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A NOTE ON PARTITION-INDUCING AUTOMORPHISM GROUPS

BY

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ABSTRACT. We consider a finite group G with a group A acting on it in such a way as to induce a partition of $G^{\#}$ (a situation which arises in the study of centralizer near-rings). With the additional hypothesis that $(|A^{\omega}|, |G|) = 1$, it is shown that either A is semiregular on $G^{\#}$ or G is an irreducible module for A.

1. **Introduction.** If A is a finite group acting on a set X, we shall say the action of A is "partitive" if the sets $C_X(C_A(x))$, $x \in X$, partition X. This is easily seen to be an extension of the more familiar notion of half-transitivity. In this note, we take X to be the set $G^{\#}$ of non-identity elements of a finite group G and A to be a group of automorphisms of G. The author's main motivation for studying this situation is a result of C. Maxson and K. Smith [4], that partitivity is equivalent to the semisimplicity of the centralizer near-ring $C(A, G)$.

Clearly the symmetric group S_3 acts partitively on itself by conjugation. On the other hand, it was shown in an earlier note [5] that if A is a nilpotent group acting partitively on $G^{\#}$, then either A is semiregular on $G^{\#}$ or G is an irreducible module for A (of dimension at most 4). It seems reasonable to ask whether weaker assumptions about the structure of A will suffice to force a similar conclusion (but without the dimension restriction). Here we observe the following:

THEOREM. *Suppose G is a finite group and $A \le \operatorname{Aut} G$ such that A acts partitively on $G^{\#}$. If $(|A^{\omega}|, |G|) = 1$, then either A is semiregular on $G^{\#}$ or G is an irreducible module for A. (A^{ω} denotes the "nilpotent residual" of A, the smallest normal subgroup of A such that A/A^{ω} is nilpotent).*

One immediate consequence of the theorem is that if $(|A^{\omega}|, |G|) = 1$ and $C(A, G)$ is semisimple but not simple, then $C(A, G)$ has the additive structure of a vector space. As a purely group theoretic result, the theorem may be regarded as a generalization of Theorem I of [3].

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2. Proof of the theorem. Let the pair (A, G) be a counterexample to the theorem with $|A|+|G|$ minimal.

(2.1) G contains a proper non-trivial A-invariant subgroup.

Proof. By [5], $A^{\omega}\neq 1$ so $C_G(A^{\omega})\neq G$. We may, therefore, assume $C_G(A^{\omega})=1$ so by [2, Theorem 6.2.2], G has a unique A^{ω}-invariant Sylow p-subgroup for each prime p. Since $A^{\omega}\trianglelefteq A$, such subgroups are A-invariant so from [5, Lemma 2.2], we conclude that G is a p-group. Hence, we may assume $G=\Omega_1(Z(G))$ so G is a $GF(p)$ $[A]$-module. Since G is not A-irreducible, (2.1) is proved.

(2.2) G contains a unique maximal A-invariant subgroup U. Moreover, either $A/C_A(U)$ is semiregular on $U^{\#}$ or U is an irreducible A-module.

Proof. See proof of (4.2) in [5].

(2.3) U is nilpotent.

Proof. From (2.2) and Thompson's theorem [2, Theorem 10.2.1].

(2.4) $U\leq Z(G)$.

Proof. Suppose first that $C_G(A^{\omega})=1$. As argued in (2.1), G is a p-group so $U\trianglelefteq G$. If $U\not\leq Z(G)$ then by (2.2), $A/C_A(U)$ is semiregular on $U^{\#}$ and $C_G(U)\leq U$. By [2, Theorem 2.2.3], $[G, C_A(U)]\leq U$. Now let $U<G_0\leq G$ with $|G_0:U|=p$ and let $A_0=C_A(G_0/U)$. If $u\in U^{\#}$, $C_A(u)=C_A(U)\leq C_A(G/U)\leq A_0$ and if $x\in G_0\backslash U$, $C_A(x)\leq C_A(G_0/U)=A_0$. It follows that $\bar{A}_0=A_0/C_{A_0}(G_0)$ acts partitively on $G_0^{\#}$ so, since this action is neither irreducible nor semiregular, the inductive hypothesis implies $G_0=G$ and $A_0=A$. But then $G=[G, A^{\omega}]\leq[G_0, A_0]\leq U$, a contradiction.

