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On a Jacobian Identity Associated with Real Hyperplane Arrangements

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Abstract. For $x \in (a_{j-1}, a_j)$ $(j = 1, \ldots, p+1;\ a_0 := -\infty,\ a_{p+1} := \infty)$ the mapping $T_j: w = x - \sum_{l=1}^{p} \lambda_l/(x - a_l)$ $(\lambda_l > 0,\ a_l \in \mathbf{R})$ is onto **R**. It was shown by G. Boole in the 1850's that $\sum_{j=1}^{p+1} [(\partial w/\partial x)^{-1}]_{x=T_j^{-1}(w)} = 1$. We give an n-dimensional analogue of this result. The proof makes use of the Griffiths–Harris residue theorem from algebraic geometry.

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1. Statement of the Result

Let $\mathcal{A}$ be a finite arrangement of hyperplanes in the n-dimensional complex affine space $\mathbf{C}^n$. Let $N(\mathcal{A})$ be the union of hyperplanes of $\mathcal{A}$ in $\mathbf{C}^n$ and $M(\mathcal{A})$ be its complement in $\mathbf{C}^n$, so that $M(\mathcal{A}) = \mathbf{C}^n - N(\mathcal{A})$. We let H_∞ denote the hyperplane at infinity in the n-dimensional complex projective space $\mathbf{CP}^n$ such that $\mathbf{CP}^n = \mathbf{C}^n \cup H_\infty$. We identify any hyperplane in $\mathbf{C}^n$ with the one which is uniquely extended in $\mathbf{CP}^n$. The set $\mathcal{A}$ can then be regarded as an arrangement of hyperplanes in $\mathbf{CP}^n$. In this article it is assumed that $\mathcal{A}$ is real, by which we mean the defining function of every hyperplane $H \in \mathcal{A}$,

$$f_H(z) = u_{H,0} + \sum_{v=1}^{n} u_{H,v} z_v, \qquad z = (z_v)_{v=1}^n \in \mathbf{C}^n, \tag{1.1}$$

has real coefficients $u_{H,0}$ and $u_{H,v}$. A connected component of $M(\mathcal{A}) \cap \mathbf{R}^n$ is called a chamber.

For a point $w = \{w_v\}_{v=1}^n \in \mathbf{R}^n$ and positive real numbers $\{\lambda_H\}_{H \in \mathcal{A}}$, we consider the level function

$$F(z) = \tfrac{1}{2}\mathrm{Re} \sum_{v=1}^{n} (z_v - w_v)^2 - \sum_{H \in \mathcal{A}} \lambda_H \log |f_H(z)| \tag{1.2}$$

and its gradient $g(z) = \operatorname{grad} F(z)$. The latter is the function $g(z) = (g_1(z), \ldots, g_n(z))$

such that

$$g_\nu(z) = z_\nu - w_\nu - \sum_{H\in\mathcal{A}} \frac{\lambda_H u_{H,\nu}}{f_H(z)}. \tag{1.3}$$

The vector field $g(z)$ is meromorphic in $\mathbf{CP}^n$ and holomorphic in $M(\mathcal{A})$.

We remark that according to the saddle point method of asymptotic analysis, the function $F(z)$ plays an important role in calculating the asymptotics in the direction λ for n-dimensional integrals of the multiplicative function

$$\Phi(z) = \exp\left[-\frac{1}{2}\sum_{\nu=1}^{n}(z_\nu - w_\nu)^2\right]\prod_{H\in\mathcal{A}} f_H{}^{\lambda'_H} \tag{1.4}$$

where $\lambda' = \{\lambda'_H\}_{H\in\mathcal{A}}$ denotes $\lambda' = t\lambda + \tilde{\lambda} \quad (t \to +\infty)$ for a fixed $\tilde{\lambda} = \{\tilde{\lambda}_H\}_{H\in\mathcal{A}}$. Indeed, these integrals give typical examples of hypergeometric integrals of irregular singularity. Their geometric and analytic structures are intimately related with the configurations of $M(\mathcal{A})$ and $N(\mathcal{A})$ (see, for example, [1, 2, 4, 12–14, 16, 17]).

Before presenting our results, we must first establish some basic properties of the function $F(z)$. In fact these properties, established in Lemma 1.1 below, are more or less known. Explicitly, in [4, Th. 4.1.1] and [17, 1.2.1], Lemma 1.1 is proved in the case of the absence of the terms $z_\nu - w_\nu$ in g_ν.

LEMMA 1.1. *The function F is strictly convex in each component $\triangle$ of $M(\mathcal{A}) \cap \mathbf{R}^n$. Furthermore, the set of the critical points for the function $F(z)$ specified by the set of points z satisfying the equalities*

$$g_1(z) = \cdots = g_n(z) = 0 \tag{1.5}$$

is finite. They lie one by one in each $\triangle$ and in $M(\mathcal{A})$ there does not exist any other point satisfying (1.5).

Proof. For completeness we sketch a proof. The function $F(x)\,(x \in \triangle)$ is strictly convex in $\triangle$ because, for real numbers $t_1, \ldots, t_n$ which do not vanish at the same time,

$$\sum_{\mu,\nu=1}^{n} \frac{\partial^2 F(x)}{\partial x_\mu \partial x_\nu} t_\mu t_\nu = \sum_{\nu=1}^{n} t_\nu^2 + \sum_{H\in\mathcal{A}} \lambda_H \frac{(\sum_{\nu=1}^{n} u_{H,\nu} t_\nu)^2}{f_H(x)^2} > 0. \tag{1.6}$$

The function $F(x)$ therefore has a unique minimum in $\triangle$ and, consequently, there exists only one point, $\mathbf{c}$ say, in $\triangle$ such that the 1-form

$$\theta = \sum_{\nu=1}^{n} g_\nu(x)\,\mathrm{d}x_\nu$$

vanishes. The fact that θ does not vanish at any complex point can be shown by contradiction. Thus if θ did vanish at a complex point $\mathbf{c}$ in $M(\mathcal{A})$, then θ would also vanish at the complex conjugate point $\bar{\mathbf{c}}$ different from $\mathbf{c}$. Consider the line $\mathbf{l}$ connecting these 2 points. This can be chosen to have real coefficients. Explicitly, every point of $\mathbf{l}$ can be parametrized as

$$z = \mathrm{Im}\,\mathbf{c} \cdot t + \mathrm{Re}\,\mathbf{c} \quad t \in [-i, i]$$

where $\mathbf{c}, \bar{\mathbf{c}}$ correspond to $t = i, -i$, respectively. The restriction of θ to this line $\mathbf{l}$ vanishes at $\mathbf{c} \in M(\mathcal{A}) \cap \mathbf{l}$ which is a contradiction, because the Lemma is obviously true in the one-dimensional case. This proves the Lemma. □

We denote by $\mathbf{c}_1, \ldots, \mathbf{c}_\kappa$ the critical points of the function $F(z)$ which lie one by one in $\triangle_1, \ldots, \triangle_\kappa$, the components of $M(\mathcal{A}) \cap \mathbf{R}^n$. The convexity property in Lemma 1 then gives

COROLLARY 1.2. *At each point* $\mathbf{c}_j$

$$\left[\det\left(\frac{\partial g_\nu}{\partial z_\mu}\right)^n_{\mu,\nu=1}\right]_{z=\mathbf{c}_j} > 0, \tag{1.7}$$

which is to say the corresponding Jacobian is positive.

Moreover, one can show the following identity, which is the main result of this article.

THEOREM 1.3

$$\sum_{j=1}^{\kappa}\left[\det\left(\frac{\partial g_\nu}{\partial z_\mu}\right)^n_{\mu,\nu=1}\right]^{-1}_{z=\mathbf{c}_j} = 1. \tag{1.8}$$

This identity can be regarded as a type of fixed point formula in $\mathbf{CP}^n$, similar to the ones which were investigated, for example, in [11, 15, 18]. There only polynomial mappings were treated, more restrictive than our rational mappings. It seems an interesting problem to ask if (1.8) can be extended for arbitrary polynomials f_H, H being irreducible hypersurfaces.

2. An Inequality Associated with Hyperplane Arrangement

Let δ be a small positive number. We consider the following subsets in $\mathbf{C}^n$,

$$V_\delta^{(0)} = \{z \in \mathbf{C}^n;\ |z_j| \leqslant \delta^{-1}\}, \tag{2.1}$$

$$V_\delta^{(k)} = \{z \in \mathbf{C}^n;\ |z_k| \geqslant \delta^{-1}, |z_k| \geqslant |z_j|(\,j = 1, \ldots, n(j \neq k))\}, \tag{2.2}$$

$(k = 1, 2, \ldots, n)$, which cover the whole of $\mathbf{C}^n$ so that $\mathbf{C}^n = \bigcup_{k=0}^n V_\delta^{(k)}$.

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We put $V^{(0)} = \cup_{\delta>0} V_\delta^{(0)}$, which coincides with $\mathbf{C}^n$, and $V^{(k)} = \cup_{\delta>0} V_\delta^{(k)}$. In $V_\delta^{(k)}$ we introduce the new affine coordinates $(\zeta_1, ..., \zeta_n)$ such that

$$z_k = 1/\zeta_1, z_1 = \zeta_2/\zeta_1, ..., z_{k-1} = \zeta_k/\zeta_1, z_{k+1} = \zeta_{k+1}/\zeta_1, ..., z_n = \zeta_n/\zeta_1. \tag{2.3}$$

We have

$$|\zeta_1| \leqslant \delta,\ |\zeta_2| \leqslant 1, \dots,\ |\zeta_n| \leqslant 1$$

and vice versa for the reciprocals. In terms of the coordinates (2.3), $f_H(z)$ can be described as

$$f_H(z) = \tilde{f}_H(\zeta)/\zeta_1. \tag{2.4}$$

where

$$\tilde{f}_H(\zeta) = u_{H,0}\zeta_1 + u_{H,k} + \sum_{v=1}^{k-1} u_{H,v}\zeta_{v+1} + \sum_{v=k+1}^{n} u_{H,v}\zeta_v. \tag{2.5}$$

For an n dimensional vector $v \in \mathbf{C}^n$, we introduce the norm of v as

$$\|v\| = \sqrt{|v_1|^2 + \cdots + |v_n|^2}.$$

In section 4 we shall prove the following proposition which plays a key role in the proof of Theorem 1.3.

PROPOSITION 2.1. *There exists a neighbourhood U of $N(\mathcal{A}) \cup \{H_\infty\}$ in $\mathbf{CP}^n$, and positive constants C_0 and C_1, such that the inequalities*

$$\|g\|^2 \geqslant C_0 \sum_{H\in\mathcal{A}} \frac{1}{|f_H(z)|^2}, \tag{2.6}$$

$$|g_v| \leqslant C_1 \sum_{H\in\mathcal{A}} \frac{1}{|f_H(z)|} \tag{2.7}$$

hold for z in $U \cap V^{(0)} \cap M(\mathcal{A})$, and

$$\|g\|^2 \geqslant C_0\left(\frac{1}{|\zeta_1|^2} + \sum_{H\in\mathcal{A}} \frac{|\zeta_1|^2}{|\tilde{f}_H(z)|^2}\right), \tag{2.8}$$

$$|g_v| \leqslant C_1\left(\frac{1}{|\zeta_1|} + \sum_{H\in\mathcal{A}} \frac{|\zeta_1|}{|\tilde{f}_H(z)|}\right) \tag{2.9}$$

hold for ζ in $U \cap V^{(k)} \cap M(\mathcal{A})$.

