{p_{x_{i_0}}}\big(x,f(x)\big) > \delta_{x_{i_0}} \ge \delta.$$

Hence we have

$$d_p\big(x,f(x)\big) > \delta \text{ for all } x \in X.$$

Now define the function $g\colon (X,W) \times (E,T) \longrightarrow \mathbb{R}$ by

$$g(x,y) = \min_{u\in X} p(u-y) - p(x-y).$$

It is easy to see that $g\colon (X,W) \times (E,T) \longrightarrow \mathbb{R}$ is continuous. Thus the condition (a) of Lemma 9 is satisfied. Clearly the condition (b) of Lemma 9 is also satisfied. By Corollary 9.6 in [11], the image $f(X) = \bigcup_{x\in X} f(x)$ of f is T-compact. As the assumptions on X, f and the graph $\mathrm{Gr}(f)$ of f remain unchanged in the completion of E, without loss of generality we may assume that E is complete. Let $Y = \overline{\mathrm{co}}\big(f(X)\big)$, then $Y \subset E$ is T-compact convex. By applying Lemma 9, we have

$$\inf_{y\in Y} \sup_{x\in X} g(x,y) \le \sup_{\substack{u\in f(x)\\ x\in X}} g(x,u).$$

Since for each $y \in Y$,

$$\sup_{x\in X} g(x,y) = \sup_{x\in X}\Big[\min_{u\in X} p(u-y) - p(x-y)\Big] = \min_{u\in X} p(u-y) - \min_{x\in X} p(x-y) = 0,$$

we have

$$\sup_{\substack{u\in f(x)\\ x\in X}} g(x,u) \ge 0.$$

Since $g\colon (X,W)\times(E,T) \longrightarrow \mathbb{R}$ is continuous and the graph $\mathrm{Gr}(f)$ of f is compact in $(X,W)\times(E,T)$, there exists $(x_0,u_0) \in \mathrm{Gr}(f)$ such that

$$g(x_0,u_0) = \min_{x\in X} p(x-u_0) - p(x_0-u_0) \ge 0.$$

It follows that $u_0 \in f(x_0)$ and

$$(*) \qquad \delta < d_p\big(x_0, f(x_0)\big) \leq p(x_0 - u_0) \leq p(x - u_0) \text{ for all } x \in X.$$

For $x \in I_X(x_0) \setminus X$, there exist $v \in X$ and $r > 1$ such that $x = x_0 + r(v - x_0)$. Suppose that $p(x - u_0) < p(x_0 - u_0)$. Since $v = \frac{1}{r}x + (1 - \frac{1}{r})x_0 \in X$, we have by $(*)$

$$\begin{aligned} p(x_0 - u_0) \leq p(v - u_0) &= p\Big(\frac{1}{r}x + \Big(1 - \frac{1}{r}\Big)x_0 - u_0\Big) \\ &\leq \frac{1}{r}p(x - u_0) + \Big(1 - \frac{1}{r}\Big)p(x_0 - u_0) < p(x_0 - u_0) \end{aligned}$$

which is a contradiction. Therefore we must have

$$p(x_0 - u_0) \leq p(x - u_0) \text{ for all } x \in I_X(x_0).$$

Since p is continuous, we have

$$0 < p(x_0 - u_0) \leq p(x - u_0) \text{ for all } x \in \overline{I_X}(x_0).$$

For the outward case, define the map $f'\colon (X, W) \longrightarrow 2^{(E,T)}$ by

$$f'(x) = 2x - f(x) \text{ for each } x \in X.$$

Then f' is an upper semi-continuous map (*e.g.* see [23, Proposition 2.2]) with non-empty T-compact convex values. Note that x is a fixed point of f' if and only if x is a fixed point of f. Therefore either f' and hence f has a fixed point in X or, by the above argument, there exist $x_0' \in X$, $u_0' \in f'(x_0')$ and a weakly continuous semi-norm p' on E such that

$$0 < p'(x_0' - u_0') \leq p'(x' - u_0') \text{ for all } x' \in I_X(x_0').$$

For each fixed $x \in O_X(x_0')$, let $x' = 2x_0' - x$ and $u_0 = 2x_0' - u_0'$, then $u_0 \in f(x_0')$ and $x' \in I_X(x_0')$ so that

$$0 < p'(x_0' - u_0) = p'(u_0' - x_0') \leq p'(x' - u_0') = p'(x - u_0).$$

By continuity of p',

$$0 < p'(x_0' - u_0) \leq p'(x_0 - u_0) \text{ for all } x \in \overline{O_X}(x_0'). \qquad \blacksquare$$

Theorem 4 improves and generalizes Theorem 3 of Ha [7], Theorem 3 of Park [13], which in turn generalizes Corollaries 1 and $1'$ of Browder [2], Theorem 3.1 of Reich [15] and Theorem 1 of Fan [5].

