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SUPERCHARACTERS AND PATTERN SUBGROUPS IN THE UPPER TRIANGULAR GROUPS

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Abstract Let $U_n(q)$ denote the upper triangular group of degree n over the finite field $\mathbb{F}_q$ with q elements. It is known that irreducible constituents of supercharacters partition the set of all irreducible characters $\mathrm{Irr}(U_n(q))$. In this paper we present a correspondence between supercharacters and pattern subgroups of the form $U_k(q) \cap {}^wU_k(q)$, where w is a monomial matrix in $\mathrm{GL}_k(q)$ for some $k < n$.

Keywords: root system; irreducible character; triangular group

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1. Introduction

Let q be a power of a prime p and let $\mathbb{F}_q$ be a field with q elements. The group $U_n(q)$ of all upper triangular $(n \times n)$-matrices over $\mathbb{F}_q$ with all diagonal entries equal to 1 is a Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$. It was conjectured by Higman [**8**] that the number of conjugacy classes of $U_n(q)$ is given by a polynomial in q with integer coefficients. Isaacs [**10**] showed that the degrees of all irreducible characters of $U_n(q)$ are powers of q. Huppert [**9**] proved that character degrees of $U_n(q)$ are precisely of the form $\{q^e : 0 \leqslant e \leqslant \mu(n)\}$, where the upper bound $\mu(n)$ was known to Lehrer [**13**]. Lehrer conjectured that each number $N_{n,e}(q)$ of irreducible characters of $U_n(q)$ of degree q^e is given by a polynomial in q with integer coefficients. Isaacs [**11**] suggested a strengthened form of Lehrer's Conjecture, stating that $N_{n,e}(q)$ is given by a polynomial in $(q-1)$ with non-negative integer coefficients. So, Isaacs's Conjecture implies Higman's and Lehrer's Conjectures.

Many efforts have been made to understand more about $U_n(q)$; see [**1, 3, 5, 7, 10, 11, 14, 15**], among others. Supercharacters arise as tensor products of some elementary characters to give a 'nice' partition of all non-principal irreducible characters of $U_n(q)$ (see [**1, 12**]). Supercharacters have been defined for Sylow p-subgroups of other finite groups of Lie type (see [**2**]), and in general for algebra groups (see [**5**]).

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Here, for $U_n(q)$ we show a natural correspondence between supercharacters and pattern subgroups (Theorem 2.8). To highlight the main idea of construction, we have deferred all of our proofs to § 3.

2. Supercharacters and pattern subgroups

Let $\Sigma = \Sigma_{n-1} = \langle \alpha_1, \dots, \alpha_{n-1} \rangle$ be the root system of $\mathrm{GL}_n(q)$ with respect to the maximal split torus equal to the diagonal group (see [**4**, Chapter 3]). Set $\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \dots + \alpha_j$ for all $0 < i \leqslant j < n$. Denote by Σ^+ the set of all positive roots. The root subgroup $X_{\alpha_{i,j}}$ is the set of all matrices of the form $I_n + c \cdot e_{i,j+1}$, where $I_n =$ the identity $(n \times n)$-matrix, $c \in \mathbb{F}_q$ and $e_{i,j+1}$ is equal to the zero matrix except for a '1' at entry $(i, j+1)$. The upper triangular group $U_n(q)$ is generated by all X_α, where $\alpha \in \Sigma^+$. We write U for $U_n(q)$ if n and q are clear from the context. For convenience when using the root system, we consider the upper triangular group as a tableaux:

$$\begin{pmatrix} 1 & * & * & * & * \\ \cdot & 1 & * & * & * \\ \cdot & \cdot & 1 & * & * \\ \cdot & \cdot & \cdot & 1 & * \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \to$$

α_1	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,4}$
	α_2	$\alpha_{2,3}$	$\alpha_{2,4}$
		α_3	$\alpha_{3,4}$
			α_4

A subset $S \subset \Sigma^+$ is called *closed* if, for each $\alpha, \beta \in S$ such that $\alpha + \beta \in \Sigma^+$, $\alpha + \beta \in S$. A *pattern* subgroup of U is a group generated by all root subgroups X_α, where $\alpha \in S$ a closed positive root subset.