Remark. The inequalities (2.6) and (2.8) do not remain true in the whole $M(\mathcal{A})$, as can be seen from the fact that the left-hand side in each case vanishes at the critical points $\mathbf{c}_j$, while the RHS is positive.

3. An Admissible System of Neighbourhoods of $N(\mathcal{A}) \cup H_\infty$ in $\mathbf{CP}^n$

Let L be an arbitrary subspace in $\mathbf{CP}^n$. We denote by $\mathcal{A}_L$ the subarrangement of hyperplanes $\mathcal{A}_L = \{H \in \mathcal{A} | H \supset L\}$. We fix a subspace L in $\mathbf{CP}^n$ such that $L \notin H_\infty$ and we also fix a point belonging to L. The latter is denoted by $z^{(0)}$ or $\zeta^{(0)}$ according to it belonging to $V^{(0)}$ or not. Let the integer r, $1 \leqslant r \leqslant n$ be such that $\dim L = n - r$. Then there exists a basis $(e_1, \ldots, e_n)$ in $\mathbf{C}^n$ such that $(e_{r+1}, \ldots, e_n)$ forms a basis of the tangent space $\mathbf{T}(L)$ of L, while $(e_1, \ldots, e_r)$ forms a complementary basis to $\mathbf{T}(L)$ at $z^{(0)}$.

Now, an arbitrary point $z \in \mathbf{C}^n$ can be represented as

$$z = \sum_{\mu=1}^{n} e_\mu z'_\mu + z^{(0)}, \qquad z' = (z'_\mu)_{\mu=1}^{n} \in \mathbf{C}^n. \tag{3.1}$$

The point z lies in L if and only if it can be expressed as

$$z = \sum_{\mu=r+1}^{n} e_\mu z'_\mu + z^{(0)}. \tag{3.2}$$

On the other hand,

$$z^* = \sum_{\mu=1}^{r} e_\mu z'_\mu \tag{3.3}$$

gives a complementary vector to L. If $H \in \mathcal{A}_L$ then $f_H(z)$ is described as

$$f_H(z) = \sum_{\nu=1}^{n} u_{H,\nu} z^*_\nu, \tag{3.4}$$

where z^* is the vector constructed from z according to (3.1)–(3.3) with $z^{(0)} \in H$.

DEFINITION 3.1. Let $(L, z^{(0)})$ be a pair consisting of a $(n-r)$-dimensional subspace L in $\mathbf{CP}^n$ and a point $z^{(0)} \in V^{(0)} \cap L$ such that $L \not\subset H_\infty$. With $\rho, \delta_1, \delta_2, \delta_3$ denoting small positive numbers, we denote by $U^{(0)}_\rho(L; z^{(0)})$ the set of points $z \in V^{(0)}$ satisfying

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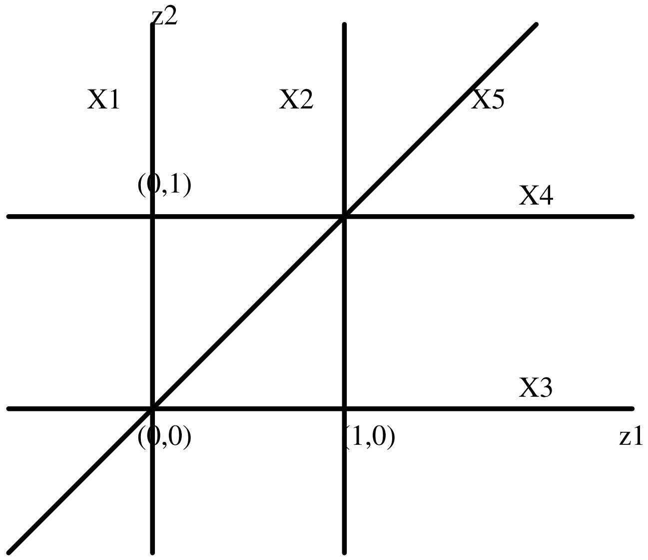

Figure 1. Real section of the line arrangement $X_1\,(z_1 = 0)$, $X_2\,(z_1 = 1)$, $X_3\,(z_2 = 0)$, $X_4\,(z_2 = 1)$ and $X_5\,(z_1 = z_2)$.

the following conditions.

(i) $\|z - z^{(0)}\| \leqslant \rho$,

(ii) $|f_H(z)| \geqslant \delta_1 \quad \text{for } H \not\ni z^{(0)}$,

(iii) $|z_1'|^2 + \cdots + |z_r'|^2 \leqslant \delta_2^2$,

(iv) $$\frac{|f_H(z)|}{\sqrt{|z_1'|^2 + \cdots + |z_r'|^2}} \geqslant \delta_2^{-1} \quad \text{for } H \notin \mathcal{A}_L,\ H \ni z^{(0)},$$

(v) $$\delta_3 \leqslant \frac{|f_H(z)|}{\sqrt{|z_1'|^2 + \cdots + |z_r'|^2}} \leqslant \delta_3^{-1} \quad \text{for } H \in \mathcal{A}_L.$$

As an illustration of the neighbourhoods $U_\rho^{(0)}(L; z^{(0)})$ specified by Definition 3.1, let $\mathcal{A}$ be the line arrangement consisting of the lines $X_1\,(z_1 = 0)$, $X_2\,(z_1 = 1)$, $X_3\,(z_2 = 0)$, $X_4\,(z_2 = 1)$ and $X_5\,(z_1 = z_2)$. The real section of these lines is drawn in Figure 1. First take $z^{(0)} = (0, 0)$. Then $\dim L$ must be equal to 0 or 1. If $\dim L = 0$, then L coincides with $\{z^{(0)}\}$. The corresponding neighbourhood $U_\rho^{(0)}(L; z^{(0)})$ is illustrated in Figure 2(A). If $\dim L = 1$, then L coincides with X_1, X_3 or X_5; the corresponding neighbourhoods $U_\rho^{(0)}(L; z^{(0)})$ are illustrated in Figures 2(B), (C) and (D) respectively. Similar neighbourhoods are obtained for $z^{(0)} = (1, 1)$.

If $z^{(0)} = (1, 0)$ and $\dim L = 0$, the neighbourhood $U_\rho^{(0)}(L; z^{(0)})$ is as in Figure 3(A). If $z^{(0)} = (1, 0)$ and $\dim L = 1$, then $U_\rho^{(0)}(L; z^{(0)})$ is as in Figure 3(B), when L coincides

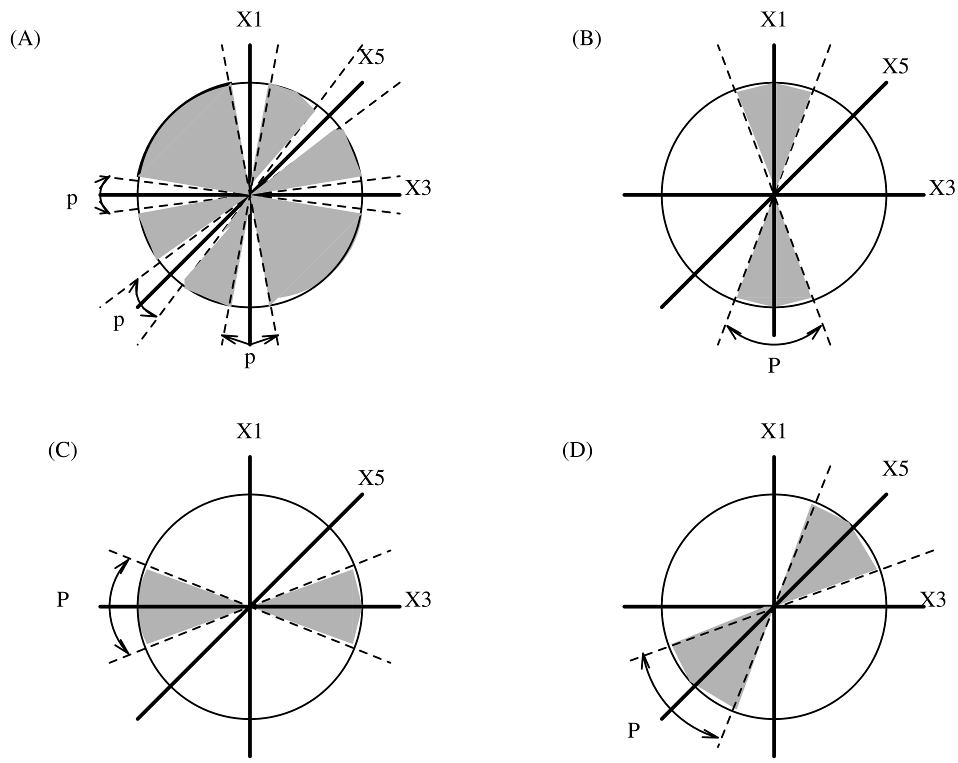

Figure 2. The neighbourhood $U_\rho^{(0)}(L; z^{(0)})$ with $z^{(0)} = (0, 0)$ and L specified in the text.

with X_2, and as in Figure 3(C) when L coincides with X_3. Similar neighbourhoods are obtained for $z^{(0)} = (0, 1)$.

If $z^{(0)} \in N(\mathcal{A})$ is different from $(0, 0), (0, 1), (1, 0), (1, 1)$, then we must have $\dim L = 1$. For example, if $z^{(0)} \in X_1$, then L coincides with X_1. $U_\rho^{(0)}(L; z^{(0)})$ is then as in Figure 3(D).