As an equivalent version of Theorem 4, we have

THEOREM 5. *Let (E, T) be a locally convex Hausdorff topological vector space and X be a non-empty W-compact convex subset of E. Suppose that $f\colon (X, W) \longrightarrow 2^{(E,T)}$ is an upper semi-continuous map with non-empty T-compact convex values. If for each weakly continuous semi-norm p on E, each $x \in X$ with $d_p\big(x, f(x)\big) > 0$ and each $u \in f(x)$,*

$$d_p\big(u, \overline{I_X}(x)\big) < p(x - u) \quad (\textit{respectively, } d_p\big(u, \overline{O_X}(x)\big) < p(x - u))$$

then f has a fixed point in X.

Theorem 5 improves and generalizes Theorem 2 of Reich [16].

As a direct consequence of Theorem 4, we have the following result:

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THEOREM 6. *Let (E, T) be a locally convex Hausdorff topological vector space, X be a non-empty W-compact convex subset of E and $f\colon (X, W) \longrightarrow 2^{(E,T)}$ be an upper semi-continuous map with non-empty T-compact convex values. Suppose that the following condition holds:*

(a) for each $x \in \partial_{(E,W)}X\setminus f(x)$ and $u \in f(x)$, there exists a number λ (real or complex, depending on whether the vector space E is real or complex) with $|\lambda| < 1$ such that $\lambda x + (1-\lambda)u \in \overline{I_X}(x)$ (respectively, $\lambda x + (1-\lambda)u \in \overline{O_X}(x)$).

Then f has a fixed point. (Here, $\partial_{(E,W)}X$ denotes the boundary of X in (E, W).)

PROOF. Suppose f has no fixed point. By Theorem 4, there exist $x_0 \in X$, $u_0 \in f(x_0)$ and a weakly continuous semi-norm p on E such that

$$0 < p(x_0 - u_0) \leq p(x - u_0) \text{ for all } x \in \overline{I_X}(x_0)$$
$$(\text{respectively, } 0 < p(x_0 - u_0) \leq p(x - u_0) \text{ for all } x \in \overline{O_X}(x_0)).$$

CASE 1. If $x_0 \in \operatorname{int}_{(E,W)} X$, then $I_X(x_0) = E$ (respectively, $O_X(x_0) = E$) and hence $x = \frac{1}{2}x_0 + \frac{1}{2}u_0 \in E = I_X(x_0)$ (respectively, $x = \frac{1}{2}x_0 + \frac{1}{2}u_0 \in O_X(x_0)$) so that

$$0 < p(x_0 - u_0) \leq p(x - u_0) = \frac{1}{2}p(x_0 - u_0)$$

which is a contradiction.

CASE 2. If $x_0 \in \partial_{(E,W)}X$, then $x_0 \in \partial_{(E,W)}X \setminus f(x_0)$ so that by (a), there exists λ with $|\lambda| < 1$ such that $x = \lambda x_0 + (1-\lambda)u_0 \in \overline{I_X}(x_0)$ (respectively, $x = \lambda x_0 + (1-\lambda)u_0 \in \overline{O_X}(x_0)$). It follows that

$$0 < p(x_0 - u_0) \leq p(x - u_0) \leq |\lambda| p(x_0 - u_0)$$

which is again a contradiction.

Therefore f must have a fixed point. ∎

Theorem 6 improves and generalizes Theorem 4 of Ha [7], Theorem 3.1 of Reich [15], Theorem 4 of Park [13] and Theorem 3 of Fan [5].

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Department of Mathematics
Sichuan Normal University
Chengdu, Sichuan
China

Department of Mathematics, Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia
Canada B3H 3J5