Let G be a group. Set $G^\times = G \setminus \{1\}$. Denote by $\mathrm{Irr}(G)$ the set of all complex irreducible characters of G, and let $\mathrm{Irr}(G)^\times = \mathrm{Irr}(G) \setminus \{1_G\}$. For $H \trianglelefteq G$, let $\mathrm{Irr}(G/H)$ denote the set of all irreducible characters of G with H in the kernel. If $K \leqslant G$ such that $G = H \rtimes K$, then for each character ξ of K we denote the inflation of ξ to G by ξ_G, i.e. ξ_G is the extension of ξ to G with $H \subset \ker(\xi_G)$. Furthermore, for $H \leqslant G$ and $\xi \in \mathrm{Irr}(H)$, we define by $\mathrm{Irr}(G, \xi) = \{\chi \in \mathrm{Irr}(G) \colon (\chi, \xi^G) \neq 0\}$ the irreducible constituent set of ξ^G, and for $\chi \in \mathrm{Irr}(G)$ we denote its restriction to H by $\chi|_H$.

For a field K, let $K^\times$ be its multiplicative group. In the whole paper, we fix a non-trivial linear character $\varphi \colon (\mathbb{F}_q, +) \to \mathbb{C}^\times$. For each $\alpha \in \Sigma^+$ and $s \in \mathbb{F}_q$, the map $\phi_{\alpha,s} \colon X_\alpha \to \mathbb{C}^\times$, $x_\alpha(d) \mapsto \varphi(ds)$ is a linear character of the root subgroup X_α, and all linear characters of X_α arise in this way.

For each $\alpha_{i,j}$, we define

$$\mathrm{arm}(\alpha_{i,j}) = \{\alpha_{i,k} \colon i \leqslant k < j\} \quad \text{and} \quad \mathrm{leg}(\alpha_{i,j}) = \{\alpha_{k,j} \colon i < k \leqslant j\}.$$

If $i = j$, $\alpha_{i,i} = \alpha_i$, then $\mathrm{arm}(\alpha_i)$ and $\mathrm{leg}(\alpha_i)$ are empty. For each $\alpha \in \Sigma^+$, we define the *hook* of α as $h(\alpha) = \mathrm{arm}(\alpha) \cup \mathrm{leg}(\alpha) \cup \{\alpha\}$, the *hook group* of α as $H_\alpha = \langle X_\beta \colon \beta \in h(\alpha) \rangle$, and the *base group* $V_\alpha = \langle X_\beta \colon \beta \in \Sigma^+ \setminus \mathrm{arm}(\alpha) \rangle$. Since $[V_\alpha, V_\alpha] \cap X_\alpha = \{1\}$, for each $s \in \mathbb{F}_q^\times$ there exists a linear $\lambda_{\alpha,s} \in \mathrm{Irr}(V_\alpha)$ such that $\lambda_{\alpha,s}|_{X_\alpha} = \phi_{\alpha,s}$ and $\lambda_{\alpha,s}|_{X_\beta} = 1_{X_\beta}$ for other root subgroups $X_\beta \subset V_\alpha$, $\beta \neq \alpha$. Denote by $\mathrm{Irr}(V_\alpha/[V_\alpha, V_\alpha])^\times$ the set of all these linear characters of V_α.

Lemma 2.1. *$\lambda^U_{\alpha,s}$ is irreducible for all $s \in \mathbb{F}_q^\times$.*

Proof. See [**1**, Lemma 2] or [**12**, Lemma 2.2]. □

We call $\lambda^U_{\alpha,s}$ an *elementary* character of U associated to α. A *basic* set D is a non-empty subset of Σ^+ in which none of the roots are in the same row or column. For each basic set D, define

$$E(D) = \bigoplus_{\alpha \in D} \mathrm{Irr}(V_\alpha/[V_\alpha, V_\alpha])^\times.$$

For each basic set D and $\phi \in E(D)$, we define a *supercharacter*, also known as *basic* character in [**1**],

$$\xi_{D,\phi} = \bigotimes_{\lambda_{\alpha,s} \in \phi} \lambda^U_{\alpha,s}.$$

It turns out that each supercharacter $\xi_{D,\phi}$ is induced from a linear character of a pattern subgroup.

Definition 2.2. We define

$$V_D = \bigcap_{\alpha \in D} V_\alpha \quad \text{and} \quad \lambda_D = \bigotimes_{\lambda_{\alpha,s} \in \phi} \lambda_{\alpha,s}|_{V_D}.$$

Lemma 2.3. *We have $\xi_{D,\phi} = \lambda_D^U$.*

Proof. See [**12**, Lemma 2.5]. □

It is easy to see that V_D is generated by all X_β, where $\beta \in \Sigma^+ \setminus (\bigcup_{\alpha \in D} \mathrm{arm}(\alpha))$, and λ_D is a linear character of V_D. For each basic set D, it can be proven that the diagonal subgroup of $\mathrm{GL}_n(q)$ acts transitively on $E(D)$ by conjugation. So it makes sense when we write λ_D here instead of $\lambda_{D,\phi}$, and it also says that the decomposition of $\xi_{D,\phi}$ is dependent only on D. To know more about supercharacters, see, for example, [**5**, **6**]. Here, we recall the main role of supercharacters as a partition of $\mathrm{Irr}(U)^\times$.