In Definition 3.1 it is required that $z^{(0)} \in V^{(0)} \cap L$. To consider the points at infinity $\zeta^{(0)} \in V^{(k)} \cap L$, first note that in $V^{(k)}$ there exists a basis $(\tilde{e}_1, \ldots, \tilde{e}_n)$ in $\mathbf{C}^n$ with respect to the affine coordinates $(\zeta_1, \ldots, \zeta_n)$ such that $(\tilde{e}_{r+1}, \ldots, \tilde{e}_n)$ forms a basis of $\mathbf{T}(L)$ at $\zeta^{(0)}$ and $(\tilde{e}_1, \ldots, \tilde{e}_r)$ forms a complementary basis to $\mathbf{T}(L)$ at $\zeta^{(0)}$. We may take $\zeta'_{r+1} = \zeta_1$. Analogous to (3.1)–(3.3), an arbitrary point $\zeta (\in \mathbf{C}^n)$ can be represented as

$$\zeta = \sum_{\mu=1}^{n} \tilde{e}_\mu \zeta'_\mu + \zeta^{(0)}, \quad \zeta' = (\zeta'_\mu)_{\mu=1}^n \in \mathbf{C}^n \tag{3.5}$$

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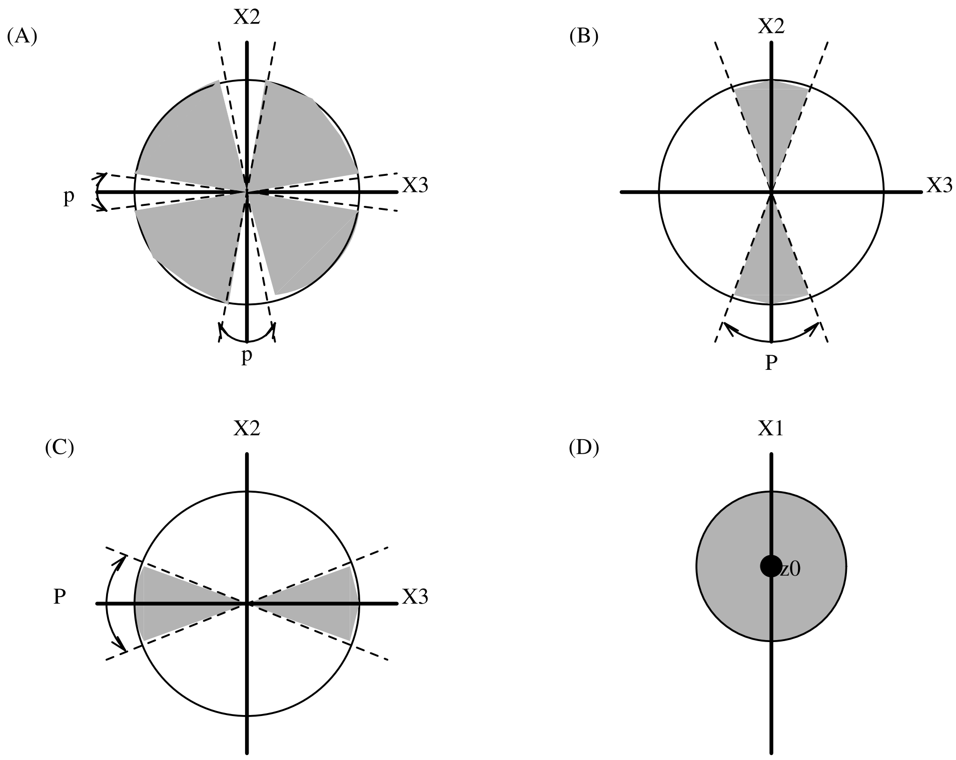

Figure 3. The neighbourhood $U_{\rho}^{(0)}(L; z^{(0)})$ with $z^{(0)} = (1, 0)$ and L specified in the text for (A)–(C) and with $z^{(0)}$ different from (0, 0), (0, 1), (1, 0), (1, 1) in (D).

such that it lies in L if and only if it is expressed as

$$\zeta = \sum_{\mu=r+1}^{n} \tilde{e}_{\mu}\zeta'_{\mu} + \zeta^{(0)}, \tag{3.6}$$

while

$$\zeta^* = \sum_{\mu=1}^{r} \tilde{e}_{\mu}\zeta'_{\mu} \tag{3.7}$$

gives a complementary vector to $\mathbf{T}(L)$. Note from (2.3) that $\zeta^* = (0, \zeta_2^*, \ldots, \zeta_n^*)$. Also, analogous to (3.4) we have the expression

$$\tilde{f}_H(\zeta) = u_{H,0}\zeta_1^* + \sum_{\nu=1}^{k-1} u_{H,\nu}\zeta_{\nu+1}^* + \sum_{\nu=k+1}^{n} u_{H,\nu}\zeta_{\nu}^* \tag{3.8}$$

provided $H \in \mathcal{A}_L$.

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DEFINITION 3.2. Let $(L, \zeta^{(0)})$ be a pair consisting of an $(n-r)$-dimensional subspace L in $\mathbf{CP}^n$ and a point $\zeta^{(0)} \in V^{(k)} \cap L$ such that $L \not\subset H_\infty$. We assume that $\zeta^{(0)} \in V^{(k)}$. We denote by $U_\rho^{(k)}(L; \zeta^{(0)})$ the set of points $z \in M(\mathcal{A}) \cap V^{(k)}$ satisfying the following conditions:

(i) $\|\zeta - \zeta^{(0)}\| \leqslant \rho$,

(ii) $|\tilde{f}_H(\zeta)| \geqslant \delta_1 \quad \text{for } H \not\ni \zeta^{(0)}$,

(iii) $|\zeta_1'|^2 + \cdots + |\zeta_r'|^2 \leqslant \delta_2^2$,

(iv) $$\frac{|\tilde{f}_H(\zeta)|}{\sqrt{|\zeta_1'|^2 + \cdots + |\zeta_r'|^2}} \geqslant \delta_2^{-1} \quad \text{for } H \notin \mathcal{A}_L, H \ni \zeta^{(0)},$$

(v) $$\frac{|\zeta_1|}{\sqrt{|\zeta_1'|^2 + \cdots + |\zeta_r'|^2}} \geqslant \delta_2^{-1},$$

(vi) $$\delta_3 \leqslant \frac{|\tilde{f}_H(\zeta)|}{\sqrt{|\zeta_1'|^2 + \cdots + |\zeta_r'|^2}} \leqslant \delta_3^{-1} \quad \text{for } H \in \mathcal{A}_L.$$

In the case of the arrangement of Figure 1, the neighbourhoods $U^{(1)}(L, \zeta^{(0)})$ and $U^{(2)}(L, \zeta^{(0)})$ with $\zeta^{(0)} = (0, 0) \in V^{(1)}$ and $V^{(2)}$ respectively, are similar to those of Figure 2, while with $\zeta^{(0)} = (1, 1) \in V^{(1)}$ or $V^{(2)}$, respectively, they are similar to those of Figure 3(A)–(C).

The remaining situation to consider is the case $\zeta^{(0)} \in L \cap H_\infty$, which was excluded in Definition 3.2.

DEFINITION 3.3. Suppose that L is as in Definition 3.2 and that $\zeta^{(0)} \in L \cap H_\infty$ in $V^{(k)}$. We can take the same basis $\tilde{e}_1, \ldots, \tilde{e}_n$ as in Definition 3.2 where ζ_1 can be identified with ζ_{r+1}'. We denote by $U_\rho^{(k)}(L \cap H_\infty; \zeta^{(0)})$ the set of points $\zeta \in M(\mathcal{A}) \cap V^{(k)}$ satisfying the properties (i) and (ii) of Definition 3.2 together with the followings properties:

(iii) $|\zeta_1'|^2 + \cdots + |\zeta_{r+1}'|^2 \leqslant \delta_2^2$,

(iv) $$\frac{|\tilde{f}_H(\zeta)|}{\sqrt{|\zeta_1'|^2 + \cdots + |\zeta_{r+1}'|^2}} \geqslant \delta_2^{-1} \quad \text{for } H \notin \mathcal{A}_L, H \ni \zeta^{(0)},$$

(v) $$\delta_3 \leqslant \frac{|\tilde{f}_H(\zeta)|}{\sqrt{|\zeta_1'|^2 + \cdots + |\zeta_{r+1}'|^2}} \leqslant \delta_3^{-1} \quad \text{for } H \in \mathcal{A}_L,$$

(vi) $$\delta_3 \leqslant \frac{|\zeta_1|}{\sqrt{|\zeta_1'|^2 + \cdots + |\zeta_{r+1}'|^2}} \leqslant \delta_3^{-1}.$$

In Definitions 3.1–3.3, $\rho, \delta_1, \delta_2, \delta_3$ depend on the choices of the pairs $(L; z^{(0)}), (L; \zeta^{(0)}), (L \cap H_\infty; \zeta^{(0)})$ respectively.

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For an arbitrary subspace L, we denote by $\mathcal{C}_L$ the finite set consisting of the critical points in $M(\mathcal{A})$ for the function

$$F_L = \frac{1}{2}\mathrm{Re}\sum_{\nu=1}^{n}(z_\nu - w_\nu)^2 - \sum_{H\in\mathcal{A}_L}\lambda_H \log|f_H(z)|. \tag{3.9}$$

We can choose a small positive number ρ such that

$$U_\rho^{(0)}(L; z^{(0)}) \cap \mathcal{C}_L = \emptyset, \tag{3.10}$$

$$U_\rho^{(k)}(L; \zeta^{(0)}) \cap \mathcal{C}_L = \emptyset, \tag{3.11}$$

and

$$U_\rho^{(k)}(L \cap H_\infty; \zeta^{(0)}) \cap \mathcal{C}_L = \emptyset, \tag{3.12}$$

for any $L, z^{(0)}, \zeta^{(0)}$. The following fact is also true.

LEMMA 3.4. *There exist small positive numbers $\delta_1, \delta_2, \delta_3$ for $\delta_2\delta_3^{-1} \gg 1$ such that the intersection of $M(\mathcal{A})$ and the union of the sets $U_\rho^{(0)}(L; z^{(0)})$, $U_\rho^{(k)}(L; \zeta^{(0)})$ and $U_\rho^{(0)}(L \cap H_\infty; \zeta^{(0)})$ includes a neighbourhood of the set $N(\mathcal{A}) \cup \{H_\infty\}$ in $\mathbf{CP}^n$.*

Note that the statement of Lemma 3.4 is illustrated in Figures 2 and 3, with the angles p and P therein being such that $p < P$.