Thus, we may assume $C_G(A^{\omega})\neq 1$ so by (2.2), $U\leq C_G(A^{\omega})$. Since $A^{\omega}\neq 1$ by [5], $U=C_G(A^{\omega})$ so by a lemma of Glauberman (Theorem 3, Corollary 1 of [1]), U controls G-fusion in itself. If P is a Sylow p-subgroup of U, then $U\leq N_G(P)$ by (2.3) so $P\trianglelefteq G$ or $N_G(P)=U$. But in the latter case, P is a Sylow subgroup of G which controls G-fusion itself and hence, G is p-nilpotent, contradicting (2.2). Thus $P\trianglelefteq G$ for every choice of P, whence again $U\trianglelefteq G$. Now by [2, Theorem 2.2.3], $[G, A^{\omega}]\leq C_G(U)$. If $U\not\leq Z(G)$ then $C_G(U)\leq U$ by (2.2) so $[G, A^{\omega}, A^{\omega}]=1$. By [2, Theorem 5.3.6], we conclude that $A^{\omega}=1$, contradicting [5]. Thus, $U\leq Z(G)$ as required.

(2.5) G is a p-group of exponent p and nilpotence class at most 2.

Proof. The argument in (4.8) of [5] shows that G has exponent p. Then $G'\neq G$ so by (2.2) and (2.4), $G'\leq Z(G)$.

(2.6) We may assume G is a module for A over $GF(p)$.

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Proof. See (4.9) of [5].

Let $K = O_{p'}(A)$ so, by hypothesis, $A^{\omega} \leq K$. Since A/K is nilpotent, it is a p-group.

(2.7) G/U is isomorphic to an A-submodule of U.

Proof. By Maschke's theorem, $G = U \oplus V$ for some K-submodule V of G. By (2.2), $V^{\alpha} \neq V$ for some $\alpha \in A$ so the projection $V^{\alpha} \to U$ (with respect to the decomposition $G = U \oplus V$) is a non-trivial K-homomorphism. Since $V^{\alpha} \simeq G/U \simeq V$ as K-modules, $\mathrm{Hom}_K(G/U, U) \neq 0$. Now A acts on the p-group $\mathrm{Hom}_K(G/U, U)$ (where, if $f \in \mathrm{Hom}_K(G/U, U)$ and $\sigma \in A$, $f^{\sigma}(x) = f(x^{\sigma^{-1}})^{\sigma}$ for all $x \in G/U$) and K is in the kernel of this action, so A/K acts on $\mathrm{Hom}_K(G/U, U)$. Since A/K is also a p-group, it fixes a non-zero element f of $\mathrm{Hom}_K(G/U, U)$. Then $f \in \mathrm{Hom}_A(G/U, U)$ and, since G/U is A-irreducible, f is injective.

(2.8) The final contradiction.

Let $f: G/U \to U$ be an A-monomorphism (by (2.7)). Then for every $x \in G$, $C_A(x) \leq C_A(xU) = C_A(f(xU))$. Since $f(xU) \in U$, partitivity implies that if $x \in G \setminus U$, then $C_A(x) = C_A(f(xU))$ so $C_A(x) = C_A(xU)$.

Now suppose $u \in U^{\#}$ and $x \in G \setminus U$. If $\alpha \in C_A(xu)$, $x^{-1}x^{\alpha} = uu^{-\alpha} \in U$ so $\alpha \in C_A(xU) = C_A(x)$. Thus, $C_A(xu) = C_A(x) \cap C_A(u)$ so by [5, Lemma 2.1], $C_A(x) = C_A(u)$. It follows that A is semiregular on $G^{\#}$, a contradiction.

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