Theorem 2.4. *For each $\chi \in \mathrm{Irr}(U)^\times$, there exist uniquely a basic set D and $\phi \in E(D)$ such that χ is an irreducible constituent of $\xi_{D,\phi}$.*

Proof. See [**1**, Theorem 1] or [**12**, Theorem 2.6]. □

Denote by $\mathrm{Irr}(\xi_{D,\phi})$ the set of all irreducible constituents of $\xi_{D,\phi}$. Here, to prove Higman's Conjecture, it suffices to prove that $|\mathrm{Irr}(\xi_{D,\phi})|$ is a polynomial in q.

Now for each basic set D of size $k = |D|$, we define an associated monomial $(k \times k)$-matrix $w_D \in \mathrm{GL}_k(q)$. First of all, we define two partial orders on Σ^+.

Definition 2.5. We define $<_{\mathrm{r}}$ and $<_{\mathrm{b}}$ on Σ^+ as follows:

(i) $\alpha_{i,j} <_{\mathrm{r}} \alpha_{l,k}$ if $j < k$ (i.e. to the right);

(ii) $\alpha_{i,j} <_{\mathrm{b}} \alpha_{l,k}$ if $i < l$ (i.e. to the bottom).

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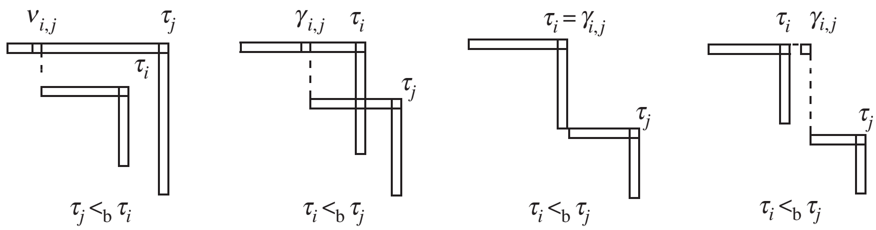


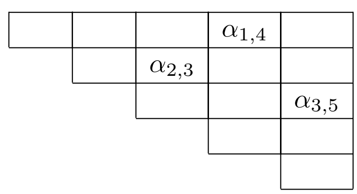
Figure 1. Positions of $\nu_{i,j}$ and $\gamma_{i,j}$.

An easy way to understand these two orders is $<_{\mathrm{r}}$ standing for left to right and $<_{\mathrm{b}}$ for top to bottom. It is noted that, on a basic set, $<_{\mathrm{r}}$ and $<_{\mathrm{b}}$ are total orders.

Now we fix a basic set D of size k ascending order of $<_{\mathrm{r}}$. Let $D = \{\tau_1, \dots, \tau_k\}$, where $\tau_i <_{\mathrm{r}} \tau_j$ if $i < j$. We define $w_D = (a_{i,j}) \in \mathrm{GL}_k(q)$ as follows:

$$a_{i,j} = \begin{cases} 1 & \text{if } \tau_j \text{ is the } i\text{th element of } D \text{ in ascending order } <_{\mathrm{b}}, \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $D = \{\alpha_{2,3}, \alpha_{1,4}, \alpha_{3,5}\}$, $|D| = 3$,



then

$$w_D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is clear that w_D is a monomial matrix in the Weyl group S_k of $\mathrm{GL}_k(q)$. Here, w_D somehow gives pivots of D by considering only rows and columns containing roots in D. Hence, it is equivalent to applying the (total) orders $<_{\mathrm{r}}$, $<_{\mathrm{b}}$ to these monomial matrices on their non-zero entries.

For each pair $0 < i < j \leqslant k$, if $\tau_i <_{\mathrm{b}} \tau_j$, let $\gamma_{i,j}$ be the root on the row of τ_i such that $\gamma_{i,j} + \tau_j \in \Sigma^+$; otherwise, i.e. $\tau_j <_{\mathrm{b}} \tau_i$, let $\nu_{i,j}$ be the root on the row of τ_j such that $\nu_{i,j} + \tau_i \in \Sigma^+$. For example, $\tau_i = \alpha_{m,i}$, $\tau_j = \alpha_{l,j}$, where $i < j$, so if $\alpha_{m,i} <_{\mathrm{b}} \alpha_{l,j}$, i.e. $m < l$, then $\gamma_{i,j} = \alpha_{m,l-1}$; otherwise, if $\alpha_{l,j} <_{\mathrm{b}} \alpha_{m,i}$, i.e. $l < m$, then $\nu_{i,j} = \alpha_{l,m-1}$. It is easy to see that $\nu_{i,j}$ exists if and only if two hooks $h(\tau_i)$ and $h(\tau_j)$ are parallel; otherwise, $\gamma_{i,j}$ exists (Figure 1).