We fix the pair $(L; z^{(0)})$ as above. There exist r hyperplanes, $H_1, \ldots, H_r$ say, such that $L = \cap_{j=1}^r H_j$. From this we see that for z in $U_\rho^{(0)}(L; z^{(0)})$ the functions

$$\sqrt{|z_1'|^2 + \cdots + |z_r'|^2}, \quad \|z^*\|, \quad \sum_{H\in\mathcal{A}_L} |f_H(z)|^2, \quad \sum_{j=1}^{r} |f_{H_j}(z)|^2$$

vanish along L in the same order, so that there exists a positive constant K such that

$$K^{-1}\|z^*\| \leqslant \sqrt{|z_1'|^2 + \cdots + |z_r'|^2} \leqslant K\|z^*\|, \tag{3.13}$$

$$K^{-1}\|z^*\| \leqslant \sqrt{\sum_{H\in\mathcal{A}_L} |f_H(z)|^2} \leqslant K\|z^*\|, \tag{3.14}$$

$$K^{-1}\|z^*\| \leqslant \sqrt{\sum_{j=1}^{r} |f_{H_j}(z)|^2} \leqslant K\|z^*\|. \tag{3.15}$$

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Similarly, for ζ in $U_\rho^{(k)}(L;\zeta^{(0)})$,

$$K^{-1}\|\zeta^*\| \leqslant \sqrt{|\zeta_1'|^2+\cdots+|\zeta_r'|^2} \leqslant K\|\zeta^*\|, \tag{3.16}$$

$$K^{-1}\|\zeta^*\| \leqslant \sqrt{\sum_{H\in\mathcal{A}_L}|\tilde{f}_H(\zeta)|^2} \leqslant K\|\zeta^*\|, \tag{3.17}$$

$$K^{-1}\|\zeta^*\| \leqslant \sqrt{\sum_{j=1}^{r}|\tilde{f}_{H_j}(\zeta)|^2} \leqslant K\|\zeta^*\|, \tag{3.18}$$

while for ζ in $U_\rho^{(k)}(L\cap H_\infty;\zeta^{(0)})$

$$K^{-1}\|\zeta^*\| \leqslant \sqrt{|\zeta_1'|^2+\cdots+|\zeta_{r+1}'|^2} \leqslant K\|\zeta^*\|, \tag{3.19}$$

$$K^{-1}\|\zeta^*\| \leqslant \sqrt{|\zeta_1|^2+\sum_{H\in\mathcal{A}_L}|\tilde{f}_H(\zeta)|^2} \leqslant K\|\zeta^*\|, \tag{3.20}$$

$$K^{-1}\|\zeta^*\| \leqslant \sqrt{|\zeta_1|^2+\sum_{j=1}^{r}|\tilde{f}_{H_j}(\zeta)|^2} \leqslant K\|\zeta^*\|, \tag{3.21}$$

$$K^{-1}\|\zeta^*\| \leqslant |\zeta_1| \leqslant K\|\zeta^*\|. \tag{3.22}$$

We denote by $\bar{S}$ the closure of a set S in $\mathbf{CP}^n$. We then have

LEMMA 3.5. *The union of the sets* $\overline{U}_\rho^{(0)}(L;z^{(0)})$, $\overline{U}_\rho^{(k)}(L;\zeta^{(0)})$ *and* $\overline{U}_\rho^{(k)}(L\cap H_\infty;\zeta^{(0)})$ *contains a neighbourhood of the set* $N(\mathcal{A})\cup\{H_\infty\}$ *in* $\mathbf{CP}^n$. *Actually a finite number of them cover a neighbourhood of* $N(\mathcal{A})\cup\{H_\infty\}$.

Proof. In fact, only finite members of $\{\overline{U}_\rho^{(0)}(L;z^{(0)})\}$ cover a neighbourhood of $z^{(0)}$ in $\mathbf{CP}^n$. Similarly only finite members of $\{\overline{U}_\rho^{(0)}(L;z^{(0)})\}$ or $\{\overline{U}_\rho^{(0)}(L\cap H_\infty;\zeta^{(0)})\}$ cover a neighbourhood of $\zeta^{(0)}$. Since $N(\mathcal{A})\cup\{H_\infty\}$ is compact, the Lemma follows due to the Heine-Borel covering theorem. □

4. Proof of the Proposition 2.1

The statement of Proposition 1 consists of the four inequalities (2.6)–(2.9). Since (2.7) and (2.9) are obvious, we have only to prove (2.6) and (2.8). First we prove (2.6) in the neighbourhood $U_\rho^{(0)}(L;z^{(0)})$, then (2.8) in the neighbourhoods $U_\rho^{(k)}(L;z^{(0)})$ and $U_\rho^{(k)}(L\cap H_\infty;z^{(0)})$

4.1. PROOF OF (2.6) IN $U_\rho^{(0)}(L; z^{(0)})$

From (1.3) and (3.4) we have

$$\sum_{\mu=1}^{n} g_\mu z_\mu^* = -\sum_{H\in\mathcal{A}_L} \lambda_H + T_1 + T_2 + T_3, \tag{4.1}$$

where

$$T_1 = \sum_{\mu=1}^{n}(z_\mu - w_\mu)z_\mu^*, \qquad T_2 = -\sum_{H\notin\mathcal{A}_L, H\ni z^{(0)}} \lambda_H \frac{\sum_{\nu=1}^{n} u_{H,\nu} z_\nu^*}{f_H(z)},$$
$$T_3 = -\sum_{H\not\ni z^{(0)}} \lambda_H \frac{\sum_{\nu=1}^{n} u_{H,\nu} z_\nu^*}{f_H(z)}.$$

We note that $\sum_{H\in\mathcal{A}_L}\lambda_H > 0$. In view of conditions (ii)–(v) of Definition 3.1 we can choose $\rho, \delta_1, \delta_2, \delta_3$ so small that

$$|T_i| < \frac{1}{6}\sum_{H\in\mathcal{A}_L} \lambda_H \quad (i = 1, 2, 3)$$

(the proportionality constant 1/6 is chosen for later convenience). Hence, after applying the triangle inequality to (4.1), we conclude

$$\left|\sum_{\mu=1}^{n} g_\mu z_\mu^*\right| \geqslant \frac{1}{2}\sum_{H\in\mathcal{A}_L} \lambda_H.$$

By the Schwarz inequality, this implies

$$\|g\|\cdot\|z^*\| \geqslant \frac{1}{2}\sum_{H\in\mathcal{A}_L} \lambda_H$$

and, thus,

$$\|g\| \geqslant \frac{1}{2}\sum_{H\in\mathcal{A}_L} \lambda_H/\|z^*\|.$$

But from condition (v) in Definition 3.1 and (3.14), there exists a positive constant K_1 such that

$$\frac{1}{|f_H(z)|} \leqslant \frac{K_1}{\|z^*\|} \quad \text{for } H\in\mathcal{A}_L.$$

Hence, there exists a positive constant C_0 such that (2.6) holds.

4.2. PROOF OF (2.8) IN $U_\rho^{(k)}(L;\zeta^{(0)})$

Without loosing generality, we may assume $k=1$. From (1.3) and (2.4) we have

$$g_1(\zeta)=\frac{1}{\zeta_1}-w_1-\sum_{H\in\mathcal{A}}\frac{\lambda_H u_{H,1}\zeta_1}{\tilde{f}_H(\zeta)}, \tag{4.2}$$

$$g_v(\zeta)=\frac{\zeta_v}{\zeta_1}-w_v-\sum_{H\in\mathcal{A}}\frac{\lambda_H u_{H,v}\zeta_1}{\tilde{f}_H(\zeta)} \quad (v\geqslant 2) \tag{4.3}$$

(here we have abused notation by writing $g_v(\zeta)$ in place of $g_v(z)$). Now we write $g^{(0)}(\zeta)=(g_1^{(0)}(\zeta),\dots,g_n^{(0)}(\zeta))$ with

$$g_1^{(0)}(\zeta)=\frac{1}{\zeta_1}-w_1-\sum_{H\in\mathcal{A}_L}\frac{\lambda_H u_{H,1}\zeta_1}{\tilde{f}_H(\zeta)}, \tag{4.4}$$

$$g_v^{(0)}(\zeta)=\frac{\zeta_v}{\zeta_1}-w_v-\sum_{H\in\mathcal{A}_L}\frac{\lambda_H u_{H,v}\zeta_1}{\tilde{f}_H(\zeta)} \quad (v\geqslant 2). \tag{4.5}$$

We first seek to prove that there exists a positive constant C_2 such that

$$\|g^{(0)}(\zeta)\|^2\geqslant C_2\left(\frac{1}{|\zeta_1|^2}+\sum_{H\in\mathcal{A}_L}\frac{|\zeta_1|^2}{|\tilde{f}_H(\zeta)|^2}\right) \tag{4.6}$$

in $U_\rho^{(1)}(L;\zeta^{(0)})$. Equivalently, if we define the function $\varphi(\zeta)$ as

$$\varphi(\zeta)=\frac{\|g^{(0)}(\zeta)\|^2}{\frac{1}{|\zeta_1|^2}+\sum_{H\in\mathcal{A}_L}\frac{|\zeta_1|^2}{|\tilde{f}_H(\zeta)|^2}} \tag{4.7}$$

then we must prove $\varphi(\zeta)\geqslant C_2$ in $U_\rho^{(1)}(L;\zeta^{(0)})$.

Suppose the contrary. Then there would exist a point sequence $\zeta^{(l)}(l=1,2,\dots)$ in $U_\rho^{(1)}(L;\zeta^{(0)})$, which converges to a point $\alpha=(\alpha_1,\dots,\alpha_n)$ in its closure $\overline{U_\rho^{(1)}(L;\zeta^{(0)})}$ in $\mathbf{CP}^n$ such that

$$\lim_{l\to\infty}\varphi(\zeta^{(l)})=0. \tag{4.8}$$

The details of the conclusion to be drawn from (4.8) depend on the value of α_1 and $\tilde{f}_H(\alpha)$.

(a) *The case where* $\alpha_1\neq 0$ *and* $\tilde{f}_H(\alpha)\neq 0$ *for every* $H\in\mathcal{A}_L$.

In this case α lies in $U_\rho^{(1)}(L;\zeta^{(0)})$. Since $\lim_{l\to\infty}\varphi(\zeta^{(l)})=\varphi(\alpha)$ we have $\|g^{(0)}(\alpha)\|=0$ and thus $g^{(0)}(\alpha)=0$.

This means that the point $z=(1/\alpha_1,\alpha_2/\alpha_1,\dots,\alpha_n/\alpha_1)\in V^{(0)}$ is a critical point for the function $F_L(z)$. However we have assumed (recall (3.11)) that there exists no such point α in $U_\rho^{(1)}(L;\zeta^{(0)})$, which is a contradiction.

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(b) *The case where* $\alpha_1 \neq 0$ *and that there exists a hyperplane* $H_0 \in \mathcal{A}_L$ *such that* $\tilde{f}_{H_0}(\alpha) = 0$.

From condition (vi) of Definition 3.2,

$$\tilde{f}_H(\alpha) = 0 \text{ for } H \in \mathcal{A}_L \tag{4.9}$$

and hence $\alpha^* = 0$. By choosing a suitable subsequence of $\{\zeta^{(l)}\}_{l \geqslant 1}$ (if necessary) we may assume that

$$\lim_{l\to\infty} \frac{(\zeta^{(l)})^*}{\|(\zeta^{(l)})^*\|} \tag{4.10}$$

exists. We denote its limit by $\beta = (\beta_1, ..., \beta_n)$. Note that $\beta_1 = 0$, because $(\zeta_1^{(l)})^* = 0$, and that we have also

$$\lim_{l\to\infty} \frac{\tilde{f}_H(\zeta^{(l)})}{\|(\zeta^{(l)})^*\|} = \sum_{v=2}^{n} u_{H,v}\beta_v \neq 0 \quad (H \in \mathcal{A}_L), \tag{4.11}$$

because of condition (vi) in Definition 3.2. We denote this value by β_H. Then, after recalling (4.4), we see that

$$\lim_{l\to\infty}\varphi(\zeta^{(l)}) = \frac{\sum_{v=1}^{n} |\sum_{H\in\mathcal{A}_L} \frac{\lambda_H u_{H,v}\alpha_1}{\beta_H}|^2}{\sum_{H\in\mathcal{A}_L} \frac{|\alpha_1|^2}{|\beta_H|^2}},$$

which, because of (4.8) implies

$$\sum_{H\in\mathcal{A}_L} \lambda_H \frac{u_{H,v}}{\beta_H} = 0 \quad (1 \leqslant v \leqslant n).$$

But this is again a contradiction because

$$\sum_{v=1}^{n}\sum_{H\in\mathcal{A}_L} \lambda_H \frac{u_{H,v}\beta_v}{\beta_H} = \sum_{H\in\mathcal{A}_L} \lambda_H \neq 0.$$

(c) *The case where* $\alpha_1 = 0$

We may again assume that (4.10) converges and that (4.9) and (4.11) hold. Thus β_1 again vanishes, while β_H does not. Since α_1 vanishes we have

$$\tilde{f}_H(\alpha) = \sum_{v=2}^{n} u_{H,v}\alpha_v + u_{H,1}. \tag{4.12}$$

Also, by the choice of an appropriate subsequence if necessary, we may assume that

$$\lim_{l\to\infty} \frac{(\zeta_1^{(l)})^2}{\|(\zeta^{(l)})^*\|} = a$$

exists or diverges to the point at infinity. The reasoning now depends on the value of a.

