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A MINIMAL REAL HYPERSURFACE OF A COMPLEX PROJECTIVE SPACE WITH NONNEGATIVE SECTIONAL CURVATURE

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Abstract

We give a characterization of a minimal real hypersurface with respect to the condition for the sectional curvature.

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1. Introduction

It is an interesting problem to study real hypersurfaces immersed in a complex projective space with additional conditions for the sectional curvature.

Let CP^n be a complex n-dimensional complex projective space of holomorphic sectional curvature 4. We denote by $\pi : S^{2n+1} \longrightarrow CP^n$ the standard fibration, where S^k is the k-dimensional unit sphere. In S^{2n+1} of curvature 1, we have the family of generalized Clifford surfaces whose fibres lie in complex subspaces:

$$M_{2p+1,2q+1} = S^{2p+1}\left(\sqrt{\frac{2p+1}{2n}}\right) \times S^{2q+1}\left(\sqrt{\frac{2q+1}{2n}}\right),$$

where $p + q = n - 1$. Then $M^C_{p,q} = \pi(M_{2p+1,2q+1})$ are connected compact minimal real hypersurfaces in CP^n (see Lawson [2]).

In [1], Kon proved that if the sectional curvature K of a compact minimal real hypersurface M in a complex projective space CP^n satisfies $K \geq 1/(2n-1)$, then M is a geodesic minimal hypersphere

$$M^C_{0,n-1} = \pi(S^1(\sqrt{1/2n}) \times S^{2n-1}(\sqrt{(2n-1)/2n})).$$

In this paper, we give a characterization for a minimal real hypersurface

$$M^C_{(n-1)/2,(n-1)/2} = \pi(S^n(\sqrt{1/2}) \times S^n(\sqrt{1/2}))$$

with respect to the condition for the sectional curvature. We prove the following theorem.

THEOREM 1.1. *Let M be a connected complete real minimal hypersurface of CP^n. If the sectional curvature K of M satisfies*

$$K(X, Y) \geq \eta(X)^2 + \eta(Y)^2$$

for any orthogonal unit tangent vectors X and Y, then M is congruent to $M^C_{(n-1)/2,(n-1)/2}$.

2. Preliminaries

Let CP^n denote the complex space form of complex dimension n (real dimension $2n$) with constant holomorphic sectional curvature 4. We denote by J the complex structure of CP^n. The Hermitian metric of CP^n will be denoted by G.

Let M be a real $(2n-1)$-dimensional real hypersurface immersed in CP^n. We denote by g the Riemannian metric induced on M by G. We take the unit normal vector field N of M in CP^n. For any vector field X tangent to M, we define ϕ, η and ξ by

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where ϕX is the tangential part of JX, ϕ is a tensor field of type (1, 1), η is a 1-form, and ξ is the unit vector field on M. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi,$$
$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi),$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Thus (ϕ, ξ, η, g) defines an almost contact metric structure on M (see [6]).

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $M^n(c)$, and by ∇ the one in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M. We call A the *shape operator* of M.

For the almost contact metric structure on M,

$$\nabla_X \xi = \phi AX, \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

We denote by R the Riemannian curvature tensor field of M. Then the *Gauss equation* is given by

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

and the *Codazzi equation* by

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi.$$

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EXAMPLE 2.1. $M^C_{p,q}$ is a connected compact real hypersurface in CP^n with three constant principal curvatures $\cot\theta$, $-\tan\theta$ and $2\cot 2\theta$ with multiplicities $2p$, $2q$ and 1, respectively. Moreover, the structure vector field ξ of $M^C_{p,q}$ is a principal curvature vector field, that is, $A\xi = 2\cot 2\theta\xi$. In particular, if a real hypersurface $M^C_{p,q}$ is minimal and the shape operator A satisfies $A\xi = 0$, then the real hypersurface turns out to be $M^C_{(n-1)/2,(n-1)/2}$ whose constant principal curvatures 1, -1 and 0 have multiplicities $n-1$, $n-1$ and 1, respectively (see Takagi [3]).

3. Proof of the theorem

First, we prove the following proposition.

PROPOSITION 3.1. *Let M be a real hypersurface of CP^n. If $A\xi = 0$, then the sectional curvature K of M is determined by $K(Z, W)$, where Z, W are orthogonal to ξ.*

PROOF. Let $\{X, Y\}$ be an orthonormal pair. We put

$$X = \eta(X)\xi + aZ, \qquad Y = \eta(Y)\xi + bW,$$

where $a = (1 - \eta(X)^2)^{1/2}$, $b = (1 - \eta(Y)^2)^{1/2}$, Z and W being orthogonal to ξ. Then Z and W are unit vectors that satisfy

$$g(Z, W) = -\frac{1}{ab}\eta(X)\eta(Y).$$

Since $A\xi = 0$, simple calculation shows that

$$\begin{aligned} K(X, Y) &= g(R(X, Y)Y, X) \\ &= g(X, X)g(Y, Y) + 3g(\phi X, Y)^2 + g(AX, X)g(AY, Y) - g(AX, Y)^2 \\ &= a^2\eta(Y)^2 + b^2\eta(X)^2 + 2\eta^2(X)\eta^2(Y) + a^2b^2 g(R(Z, W)W, Z). \end{aligned}$$