Let Γ_D be the set of all $\gamma_{i,j}$, let Λ_D be the set of all $\nu_{i,j}$ and let $\Delta_D = \Gamma_D \cup \Lambda_D$. Hence, by the definitions for the existence of $\gamma_{i,j}$ and $\nu_{i,j}$, $\Gamma_D \cap \Lambda_D = \emptyset$.

Definition 2.6. We define $R_D = \langle X_\gamma \colon \gamma \in \Gamma_D \rangle$ and $C_D = \langle X_\nu \colon \nu \in \Lambda_D \rangle$.

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The next lemma provides interesting correspondences between the size of D and Δ_D, and between w_D and Γ_D or Λ_D. Moreover, it shows that $\langle V_D, R_D\rangle = V_D R_D$, and the pattern subgroups R_D, C_D are only determined by w_D in a natural way.

Lemma 2.7. *Let D be a basic set of size k. The following are true.*

(i) *Δ_D is closed and $\langle X_\alpha \colon \alpha \in \Delta_D\rangle$ is isomorphic to $U_k(q)$.*

(ii) *Γ_D is closed. For each pair $i < j$, if $\gamma_{i,s}, \gamma_{j,r}$ exist and $\gamma_{i,s} + \gamma_{j,r} \in \Sigma^+$, then $s = j$ and $\gamma_{i,j} + \gamma_{j,r} = \gamma_{i,r}$.*

(iii) *Λ_D is closed. For each pair $i < j$, if $\nu_{i,s}, \nu_{j,r}$ exist and $\nu_{i,s} + \nu_{j,r} \in \Sigma^+$, then $s = j$ and $\nu_{i,j} + \nu_{j,r} = \nu_{i,r}$.*

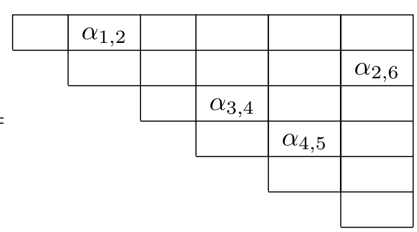
(iv) *R_D is isomorphic to $U_k(q) \cap {}^{w_D}U_k(q)$ and C_D is isomorphic to $U_k(q) \cap {}^{w_0 w_D}U_k(q)$, where*

$$w_0 = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}$$

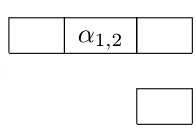
is the longest element in S_k.

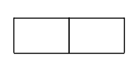
(v) *$V_D R_D$ is a pattern subgroup of U and R_D normalizes V_D.*

For example, let $D = \{\alpha_{1,2}, \alpha_{3,4}, \alpha_{4,5}, \alpha_{2,6}\}$ be a basic set in Σ_6^+:

$U_7(q) =$ 

and

$R_D =$ 

, $C_D =$  .

The next result is the main theorem, which provides a correspondence between supercharacters $\xi_{D,\phi}$ and pattern subgroups R_D.

Theorem 2.8. *Let $\xi_{D,\phi}$ be a supercharacter. The following are true.*

(i) *$\xi_{D,\phi} = (\lambda_D^{V_D R_D})^U$.*

(ii) *For each $\chi \in \mathrm{Irr}(V_D R_D, \lambda_D)$, $\chi^U \in \mathrm{Irr}(\xi_{D,\phi})$.*

(iii) *If $\chi_1 \neq \chi_2 \in \mathrm{Irr}(V_D R_D, \lambda_D)$, then $\chi_1^U \neq \chi_2^U$.*

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Therefore, to decompose $\xi_{D,\phi}$, it suffices to decompose $\lambda_D^{V_D R_D}$. Moreover, the induced character $\lambda_D^{V_D R_D}$ is equal to

$$(\lambda_D|_{V_D \cap R_D}^{R_D})_{V_D R_D} \otimes \theta,$$

where θ is some linear character of $V_D R_D$ (in Lemma 3.1). We see that $\lambda_D|_{V_D \cap R_D}^{R_D}$ is a 'very special' constituent of the regular character 1^{R_D}. Hence, the decomposition method of all supercharacters $\xi_{D,\phi}$ of $U_n(q)$ with the same w_D is generally restricted to the one of the regular character 1^{R_D}.