$a = 0$
Then we have

$$\lim_{l\to\infty}\varphi(\zeta^{(l)}) = 1 + \sum_{v=2}^{n} |\alpha_v|^2 \neq 0,$$

which is an immediate contradiction.

$a \neq 0$
Then

$$\lim_{l\to\infty}\varphi(\zeta^{(l)}) = \frac{\left|1 - \sum_{H\in\mathcal{A}_L} \frac{\lambda_H u_{H,1} a}{\beta_H}\right|^2 + \sum_{v=2}^{n} \left|\alpha_v - \sum_{H\in\mathcal{A}_L} \frac{\lambda_H u_{H,v} a}{\beta_H}\right|^2}{1 + \sum_{H\in\mathcal{A}_L} \frac{|a|^2}{|\beta_H|^2}}.$$

Hence the assumption (4.8) gives

$$1 - a \sum_{H\in\mathcal{A}_L} \frac{\lambda_H u_{H,1}}{\beta_H} = 0,$$

$$\alpha_v - a \sum_{H\in\mathcal{A}_L} \frac{\lambda_H u_{H,v}}{\beta_H} = 0.$$

But these equations together with (4.9) imply

$$1 + \sum_{v=2}^{n} |\alpha_v|^2 = a \sum_{H\in\mathcal{A}_L} \frac{\lambda_H (u_{H,1} + \sum_{v=2}^{n} u_{H,v}\bar{\alpha}_v)}{\beta_H} = 0,$$

which is again a contradiction.

$a = \infty$
Then

$$\lim_{l\to\infty}\varphi(\zeta^{(l)}) = \frac{\sum_{v=1}^{n} \left|\sum_{H\in\mathcal{A}_L} \frac{\lambda_H u_{H,v}}{\beta_H}\right|^2}{\sum_{H\in\mathcal{A}_L} \frac{1}{|\beta_H|^2}}.$$

Hence

$$\sum_{H\in\mathcal{A}_L} \frac{\lambda_H u_{H,v}}{\beta_H} = 0 \quad (1 \leqslant v \leqslant n),$$

which implies

$$\sum_{H\in\mathcal{A}_L} \lambda_H = 0$$

and thus gives another contradiction.

Considering the conclusion of cases (a),(b) and (c) together, we see that in all situations the hypothesis (4.8) is false. Since this hypothesis is equivalent to the assumption that the inequality (4.6) is invalid, we must have that (4.6) is in fact true. Furthermore, the above working shows that there exists a positive constant C_2' such that

$$\|g^{(0)}\|^2 \geqslant C_2'\left(\frac{|\zeta_1|^2}{\|\zeta^*\|^2} + \frac{1}{|\zeta_1|^2}\right), \tag{4.13}$$

and that this is equivalent to (4.6).

With this preliminary result established, we now proceed to prove the inequality (2.8). $\|g\|^2$ can be described as

$$\|g\|^2 = \|g^{(0)} - g - g^{(0)}\|^2 = \|g^{(0)}\|^2 + \|g - g^{(0)}\|^2 + 2\mathrm{Re}\{g^{(0)} \cdot \overline{(g - g^{(0)})}\}, \tag{4.14}$$

which gives

$$\|g\|^2 \geqslant \|g^{(0)}\|^2 - 2\|g^{(0)}\|\|g - g^{(0)}\|. \tag{4.15}$$

Now, from Definitions 3.2 (ii) and (iii) we have

$$\frac{|\zeta_1|}{\tilde{f}_H(\zeta)} \leqslant \frac{|\zeta_1|}{\|\zeta^*\|}\delta_2 \quad H \notin \mathcal{A}_L$$

which, according to (4.2)–(4.5), implies for some positive constant C_3

$$\|g - g^{(0)}\| \leqslant C_3\delta_2\frac{|\zeta_1|}{\|\zeta^*\|}.$$

Comparison with (4.13) shows that we can choose δ_2 small enough so that

$$\|g - g^{(0)}\| \leqslant \frac{1}{4}\|g^{(0)}\|. \tag{4.16}$$

Substituting (4.16) in (4.15) gives

$$\|g\| \geqslant \frac{1}{2}\|g^{(0)}\|,$$

and use of (4.6) immediately establishes (2.8).

4.3. PROOF OF (2.8) IN $U_\rho^{(k)}(L \cap H_\infty; \zeta^{(0)})$

Indeed from (1.3) and (2.4)

$$g_1(\zeta)\zeta_1 = 1 - w_1\zeta_1 - \sum_{H\in\mathcal{A}}\frac{\lambda_H u_{H,1}\zeta_1^2}{\tilde{f}_H(\zeta)}.$$

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But from Definition 3(ii)–(v), we may choose $\rho, \delta_1, \delta_2, \delta_3$ so small that

$$|w_1\zeta_1| + \sum_{H\in\mathcal{A}} \frac{\lambda_H |u_{H,1}\zeta_1^2|}{|\tilde{f}_H(\zeta)|} \leqslant \frac{1}{2}.$$

Hence $|g_1\zeta_1| \geqslant \frac{1}{2}$, which implies

$$\|g\| \geqslant |g_1| \geqslant \frac{1}{2|\zeta_1|}.$$

Since $|\zeta_1|/|\tilde{f}_H(\zeta)|$ is always bounded in $U_\rho^{(k)}(L \cap H_\infty; \zeta^{(0)})$, the inequality (2.8) holds there for a suitable positive constant C_0.

By (1)–(3), the inequalities (2.6) and (2.8) have been proved in each of the neighbourhoods $U_\rho^{(0)}(L; z^{(0)})$, $U_\rho^{(k)}(L; \zeta^{(0)})$ and $U_\rho^{(k)}(L \cap H_\infty; \zeta^{(0)})$. But from Lemma 3.5, the union of a finite number of their closures includes a neighbourhood of $N(\mathcal{A}) \cup H_\infty$ in $\mathbf{CP}^n$. By taking the constant C_0 as the minimal one among the C_0 in each of the regions, we see that (2.6) and (2.8) hold in a certain neighbourhood of $N(\mathcal{A}) \cup H_\infty$.

Hence we have proved Proposition 2.1.

5. Griffiths–Harris Formula

To prove Theorem 1.3, we first define a meromorphic form on $\mathbf{CP}^n$ and then apply the Griffiths–Harris residue formula to it in $M(\mathcal{A})$.

Let $\varepsilon_1, \varepsilon_2$ be small positive numbers. We consider the closed subset $M_{\varepsilon_1,\varepsilon_2}$ of $M(\mathcal{A})$ defined by

$$|z_1| \leqslant \varepsilon_1^{-1}, \dots, |z_n| \leqslant \varepsilon_1^{-1}, \quad |f_H(z)| \geqslant \varepsilon_2 \quad \text{for all} \quad H \in \mathcal{A}. \tag{5.1}$$

Let $\Omega(\lambda)$ be the meromorphic n-form on $\mathbf{C}^n$

$$\Omega(\lambda) = \frac{1}{(2\pi i)^n} \frac{\mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_n}{g_1(z) \cdots g_n(z)}. \tag{5.2}$$

Note from (1.3) that $\Omega(\lambda)$ depends on $\lambda = \{\lambda_H\}_{H\in\mathcal{A}}$, and when $\lambda = 0$, $\Omega(\lambda)$ reduces to Cauchy kernel

$$\Omega(0) = \frac{1}{(2\pi i)^n} \frac{\mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_n}{\prod_{\nu=1}^n (z_\nu - w_\nu)}. \tag{5.3}$$

Let $\Psi(\lambda)$ be the $(2n-1)$-form of type $(n, n-1)$ defined as

$$\Psi(\lambda) = \sigma_n \frac{\sum_{\nu=1}^n (-1)^{\nu-1} \bar{g}_\nu(z) \mathrm{d}\bar{g}_1(z) \wedge \cdots \wedge d\bar{g}_{\nu-1}(z) \wedge \mathrm{d}\bar{g}_{\nu+1}(z) \cdots \wedge \mathrm{d}\bar{g}_n(z)}{\|g(z)\|^{2n}} \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \tag{5.4}$$

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with

$$\sigma_n = \frac{(n-1)!}{(2i)^n (2\pi)^{2n}}.$$

When $\lambda = 0$, $\Psi(\lambda)$ reduces to

$$\Psi(0) = \sigma_n \frac{\sum_{\nu=1}^{n} (-1)^{\nu-1} (\bar{z}_\nu - \bar{w}_\nu) \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_{\nu-1} \wedge \mathrm{d}\bar{z}_{\nu+1} \cdots \wedge \mathrm{d}\bar{z}_n}{\|z - w\|^{2n}} \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \tag{5.5}$$

which coincides with the Bochner–Martinelli kernel on $\mathbf{C}^n$.

The residue of $\Omega(\lambda)$ in $M(\mathcal{A})$ at each critical point $\mathbf{c}_j$ (we denote it by $\mathrm{Res}_{\mathbf{c}_j}\Omega(\lambda)$) is given by

$$\left\{ \frac{\partial(g_1, \ldots, g_n)}{\partial(z_1, \ldots, z_n)} \right\}^{-1} = \left[\det \left(\frac{\partial g_\nu}{\partial z_\mu} \right)^n_{\mu,\nu=1} \right]^{-1}_{z=\mathbf{c}_j}$$

at $\mathbf{c}_j$ and vanishes elsewhere. The residue theorem due to Griffiths–Harris [10,11] can be stated as follows.