Noticing that

$$\begin{aligned} g(R(Z, W)W, Z) &= (1 - g(Z, W)^2)K(Z, W) \\ &= \left(1 - \frac{1}{a^2b^2}\eta(X)^2\eta(Y)^2\right)K(Z, W), \end{aligned}$$

we obtain

$$K(X, Y) = \eta(X)^2 + \eta(Y)^2 + (1 - \eta(X)^2 - \eta(Y)^2)K(Z, W). \qquad (*)$$

Therefore, $K(Z, W)$ determines $K(X, Y)$. □

REMARK 3.2. Generally, for a Sasakian manifold, Equation $(*)$ is always satisfied (see [6, p. 280]). For a real hypersurface M of CP^n we see that M is a Sasakian manifold if and only if $AX = X - \eta(X)\xi$. Then $A\xi = 0$.

On the other hand, there exists a real hypersurface M of CP^n which satisfies Equation $(*)$ and is not a Sasakian manifold. For example, $M^C_{(n-1)/2,(n-1)/2}$ satisfies Equation $(*)$ and is not Sasakian. We notice that $M^C_{(n-1)/2,(n-1)/2}$ satisfies $A\xi = 0$

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and $K(Z, W) \geq 0$ for any Z and W orthogonal to ξ. From Proposition 3.1, if $A\xi = 0$ and $K(Z, W) \geq 0$ for any Z and W orthogonal to ξ, then the sectional curvature K satisfies $K(X, Y) \geq \eta(X)^2 + \eta(Y)^2$.

Finally, we prove our theorem.

PROOF OF THEOREM 1.1. An orthonormal basis $\{e_1, \ldots, e_{2n-2}, e_{2n-1} = \xi\}$ of $T_x(M)$ can be chosen such that the shape operator A is represented by a matrix,

$$A = \begin{pmatrix} a_1 & \cdots & 0 & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & a_{2n-2} & h_{2n-2} \\ h_1 & \cdots & h_{2n-2} & \alpha \end{pmatrix}$$

where we have put $h_i = g(Ae_i, \xi)$, $i = 1, \ldots, 2n-2$, and $\alpha = g(A\xi, \xi)$. By the Gauss equation and the assumption for the sectional curvature,

$$K(e_i, e_j) = 1 + 3g(\phi e_i, e_j)^2 + a_i a_j \geq 0$$

for $i, j = 1, \ldots, 2n-2$. On the other hand, we obtain

$$K(e_i, \xi) = 1 + a_i\alpha - h_i^2 \geq 1$$

for any $i = 1, \ldots, 2n-2$, from which we have $a_i\alpha \geq h_i^2$. Thus

$$\left(\sum_i a_i\right)\alpha \geq \sum_i h_i^2 \geq 0.$$

Since M is minimal, it follows that $(\sum_i a_i)\alpha = -\alpha^2 \leq 0$. Hence we have $\alpha = 0$ and $h_i = 0$ for $i = 1, \ldots, 2n-2$. So we obtain $A\xi = 0$ and $Ae_i = a_i e_i$. This implies that $g((\nabla_X A)Y, \xi) = -g(A\phi AX, Y)$. Using the Codazzi equation,

$$-2g(\phi X, Y) = g((\nabla_X A)Y, \xi) - g((\nabla_Y A)X, \xi) = -2g(A\phi AX, Y).$$

Thus $A\phi AX = \phi X$ for any vector X orthogonal to ξ. Therefore, if $AX = aX$, then $A\phi X = (1/a)\phi X$.

We now choose an orthonormal basis $\{e_1, \ldots, e_{n-1}, \phi e_1, \ldots, \phi e_{n-1}, \xi\}$ that satisfies

$$Ae_i = a_i e_i, \quad A\phi e_i = (1/a_i)\phi e_i \quad (i = 1, \ldots, n-1).$$

Since M is minimal, there exist i and j such that $a_i a_j < 0$. By the assumption on K,

$$0 \leq K(e_i, \phi e_j) = 1 + \frac{a_i}{a_j} \leq 1, \quad 0 \leq K(e_j, \phi e_i) = 1 + \frac{a_j}{a_i} \leq 1.$$

Hence,

$$-1 \leq \frac{a_i}{a_j} \leq 0, \quad -1 \leq \frac{a_j}{a_i} \leq 0.$$

From these inequalities we see that $a_i^2 = a_j^2 = 1$, and hence $a_i = \pm 1$, $a_j = \mp 1$. Since M is minimal and $A\xi = 0$, M has three principal curvatures 1, -1, 0 with multiplicities $n-1$, $n-1$, 1, respectively. By the theorem of Takagi [4] and Wang [5], we have our assertion. □

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