Here, we attempt to make a link for this special pattern $R_D = U_k(q) \cap {}^{w_D}U_k(q)$ in Lemma 2.7. Denoting $U \cap {}^{w}U$ by U_w, where $U = U_n(q)$ and $w \in S_n$ is the Weyl group of $\mathrm{GL}_n(q)$, Thompson [**16**] conjectured that, for each pair $r, s \in S_n$, the cardinality of the double coset $U_r \setminus U / U_s$ is a polynomial in q with integer coefficients. In addition, U_w also takes an important role when one studies $\mathrm{GL}_n(q)$ as groups with a (B, N)-pair, such as, for example, the Bruhat decomposition.

From Theorem 2.8 and Lemma 2.7 (v), we obtain a nice decomposition of $\xi_{D,\phi}$.

Corollary 2.9. *Let $\xi_{D,\phi}$ be a supercharacter. The following are true:*

(i) $\mathrm{Irr}(\xi_{D,\phi}) = \{\chi^U : \chi \in \mathrm{Irr}(V_D R_D, \lambda_D)\}$;

(ii) $\xi_{D,\phi} = \sum_{\chi \in \mathrm{Irr}(V_D R_D, \lambda_D)} \chi(1)\chi^U$.

Theorem 2.4, Lemma 2.7 and Corollary 2.9 give a clear proof for the following corollary, which is a different version of [**1**, Theorem 1.4].

Corollary 2.10.

$$(\xi_{D,\phi}, \xi'_{D',\phi'}) = \begin{cases} [V_D R_D : V_D] & \text{if } (D,\phi) = (D',\phi'), \\ 0 & \text{otherwise.} \end{cases}$$

3. All proofs

In this section, we prove Theorem 2.8 mainly to give a correspondence between supercharacters $\xi_{D,\phi}$ and pattern subgroups $U_k(q) \cap {}^{w_D}U_k(q)$, where $k = |D|$. First, we shall prove Lemma 2.7.

Proof of Lemma 2.7. Suppose that $D = \{\tau_1, \dots, \tau_k\}$ in ascending order $<_{\mathrm{r}}$.

(i) If we rearrange D in ascending order of $<_{\mathrm{b}}$ to be $\{\theta_1, \dots, \theta_k\}$, it is clear that, on the row of θ_i, Δ_D has $(k-i)$ roots and the row of θ_k does not have any root in Δ_D.

For each pair $i < j \in [1,k]$, let $\omega_{i,j} \in \Delta_D$ be the root on the row of τ_i such that $\omega_{i,j} + \tau_j \in \Sigma^+$. (Note that $\omega_{i,j}$ is either $\gamma \in \Gamma_D$ or $\nu \in \Lambda_D$.) Hence, if $\tau_i = \alpha_{i_1,i_2} <_{\mathrm{b}} \tau_j = \alpha_{j_1,j_2}$, i.e. $i_1 < j_1$, we have $\omega_{i,j} = \alpha_{i_1,j_1-1}$. Therefore, for each $\omega_{i,j} = \alpha_{i_1,j_1-1} <_{\mathrm{r}} \omega_{m,l} = \alpha_{m_1,l_1-1} \in \Delta_D$, if $\omega_{i,j} + \omega_{m,l} \in \Sigma^+$, then j_1 must equal m_1, and $\omega_{i,j} + \omega_{j,l} = \alpha_{i_1,l_1-1} = \omega_{i,l}$. This shows that Δ_D is closed, and the longest root in Δ_D is $\omega_{1,2} + \cdots + \omega_{k-1,k} = \omega_{1,k}$. So $\omega_{i,j}$ corresponds to $\alpha_{i,j-1}$ in the positive root set Σ_{k-1}^+. Therefore, $\langle X_\alpha : \alpha \in \Delta_D \rangle$ is a pattern subgroup isomorphic to $U_k(q)$.

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(ii) With the same argument as in (i), by the definition of $\gamma_{i,s}$ and $\gamma_{j,r}$, if $\gamma_{i,s}+\gamma_{j,r}\in\Sigma^+$, then $s=j$. By the transitive property of $<_{\mathrm{r}}$ and $<_{\mathrm{b}}$ on τ_i, τ_j, τ_r, from $\tau_i <_{\mathrm{r}}, <_{\mathrm{b}} \tau_j$ and $\tau_j <_{\mathrm{r}}, <_{\mathrm{b}} \tau_k$ we have $\tau_i <_{\mathrm{r}}, <_{\mathrm{b}} \tau_r$. So $\gamma_{i,r}$ exists and $\gamma_{i,j}+\gamma_{j,r}=\gamma_{i,r}$ follows.

(iii) The argument of (ii) holds for $\nu_{i,s}$ and $\nu_{j,r}\in\Lambda_D$.