THEOREM 5.1

$$\sum_{j=1}^{\kappa} \mathrm{Res}_{z=\mathbf{c}_j} \Omega(\lambda) = \int_{\partial M_{\varepsilon_1,\varepsilon_2}} \Psi(\lambda). \tag{5.6}$$

Since the left-hand side of (5.6) is identical to the left-hand side of (1.8), our task is to prove that the right-hand side is equal to 1. The right-hand side of (5.6) does not depend on either ε_1 or ε_2, so it is sufficient to prove that

$$\lim_{\varepsilon_1 \downarrow 0} \lim_{\varepsilon_2 \downarrow 0} \int_{\partial M_{\varepsilon_1,\varepsilon_2}} \Psi(\lambda) = 1 \tag{5.7}$$

or, equivalently,

$$\lim_{\varepsilon_1 \downarrow 0} \lim_{\varepsilon_2 \downarrow 0} \int_{\partial M_{\varepsilon_1,\varepsilon_2}} (\Psi(\lambda) - \Psi(0)) = 0. \tag{5.8}$$

We now fix a small positive number δ. Let $\mathcal{B}$ be a subset of $\mathcal{A}$. We denote by $V_\delta^{(0)}(\mathcal{B})$ the subset of $V^{(0)} = \mathbf{C}^n$ defined by the inequalities

$$|z_1| \leqslant \delta^{-1}, \ldots, |z_n| \leqslant \delta^{-1}, \qquad |f_H(z)| \leqslant \delta \quad (H \in \mathcal{B}), \qquad |f_H(z)| \geqslant \delta \quad (H \notin \mathcal{B}).$$

Similarly we denote by $V_\delta^{(k)}(\mathcal{B})$ the subset of $M(\mathcal{A}) \cap V_\delta^{(k)}$ (recall (2.2)) defined by

$$|\zeta_1| \leqslant \delta, |\zeta_2| \leqslant 1, \ldots, |\zeta_n| \leqslant 1,$$
$$|\tilde{f}_H(\zeta)| \leqslant \delta \quad (H \in \mathcal{B}), \qquad |\tilde{f}_H(\zeta)| \geqslant \delta \quad (H \notin \mathcal{B}).$$

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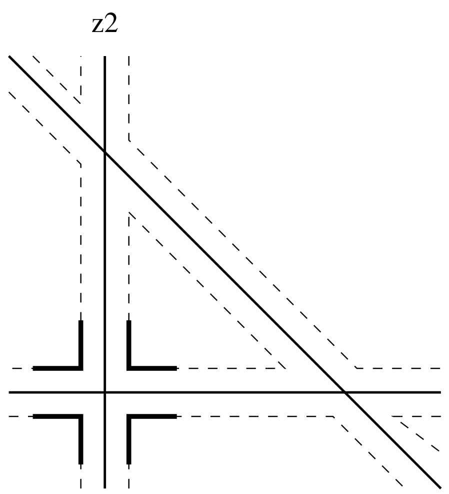

Figure 4. Real section of the line arrangement $z_1 = 0$, $z_2 = 0$, $z_2 = -z_1 + 1$ in $\mathbf{R}^2$. $\partial M_{\varepsilon_1,\varepsilon_2}$ is the dotted line. Superimposed on $\partial M_{\varepsilon_1,\varepsilon_2}$ is $\Gamma_\rho^{(0)}((0,0); \{z_1 = 0, z_2 = 0\})$.

Then the union of $V_\delta^{(0)}(\mathcal{B})$ and the $V_\delta^{(k)}(\mathcal{B})$ cover a neighbourhood of $N(\mathcal{A}) \cup H_\infty$ in $\mathbf{CP}^n$.

We take a point $z^{(0)} \in V^{(0)} \cap N(\mathcal{A})$ or $\zeta^{(0)} \in V^{(k)} \cap N(\mathcal{A}) \cap H_\infty$. Let $\mathcal{B}$ be the subset of $\mathcal{A}$ defined as

$$\mathcal{B} = \{H \in \mathcal{A};\, H \ni z^{(0)}(\text{or } \zeta^{(0)})\}. \tag{5.9}$$

We define $\Gamma_\rho^{(0)}(z^{(0)}; H_0)$ or $\Gamma_\rho^{(k)}(\zeta^{(0)}; H_0)$ for $H_0 \in \mathcal{B}$ as a piece of the boundary $\partial M_{\varepsilon_1,\varepsilon_2}$ in each neighbourhood $V_\delta^{(0)}(\mathcal{B})$ or $V_\delta^{(k)}(\mathcal{B})$ as follows: $\Gamma_\rho^{(0)}(z^{(0)}; H_0)$ is the set of $z \in \partial M_{\varepsilon_1,\varepsilon_2} \cap V_\delta^{(0)}(\mathcal{B})$ satisfying

$$\begin{aligned} &|f_{H_0}(z)| = \varepsilon_2, \qquad \delta \geqslant |f_H(z)| \geqslant \varepsilon_2 \ \text{ for } H \in \mathcal{B} - \{H_0\},\\ &|f_H(z)| \geqslant \delta \text{ for } H \notin \mathcal{B} \qquad \|z - z^{(0)}\| \leqslant \rho. \end{aligned} \tag{5.10}$$

An example of this construction is given in Figure 4.

$\Gamma_\rho^{(k)}(\zeta^{(0)}; H_0)$ is the set of $z \in \partial M_{\varepsilon_1,\varepsilon_2} \cap V_\delta^{(k)}(\mathcal{B})$ satisfying

$$\begin{aligned} &|\tilde{f}_{H_0}(\zeta)| = \varepsilon_2|\zeta_1|,\ \delta \geqslant |\zeta_1| \geqslant \varepsilon_1, \qquad \delta \geqslant |\tilde{f}_H(z)| \geqslant \varepsilon_2|\zeta_1| \quad \text{for } H \in \mathcal{B} - \{H_0\},\\ &|\tilde{f}_H(\zeta)| \geqslant \delta \quad \text{for } H \notin \mathcal{B}, \qquad \|\zeta - \zeta^{(0)}\| \leqslant \rho. \end{aligned} \tag{5.11}$$

Finally $\Gamma_\rho^{(k)}(\zeta^{(0)}; H_\infty)$ is the set of $z \in \partial M_{\varepsilon_1,\varepsilon_2} \cap V_\delta^{(k)}(\mathcal{B} \cup \{H_\infty\})$ satisfying the conditions

$$|\zeta_1| = \varepsilon_1, \qquad \varepsilon_1\varepsilon_2 \leqslant |\tilde{f}_H(\zeta)| \leqslant \delta \quad \text{for } H \in \mathcal{B}, \\ |\tilde{f}_H(\zeta)| \geqslant \delta \quad \text{for } H \notin \mathcal{B}, \qquad \|\zeta - \zeta^{(0)}\| \leqslant \rho. \tag{5.12}$$

We remark that $\varepsilon_1', \varepsilon_2'$ can be chosen so small that $\partial M_{\varepsilon_1,\varepsilon_2}$, for $0 < \varepsilon_1 \leqslant \varepsilon_1', 0 < \varepsilon_2 \leqslant \varepsilon_2'$ is contained in the union $\cup_{\mathcal{B}}\{V_\delta^{(0)}(\mathcal{B}) \cup_{k=1}^n V_\delta^{(k)}(\mathcal{B})\}$. Also, by the Heine–Borel theorem, only a finite number of the sets $\Gamma_\rho^{(0)}(z^{(0)}; H_0)$, $\Gamma_\rho^{(k)}(\zeta^{(0)}; H_0)$, $\Gamma_\rho^{(k)}(\zeta^{(0)}; H_\infty)$ are needed to cover $\partial M_{\varepsilon_1,\varepsilon_2}$

Seeing from (1.3) that

$$d\bar{g}_\nu = d\bar{z}_\nu + \sum_{H \in \mathcal{A}} \lambda_H \frac{u_{H,\nu}}{\overline{f_H(z)}^2} d\overline{f_H(z)}, \tag{5.13}$$

the $(n-1)$-form

$$\sum_{\nu=1}^{n} (-1)^{\nu-1} \bar{g}_\nu d\bar{g}_1 \wedge \cdots \wedge d\bar{g}_{\nu-1} \wedge d\bar{g}_{\nu+1} \wedge \cdots \wedge d\bar{g}_n,$$

being a polynomial of degree $n-1$ in λ, is represented as

$$G_0 + \sum_{p=1}^{n} \sum_{H_1,\cdots,H_p \subset \mathcal{A}} \lambda_{H_1} \cdots \lambda_{H_p} G_{H_1,\ldots,H_p},$$

where the sum with respect to $\{H_1, \ldots, H_p\}$ is over the set of all p tuples of members of $\mathcal{A}$ such that $\dim \cap_{j=1}^p H_j = n - p$. Substitution into (5.4) shows $\Psi(\lambda)$ can be represented as

$$\Psi(\lambda) = \Psi_0(\lambda) + \sum_{p=1}^{n} \sum_{H_1,\cdots,H_p \subset \mathcal{A}} \lambda_{H_1} \cdots \lambda_{H_p} \Psi_{H_1,\ldots,H_p}(\lambda), \tag{5.14}$$

where

$$\Psi_0(\lambda) = \frac{G_0}{\|g\|^{2n}} \wedge dz_1 \wedge \cdots \wedge dz_n, \tag{5.15}$$

$$\Psi_{H_1,\ldots,H_p}(\lambda) = \frac{G_{H_1,\ldots,H_p}}{\|g\|^{2n}} \wedge dz_1 \wedge \cdots \wedge dz_n. \tag{5.16}$$

Obviously $\Psi(0)$ as defined by (5.3) coincides with $\Psi_0(0)$. This is important because we already know that due to the Cauchy formula

$$\int_{\partial M_{\varepsilon_1,\varepsilon_2}} \Psi(0) = 1.$$

In the next section we shall show the following identities:

(I) $\lim_{\varepsilon_1\downarrow 0}\lim_{\varepsilon_2\downarrow 0}\int_{\Gamma_\rho^{(0)}(z^{(0)};H_0)}\Psi(\lambda)=0,$

(I′) $\lim_{\varepsilon_1\downarrow 0}\lim_{\varepsilon_2\downarrow 0}\int_{\Gamma_\rho^{(0)}(z^{(0)};H_0)}\Psi(0)=0,$

(II) $\lim_{\varepsilon_1\downarrow 0}\lim_{\varepsilon_2\downarrow 0}\int_{\Gamma_\rho^{(k)}(\zeta^{(0)};H_0)}\Psi(\lambda)=0,$

(II′) $\lim_{\varepsilon_1\downarrow 0}\lim_{\varepsilon_2\downarrow 0}\int_{\Gamma_\rho^{(k)}(\zeta^{(0)};H_0)}\Psi(0)=0,$

(III) $\lim_{\varepsilon_1\downarrow 0}\lim_{\varepsilon_2\downarrow 0}\int_{\Gamma_\rho^{(k)}(\zeta^{(0)};H_\infty)}(\Psi(\lambda)-\Psi_0(\lambda))=0,$

(IV) $\lim_{\varepsilon_1\downarrow 0}\lim_{\varepsilon_2\downarrow 0}\int_{\Gamma_\rho^{(k)}(\zeta^{(0)};H_\infty)}(\Psi_0(\lambda)-\Psi_0(0))=0.$

(I)–(IV) imply that

$$\int_{\partial M_{\varepsilon_1,\varepsilon_2}}\Psi(\lambda)=\int_{\partial M_{\varepsilon_1,\varepsilon_2}}\Psi(0)=1 \tag{5.17}$$

which proves Theorem 1.3.

6. Proof of the Theorem

In this section $C, C_0, C_1, C_2, \ldots$ will denote suitable positive constants. We seek to prove (I)–(IV) and thus Theorem 1.3.