(iv) Let $w_D=(w_{i,j})\in S_k\subset \mathrm{GL}_k(q)$. Since w_D is a monomial matrix, $w_D^{-1}=w_D^{\mathrm{T}}$, the transpose of w_D. For each $X=(x_{i,j})\in U_k(q)$, we observe $Y:=w_D\cdot X\cdot w_D^{-1}$. Let $Y=(y_{i,j})$. For each pair $i<j$, we have

$$y_{i,j}=\sum_{s,r\in[1,k]} w_{i,s}x_{s,r}w_{j,r}.$$

Since i, j are fixed, there exist unique $1\leqslant f,h\leqslant k$ such that $w_{i,f}=1=w_{j,h}$, and others $w_{i,s}=0=w_{j,r}$. Hence, $y_{i,j}=w_{i,f}x_{f,h}w_{j,h}$.

Since $h\neq f$ and all $x_{s,r}=0$ if $r<s$, we have the following:

- $y_{i,j}=0$ if $f>h$, i.e. $w_{i,f}<_{\mathrm{b}} w_{j,h}$ and $w_{j,h}<_{\mathrm{r}} w_{i,f}$;
- $y_{i,j}$ has non-zero value if $f<h$, i.e. $w_{i,f}<_{\mathrm{b}} w_{j,h}$ and $w_{i,f}<_{\mathrm{r}} w_{j,h}$.

So R_D is isomorphic to $U_k(q)\cap {}^{w_D}U_k(q)$ by the definition of $\gamma_{i,j}\in\Gamma_D$. And, hence, C_D is isomorphic to $U_k(q)\cap {}^{w_0\cdot w_D}U_k(q)$ by (i)–(iii) and $\Delta_D=\Gamma_D\cup\Lambda_D$.

(v) From the definition of $\gamma_{i,j}$, it is easy to check that R_D normalizes V_D. Hence, V_DR_D is a pattern subgroup of U. □

Set

$$K_D=\langle X_\alpha\colon X_\alpha\subset V_D \text{ and } \alpha\notin D\rangle=\langle X_\alpha\colon X_\alpha\subset V_D\cap\ker(\lambda_D)\rangle.$$

It is clear that K_D is normal in V_D, $[V_D:K_D]=q^{|D|}$ and $V_D=K_D\cdot\prod_{\tau\in D}X_\tau$. To prove Theorem 2.8, we need the following lemma.

Lemma 3.1. *Let $\xi_{D,\phi}$ be a supercharacter. The following are true.*

(i) $K_D\subset\ker(\lambda_D^{V_DR_D})$. *Moreover,* $\lambda_D^{V_DR_D}(x)=[V_DR_D\colon V_D]\lambda_D(x)$ *for all* $x\in V_D$.

(ii) $(K_D\cap R_D)\trianglelefteq R_D$ *and* $(V_D\cap R_D)/(K_D\cap R_D)\subset Z(R_D/(K_D\cap R_D))$.

(iii) *Let* $\bar\phi_D=\{\lambda_{\alpha,s}\in\phi\colon X_\alpha\not\subseteq R_D\}$. *We have*

$$\lambda_D^{V_DR_D}=(\lambda_D|^{R_D}_{V_D\cap R_D})_{V_DR_D}\otimes\Bigl(\bigotimes_{\lambda_{\alpha,s}\in\bar\phi_D}(\lambda_{\alpha,s}|_{V_D})_{V_DR_D}\Bigr).$$

Proof. (i) It is enough to show the statement for all $X_\alpha\subset V_D$. By Lemma 2.7 (v) $V_D\trianglelefteq V_DR_D$, we have

$$\lambda_D^{V_DR_D}(x)=\frac{1}{|V_D|}\sum_{y\in V_DR_D}\lambda_D(x^y)$$

for all $x\in V_D$. For each $x\in X_\alpha$, we suppose that there is $X_\beta\subset V_DR_D$ such that $\alpha+\beta\in\Sigma^+$, and hence $X_{\alpha+\beta}\subset V_D$. We shall show that $\lambda_D(x^y)=\lambda_D(x)$ for all $y\in X_\beta$.

Since $X_\tau\cap[V_D,V_D]=\{1\}$ for all $\tau\in D$, we have $X_{\alpha+\beta}\subset K_D\subset\ker(\lambda_D)$. Thus, $[\lambda_D(x),\lambda_D(y)]=\lambda_D([x,y])=1$ since $[x,y]\in X_{\alpha+\beta}$, i.e. $\lambda_D(x)^{-1}\lambda_D(x^y)=1$.

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(ii) By the definition of $K_D \trianglelefteq V_D$ and $V_D = K_D \cdot \prod_{\tau \in D} X_\tau$, it suffices to show that $(K_D \cap R_D) \trianglelefteq R_D$. This is clear because for all $X_\alpha \subset K_D \cap R_D$ and all $X_\beta \subset R_D$ either $\alpha + \beta \notin \Sigma^+$ or $X_{\alpha+\beta} \subset K_D \cap R_D$.

(iii) The inflations to $V_D R_D$ of $\lambda_D|^{R_D}_{V_D \cap R_D}$ and $\lambda_{\alpha,s}|_{V_D}$, for all $\lambda_{\alpha,s} \in \bar{\phi}_D$, follow directly from (i). □

By Lemma 3.1 (iii), if $R_D \cap V_D = \{1\}$, $\lambda_D^{V_D R_D}$ is equivalent to 1^{R_D}, the regular character of R_D. In general, $\lambda_D^{V_D R_D}$ is equivalent to a constituent of 1^{R_D} with $R_D \cap K_D$ in the kernel. Now we prove Theorem 2.8.

Proof of Theorem 2.8. (i) This is clear by the transitivity of induction.

(ii) Suppose that $D = \{\tau_1, \dots, \tau_k\}$ in ascending order $<_{\mathrm{r}}$ and

$$\lambda_D = \bigotimes_{\tau_i \in D} \lambda_{\tau_i, s_i}|_{V_D},$$

where $s_i \in \mathbb{F}_q^\times$.

First, we show that, for each $\chi \in \mathrm{Irr}(V_D R_D, \lambda_D)$, χ^U is irreducible. By the transitive property of induction, we shall induce χ from $V_D R_D$ to U by a sequence of inductions along the arms of $\tau_1, \tau_2, \dots, \tau_k$ respectively by $<_{\mathrm{r}}$ order. Now we setup these such induction steps.

For each $\tau_i \in D$, let $A(\tau_i) = \{\alpha \in \mathrm{arm}(\tau_i) \colon X_\alpha \not\subseteq V_D R_D\}$, and $c_i = |A(\tau_i)|$. Let $d_0 = 0$ and $d_i = d_{i-1} + c_i$ for all $i \in [1, k]$. Now, if $c_i > 0$, $i \in [1, k]$, we arrange $A(\tau_i)$ in decreasing order $<_{\mathrm{r}}$ to be $\{\beta_{d_{i-1}+1}, \dots, \beta_{d_{i-1}+c_i}\}$. Let $M_0 = V_D R_D$, $M_{i+1} = M_i \rtimes X_{\beta_i}$ for all $i \in [1, d_k]$. It is clear that $M_{d_k+1} = U$ and X_{β_j} normalizes M_j; hence, this sequence of pattern subgroups is well defined.

For each $\beta_j \in \mathrm{arm}(\tau_i)$, $j \in [1, d_k]$, there exists a unique $\delta \in \mathrm{leg}(\tau_i)$ such that $\beta_j + \delta = \tau_i$ and $X_\delta \subset K_D$, since if $X_\delta \not\subseteq K_D$, there exists $\tau_m \in D$ such that $\delta \in \mathrm{arm}(\tau_m)$, so $\tau_i <_{\mathrm{r}} \tau_m$, $\tau_i <_{\mathrm{b}} \tau_m$, and this implies $\beta_j = \gamma_{i,m}$. We number this δ as δ_j, and let $L(D) = \{\delta_j \colon j \in [1, d_k]\}$. By Lemma 3.1 (i), $X_\delta \subset \ker(\chi)$ for all $\delta \in L(D)$. Now we proceed the induction of χ from $V_D R_D$ to U via a sequence of pattern subgroups along the arms of all $\tau_i \in D$, namely from M_0 to $M_1, \dots, M_{d_k+1} = U$.

Suppose that $\chi^{M_j} \in \mathrm{Irr}(M_j)$ for some M_j, $j \in [1, d_k + 1]$, and $X_{\delta_t} \subset \ker(\chi^L)$ for all $t \in [j, d_k]$. If $j = d_k + 1$, the proof is complete. Otherwise, the next induction step is from M_j to $M_{j+1} = M_j X_{\beta_j}$, and we suppose that it happens on the arm of τ_i. For each $x \in X_{\beta_j}^\times$, since $[X_{\delta_j}, x] = X_{\tau_i}$, there is some $y \in X_{\delta_j}$ such that $\lambda_{\tau_i, s_i}([y, x]) \neq 1$ and

$${}^x(\chi^{M_j})(y) = \chi^{M_j}(y^x) = \chi^{M_j}([y, x]y) = \lambda_{\tau_i, s_i}([y, x])\chi^{M_j}(y) \neq \chi^{M_j}(y) = \chi^{M_j}(1).$$

Hence, $X_{\delta_j} \not\subseteq \ker({}^x(\chi^{M_j}))$, and

$${}^x(\chi^{M_j}) \neq \chi^{M_j} \quad \text{for all } x \in X_{\beta_j}^\times.$$

This shows that the inertia group $I_{M_j X_{\beta_j}}(\chi) = M_j$ and $\chi^{M_j X_{\beta_j}} \in \mathrm{Irr}(M_j X_{\beta_j}, \lambda_D)$.