6.1. PROOF OF (I) AND (I′): ESTIMATE ON $\Gamma_\rho^{(0)}(z^{(0)}; H_0)$

We fix $H_0 \in \mathcal{A}$ and $z^{(0)} \in V^{(0)} = \mathbf{C}^n$ such that $z^{(0)} \in H_0$. We denote by $L(\subset H_0)$ the subspace $\cap_{H\ni z^{(0)}} H$ which, we assume, has dimension $n-r$ $(1 \leqslant r \leqslant n)$. In this case $\mathcal{B}$ coincides with $\mathcal{A}_L$. We can choose the coordinates $z'_1, \ldots, z'_n$ as in Definition 3.1 such that $z'_1 = f_{H_0}(z)$.

In $\Gamma_\rho^{(0)}(z^{(0)}; H_0)$ we can write

$$\begin{aligned}(-1)^{\nu-1}&\mathrm{d}\bar{g}_1\wedge\cdots\mathrm{d}\bar{g}_{\nu-1}\wedge\mathrm{d}\bar{g}_{\nu+1}\wedge\cdots\wedge\mathrm{d}\bar{g}_n\wedge\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n\\ &=\psi^{(\nu)}(z)\mathrm{d}\bar{z}'_2\wedge\cdots\wedge\mathrm{d}\bar{z}'_n\wedge\mathrm{d}z'_1\wedge\cdots\wedge\mathrm{d}z'_n\end{aligned} \tag{6.1}$$

for a suitable function $\psi^{(\nu)}(z)$, since $\mathrm{d}\bar{z}'_1\wedge\mathrm{d}z'_1$ vanishes on $\Gamma_\rho^{(0)}(z^{(0)}; H_0)$. Substituting in (5.4) gives

$$\Psi(\lambda)=\frac{\psi(z)}{\|g\|^{2n}}\mathrm{d}\bar{z}'_2\wedge\cdots\wedge\mathrm{d}\bar{z}'_n\wedge\mathrm{d}z'_1\cdots\wedge\mathrm{d}z'_n \tag{6.2}$$

for $\psi(z)=\sum_{\nu=1}^n \bar{g}_\nu\psi^{(\nu)}(z)$.

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Let us denote by $|\Psi(\lambda)|_{H_0,z^{(0)}}$ the maximum of the absolute values $|\psi(z)|/\|g\|^{2n}$ on $\Gamma_\rho^{(0)}(z^{(0)}; H_0)$. We seek a bound on this quantity. Now

$$\mathrm{d}\bar{f}_{H_1}(z) \wedge \cdots \wedge \mathrm{d}\bar{f}_{H_s}(z) = 0 \tag{6.3}$$

if $s > r$ and $\{H_1, \ldots, H_s\} \subset \mathcal{B}$, while

$$|f_H(z)| \geqslant \delta \quad \text{for } H \notin \mathcal{B}, \tag{6.4}$$

$$\delta \geqslant |f_H(z)| \geqslant \varepsilon_2 \quad \text{for } H \in \mathcal{B}. \tag{6.5}$$

Also from (2.6) and (2.7)

$$\|g\|^2 \geqslant C_0 \varepsilon_2^{-2}, \tag{6.6}$$

$$|g_\nu| \leqslant C_1 \varepsilon_2^{-1}. \tag{6.7}$$

The results (6.3)–(6.7) imply

LEMMA 6.1. *ρ and δ being fixed*

$$|\Psi(\lambda)|_{H_0,z^{(0)}} \leqslant \mathrm{O}(\varepsilon_2^{2n-2r+1}). \tag{6.8}$$

The identity (I) follows. Since (6.8) holds independent of λ, (I′) follows also.

6.2. PROOF OF (II) AND (II′): ESTIMATE ON $\Gamma_\rho^{(k)}(\zeta^{(0)}; H_0)$

For definiteness we assume $k = 1$. Let $\zeta^{(0)} \in H_\infty$ and $\zeta^{(0)} \in H_0 \in \mathcal{A}$. With $\Gamma_\rho^{(k)}(\zeta^{(0)}; H_0)$ defined as in (5.11), choose the local affine coordinates $(\zeta_1, \ldots, \zeta_n)$ in $V_\delta^{(k)}(\mathcal{B})$ such that

$$|\zeta_1| \leqslant \delta, |\zeta_2| \leqslant 1, \cdots, |\zeta_n| \leqslant 1.$$

We denote by L the subspace $\cap_{H \in \mathcal{B}} H$, which we take to have dimension $n - r$. $\mathcal{B}$ coincides again with $\mathcal{A}_L$. Then the coordinates $(\zeta'_1, \cdots, \zeta'_n)$ are related to $(\zeta_1, \cdots, \zeta_n)$ as in (3.5) and (3.6). We may assume that $\tilde{f}_{H_0}(\zeta) = \zeta'_1$ and $\zeta_1 = \zeta'_{r+1}$. Furthermore, ζ_ν and $\tilde{f}_H(\zeta)$ can be written as

$$\zeta_\nu = \sum_{j=1}^{n} \tilde{e}_{j,\nu} \zeta'_j + \zeta_\nu^{(0)}, \tag{6.9}$$

$$\tilde{f}_H(\zeta) = \sum_{\nu=1}^{n} v_{H,\nu} \zeta'_\nu + v_{H,0} \tag{6.10}$$

for suitable real constants $\tilde{e}_{j,\nu}$ and $v_{H,\nu}$, $v_{H,0}$. Note that the $\tilde{e}_j = (\tilde{e}_{j,\nu})_{\nu=1}^n$ are tangent to

L for $r+1 \leqslant j \leqslant n$. Thus for $H \in \mathcal{B}$

$$\tilde{f}_H(\zeta) = \sum_{\nu=1}^{r} \nu_{H,\nu}\zeta'_\nu. \tag{6.11}$$

Next we introduce the new coordinates $(\zeta''_1, \ldots, \zeta''_n)$ blowing up the coordinates $(\zeta'_1, \ldots, \zeta'_n)$ along L according to

$$\zeta''_1 = \frac{\zeta'_1}{\zeta'_{r+1}}, \cdots, \qquad \zeta''_r = \frac{\zeta'_r}{\zeta'_{r+1}}, \qquad \zeta''_j = \zeta'_j\,(n \geqslant j \geqslant r+1). \tag{6.12}$$

Then from (5.11) we see that $\Gamma^{(1)}_\rho(\zeta^{(0)}; H_0)$ is included in the set

$$\begin{aligned} &|\zeta''_1| = \varepsilon_2, \qquad \varepsilon_1 \leqslant |\zeta''_{r+1}| \leqslant \delta, \qquad \varepsilon_2 \leqslant |\zeta''_j| \leqslant \delta/|\zeta''_{r+1}|\ (2 \leqslant j \leqslant r), \\ &|\zeta''_j| \leqslant K\,(r+2 \leqslant \nu \leqslant n), \end{aligned} \tag{6.13}$$

K being a suitable constant. Also, we can write

$$\begin{aligned} &(-1)^{\nu-1}\mathrm{d}\bar{g}_1 \wedge \cdots \mathrm{d}\bar{g}_{\nu-1} \wedge \mathrm{d}\bar{g}_{\nu+1} \wedge \cdots \wedge \mathrm{d}\bar{g}_n \\ &\quad \equiv \psi^{(\nu)}(\zeta'')\mathrm{d}\bar{\zeta}''_2 \wedge \cdots \wedge \mathrm{d}\bar{\zeta}''_n (\mathrm{mod}\ \mathrm{d}\bar{\zeta}''_1) \end{aligned} \tag{6.14}$$

so that

$$\Psi(\lambda) = \frac{\psi(\zeta)}{\|g\|^{2n}}\mathrm{d}\bar{\zeta}''_2 \wedge \cdots \wedge \mathrm{d}\bar{\zeta}''_n \wedge \mathrm{d}\zeta''_1 \wedge \cdots \wedge \mathrm{d}\zeta''_n \tag{6.15}$$

on $\Gamma^{(1)}_\rho(\zeta^{(0)}; H_0)$, $\psi(\zeta)$ being defined by $\sum_{\nu=1}^n \bar{g}_\nu \psi^{(\nu)}(\zeta)$. In fact this follows because $\mathrm{d}\bar{\zeta}_1{}'' \wedge \mathrm{d}\zeta''_1$ vanishes on $\Gamma^{(1)}_\rho(\zeta^{(0)}; H_0)$.

As is seen from (6.1)–(6.2), $\psi^{(\nu)}(\zeta)$ can be described as a polynomial in λ of degree $n-1$,

$$\psi^{(\nu)}(\zeta) = \psi^{(\nu)}_0(\zeta) + \sum_{p=1}^{n-1} \sum_{\{H_j\}_{j=1}^p \subset \mathcal{A}, H_j \neq H_0} \lambda_{H_1} \cdots \lambda_{H_p} \psi^{(\nu)}_{H_1,\ldots,H_p}(\zeta) \tag{6.16}$$

such that $\dim \cap_{j=1}^p H_j = n-p$, where $\psi^{(\nu)}_0(\zeta), \psi^{(\nu)}_{H_1,\ldots,H_p}(\zeta)$ do not depend on λ. The quantities $\psi^{(\nu)}_0(\zeta)$ and , $\psi^{(\nu)}_{H_1,\ldots,H_p}(\zeta)$ can be bounded above, as we will now show.