It is easy to check directly that $X_{\delta_t} \subset \ker(\chi^{M_j X_{\beta_j}})$ for all $t \in [j + 1, d_k]$ by using $[X_{\beta_j}, X_{\delta_t}] \subset \ker(\chi^{M_j})$. Therefore, we have χ^U is irreducible for all $\chi \in \mathrm{Irr}(V_D R_D, \lambda_D)$ by induction on j.

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(iii) Now suppose $\chi_1 \neq \chi_2 \in \mathrm{Irr}(V_D R_D, \lambda_D)$ and $\chi_1^{M_j} \neq \chi_2^{M_j}$ for some M_j. As above, it is enough to show that

$$\chi_1^{M_j X_{\beta_j}} \neq \chi_2^{M_j X_{\beta_j}},$$

where $\beta_j \in \mathrm{arm}(\tau_i)$. Note that

$$X_{\delta_j} \subset \ker(\chi_1^{M_j}) \cap \ker(\chi_2^{M_j}).$$

By the Mackey Formula with the double coset $M_j \setminus M_j X_{\beta_j} / M_j$ represented by X_{β_j},

$$(\chi_1^{M_j X_{\beta_j}}, \chi_2^{M_j X_{\beta_j}}) = \sum_{x \in X_{\beta_j}} (\chi_1^{M_j}, {}^x(\chi_2^{M_j})).$$

By using the same argument as in (ii),

$$X_{\delta_j} \nsubseteq \ker({}^x(\chi_2^{M_j})) \quad \text{for all } x \in X_{\beta_j}^{\times}.$$

Hence, ${}^x(\chi_2^{M_j}) \neq \chi_1^{M_j}$ for all $x \in X_{\beta_j}^{\times}$ since $X_{\delta_j} \subset \ker(\chi_1^{M_j})$. Therefore,

$$(\chi_1^{M_j X_{\beta_j}}, \chi_2^{M_j X_{\beta_j}}) = (\chi_1^{M_j}, \chi_2^{M_j}) = 0,$$

since $\chi_1^{M_j} \neq \chi_2^{M_j}$ by the above assumption on M_j. □

Note that $V_D R_D$ is not normal in U. In the proof of Theorem 2.8, although all inductions from $V_D R_D$ to U are irreducible, Clifford correspondence cannot be applied. The technique of a sequence of inductions from M_j to $M_{j+1} \subset N_U(M_j)$ has been used to control distinct induced characters.

Since V_D is normal in $V_D R_D$ and $V_D R_D / V_D \cong R_D/(V_D \cap R_D)$, by Theorem 2.8 and Lemma 3.1 (iii), we only need to decompose $\lambda_D|_{V_D \cap R_D}^{R_D}$ instead of decomposing the supercharacter $\xi_{D,\phi} = \lambda_D^U$. Hence, all work is restricted to a pattern subgroup of $U_k(q)$, where $k = |D| < n$.

Proof of Corollary 2.9. Theorem 2.8 gives a one-to-one correspondence on the multiplicities and degrees between $\mathrm{Irr}(V_D R_D, \lambda_D)$ and $\mathrm{Irr}(\xi_{D,\phi})$, i.e.

$$|\mathrm{Irr}(V_D R_D, \lambda_D)| = |\mathrm{Irr}(\xi_{D,\phi})|,$$

and if $\chi \in \mathrm{Irr}(V_D R_D, \lambda_D)$ has multiplicity t, then $\chi^U \in \mathrm{Irr}(\xi_{D,\phi})$ also has multiplicity t, and

$$\chi^U(1) = [U : V_D R_D]\chi(1).$$

Therefore, it is enough to show that $\chi \in \mathrm{Irr}(R_D, \lambda_D|_{V_D \cap R_D})$ has multiplicity $\chi(1)$.

By Lemma 3.1 (i),

$$K_D \cap R_D \subset \ker(\lambda_D|_{V_D \cap R_D}) \cap \ker(\lambda_D|_{V_D \cap R_D}^{R_D})$$

is normal in R_D. So $\lambda_D|_{V_D \cap R_D}$ can be considered as a linear character of the quotient group $R_D/(K_D \cap R_D)$. By Lemma 3.1 (ii), $(V_D \cap R_D)/(K_D \cap R_D) \subset Z(R_D/(K_D \cap R_D))$, $\lambda_D|_{V_D \cap R_D}$ is a linear character of the centre and the claim holds. □

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