LEMMA 6.2. *Assume that* $H_1, \ldots, H_q \in \mathcal{B}$ *while* $H_{q+1}, \ldots, H_p \notin \mathcal{B}$ *for* $0 \leqslant p \leqslant n-1$. *Assume furthermore that each of* $H_1, \ldots, H_q$ *differs from* H_0. *Then*

$$\max_{\zeta \in \Gamma^{(1)}_\rho(\zeta^{(0)}; H_0)} |\psi^{(\nu)}_{H_1,\ldots,H_p}(\zeta)| \leqslant C|\zeta''_{r+1}|^{-n+r-1} \prod_{j=1}^{q} \left| \sum_{\nu=1}^{r} \nu_{H_j,\nu}\zeta''_\nu \right|^{-2} \tag{6.17}$$

for $0 \leqslant q \leqslant r-1$. *Similarly*

$$\max_{\zeta \in \Gamma^{(1)}_\rho(\zeta^{(0)}; H_0)} |\psi^{(\nu)}_0(\zeta)| \leqslant C|\zeta''_{r+1}|^{-n+r-1}. \tag{6.18}$$

Proof. In terms of the coordinates $\zeta_1'', \ldots, \zeta_n''$ ($\zeta_{r+1}'' = \zeta_1$), we have from (6.9) the expressions

$$d\left(\frac{1}{\bar\zeta_1}\right) = -\frac{1}{\bar\zeta_{r+1}''^2}\, d\bar\zeta_{r+1}'',$$

$$d\left(\frac{\bar\zeta_\nu}{\bar\zeta_1}\right) = -\frac{\bar\zeta_\nu}{\bar\zeta_{r+1}''^2}\, d\bar\zeta_{r+1}'' + \frac{1}{\bar\zeta_{r+1}''} \times \\ \times \left[\left(\sum_{j=1}^{r} \tilde e_{j,\nu}\bar\zeta_j''\right) d\bar\zeta_{r+1}'' + \left(\sum_{j=1}^{r} \tilde e_{j,\nu}\, d\bar\zeta_j''\right)\bar\zeta_{r+1}'' + \sum_{\mu=r+1}^{n} \tilde e_{\mu,\nu}\, d\bar\zeta_\mu''\right].$$

On the right-hand side the coefficients with respect to $d\bar\zeta_1'', \ldots, d\bar\zeta_r''$ are bounded, the one with respect to $d\bar\zeta_{r+1}''$ is of order $O(1/|\zeta_{r+1}''|^2)$, while the ones with respect to $d\bar\zeta_{r+2}'', \ldots, d\bar\zeta_n''$ are $O(1/|\zeta_{r+1}''|)$. On the other hand, from (6.10), we have

$$d\left(\frac{\bar\zeta_1}{\bar{\tilde f}_H(\zeta)}\right) \\ = \frac{d\bar\zeta_{r+1}''}{\bar{\tilde f}_H(\zeta)} - \frac{\bar\zeta_{r+1}''}{\bar{\tilde f}_H(\zeta)^2}\left[\left(\sum_{\nu=1}^{r} v_{H,\nu}\bar\zeta_\nu''\right) d\bar\zeta_{r+1}'' + \left(\sum_{\nu=1}^{r} v_{H,\nu}\, d\zeta_\nu''\right)\bar\zeta_{r+1}'' + \sum_{\mu=r+1}^{n} v_{H,\mu}\, d\bar\zeta_\mu''\right],$$

where $v_{H,\mu} = 0\,(r+1 \leqslant \mu \leqslant n)$ if $H \in \mathcal{B}$. When $H \notin \mathcal{B}$, in the right-hand side the coefficients with respect to $d\bar\zeta_1'', \ldots, d\bar\zeta_r''$ are of order $O(|\zeta_{r+1}''|^2)$, the one with respect to $d\bar\zeta_{r+1}''$ is bounded and the ones with respect to $d\bar\zeta_{r+2}'', \ldots, d\bar\zeta_n''$ are of order $O(|\zeta_{r+1}''|)$. On the contrary when $H \in \mathcal{B}$, the coefficients with respect to $d\bar\zeta_1'', \ldots, d\bar\zeta_r''$ are all $O(|\sum_{\nu=1}^{r} v_{H,\nu}\zeta_\nu''|^{-2})$ the one with respect to $d\bar\zeta_{r+1}''$ is of order $O(1/|\tilde f_H(\zeta)|)$. (6.17) and (6.18) then follow in view of Definition 3.2. □

We denote by

$$|\Psi(\lambda)|_{H_0,\zeta^{(0)}} = \max_{\zeta\in\Gamma_p^{(1)}(\zeta^{(0)};H_0)} \frac{|\psi(\zeta)|}{\|g\|^{2n}}, \tag{6.19}$$

$$|\Psi_0(\lambda)|_{H_0,\zeta^{(0)}} = \max_{\zeta\in\Gamma_p^{(1)}(\zeta^{(0)};H_0)} \frac{|\sum_{\nu=1}^{n} \bar g_\nu \psi_0^{(\nu)}(\zeta)|}{\|g\|^{2n}}, \tag{6.20}$$

$$|\Psi_{H_1,\ldots,H_p}(\lambda)|_{H_0,\zeta^{(0)}} = \max_{\zeta\in\Gamma_p^{(1)}(\zeta^{(0)};H_0)} \frac{|\sum_{\nu=1}^{n} \bar g_\nu \psi_{H_1,\ldots,H_p}^{(\nu)}(\zeta)|}{\|g\|^{2n}}. \tag{6.21}$$

To obtain bounds on these quantities, note from (2.8) and (2.9) that

$$\|g\|^2 \geqslant C_0\left(\sum_{H\in\mathcal{B},H\neq H_0}\frac{1}{|\sum_{\nu=1}^r v_{H,\nu}\zeta''_\nu|^2}+\frac{1}{|\zeta''_1|^2}+\frac{1}{|\zeta''_{r+1}|^2}\right)\geqslant C'_0\left(\varepsilon_2^{-2}+\frac{1}{|\zeta''_{r+1}|^2}\right),$$

$$|g_\nu| \leqslant C_1\left(\sum_{H\in\mathcal{B},H\neq H_0}\frac{1}{|\sum_{\nu=1}^r v_{H,\nu}\zeta''_\nu|}+\frac{1}{|\zeta''_1|}+\frac{1}{|\zeta''_{r+1}|}\right)\leqslant C'_1\left(\varepsilon_2^{-1}+\frac{1}{|\zeta''_{r+1}|}\right).$$

Using the inequality $m\sum_{j=1}^m x_j^2 \geqslant (\sum_{j=1}^m x_j)^2$ for $x_j\in\mathbf{R}$, we then have

$$\frac{|g_\nu|}{\|g\|^{2n}} \leqslant C_2\left(\sum_{H\in\mathcal{B},H\neq H_0}\frac{1}{|\sum_{\nu=1}^r v_{H,\nu}\zeta''_\nu|^2}+\frac{1}{|\zeta''_1|^2}+\frac{1}{|\zeta''_{r+1}|^2}\right)^{-n+\frac{1}{2}}.$$

To make use of these inequalities, we enlarge $\{H_0, H_1, \dots, H_q\}$ to a system of r hyperplanes $\{H_0, \dots, H_q, H^*_{q+1}, \dots, H^*_{r-1}\}$ such that it becomes possible to write $L = H_0 \cap \bigcap_{j=1}^q H_j \cap_{j=q+1}^{r-1} H^*_j$. Then we see that

$$\begin{aligned}
&|\Psi_{H_1,\dots,H_p}(\lambda)|_{H_0,\zeta^{(0)}}\\
&\quad\leqslant C_3|\zeta''_{r+1}|^{-1}\left(\sum_{H\in\mathcal{B},H\neq H_0}\frac{1}{|\sum_{\nu=1}^r v_{H,\nu}\zeta''_\nu|^2}+\frac{1}{|\zeta''_1|^2}+\frac{1}{|\zeta''_{r+1}|^2}\right)^{-r+1}\times \qquad (6.22)\\
&\quad\times\left(\prod_{j=1}^q|\sum_{\nu=1}^r v_{H_j,\nu}\zeta''_\nu|\right)^{-2}\\
&\quad\leqslant C_3|\zeta''_{r+1}|^{-1}\prod_{j=1}^q\left(\frac{|\sum_{\nu=1}^r v_{H_j,\nu}\zeta''_\nu|^2}{|\zeta''_{r+1}|^2}+1\right)^{-1}\times\\
&\quad\times\prod_{j=q+1}^{r-1}\left(\frac{1}{|\sum_{\nu=1}^r v_{H^*_j,\nu}\zeta''_\nu|^2}+\frac{1}{|\zeta''_1|^2}+\frac{1}{|\zeta''_{r+1}|^2}\right)^{-1}. \qquad (6.23)
\end{aligned}$$

Similarly

$$|\Psi_0(\lambda)|_{H_0,\zeta^{(0)}} \leqslant C_3|\zeta''_{r+1}|^{-1}\prod_{j=1}^{r-1}\left(\frac{1}{|\sum_{\nu=1}^r v_{H^*_j,\nu}\zeta''_\nu|^2}+\frac{1}{|\zeta''_1|^2}+\frac{1}{|\zeta''_{r+1}|^2}\right)^{-1} \qquad (6.24)$$

since $p = q = 0$ in this case.

To establish (II) and (II′) we must estimate the integrals of $\Psi_{H_1,\dots,H_p}(\lambda)$ and $\Psi_0(\lambda)$ with respect to $d\zeta''_2, \dots, d\zeta''_r, d\bar{\zeta}''_2, \dots, d\bar{\zeta}''_r$, with ζ''_1 being fixed such that $|\zeta''_1| = \varepsilon_2$. The following lemma is useful for this purpose.

LEMMA 6.3. *We denote by* $|d\bar{\tau} \wedge d\tau|$ *the positive measure on* $\mathbf{C}(\tau \in \mathbf{C})$ *defined by the 2-form* $d\bar{\tau} \wedge d\tau$. *We have*

$$\int_{\varepsilon_2 \leqslant |\tau| \leqslant \delta/|\zeta''_{r+1}|} |\mathrm{d}\bar{\tau} \wedge \mathrm{d}\tau| \left(\frac{1}{1 + \frac{|\tau|^2}{|\zeta''_{r+1}|^2}} \right) \leqslant \pi |\zeta''_{r+1}|^2 \log\left(1 + \frac{\delta^2}{|\zeta''_{r+1}|^4}\right),$$
$$\int_{\varepsilon_2 \leqslant |\tau| \leqslant \delta/|\zeta''_{r+1}|} \frac{|\mathrm{d}\bar{\tau} \wedge \mathrm{d}\tau|}{\frac{1}{|\tau|^2} + \frac{1}{|\zeta''_{r+1}|^2}} \leqslant \pi\delta^2.$$

It follows from (6.23) and Lemma 6.3 that the integral of $|\bar{g}_\nu \psi^{(\nu)}_{H_1,\ldots,H_p}(\zeta)|/\|g\|^{2n}$ over $\Gamma^{(1)}_\rho(\zeta^{(0)}; H_0) \cap \{\zeta''_1; |\zeta''_1| = \varepsilon_2\}$ has a majorization

$$\begin{aligned} &\int_{\Gamma^{(1)}_\rho(\zeta^{(0)};H_0)\cap\{|\zeta''_1|=\varepsilon_2\}} \frac{|\bar{g}_\nu \psi^{(\nu)}_{H_1,\ldots,H_p}(\zeta)|}{\|g\|^{2n}} |\mathrm{d}\bar{\zeta}''_2 \wedge \cdots \wedge \mathrm{d}\bar{\zeta}''_n \wedge \mathrm{d}\zeta''_2 \wedge \cdots \wedge \mathrm{d}\zeta''_n| \\ &\quad \leqslant C\delta^{2(r-1-q)} \int \frac{1}{|\zeta''_{r+1}|} \left\{ |\zeta''_{r+1}|^2 \log\left(1 + \frac{\delta^2}{|\zeta''_{r+1}|^4}\right)\right\}^q \times \\ &\qquad \times |\mathrm{d}\bar{\zeta}''_{r+1} \wedge \cdots \wedge \mathrm{d}\bar{\zeta}''_n \wedge \mathrm{d}\zeta''_{r+1} \wedge \cdots \wedge \mathrm{d}\zeta''_n|. \end{aligned} \tag{6.25}$$

The last integral is done over the region $\varepsilon_1 \leqslant |\zeta''_{r+1}| \leqslant \delta, |\zeta''_{r+2}| \leqslant 1, \ldots, |\zeta''_n| \leqslant 1$. Its value is bounded by a constant which does not depend on either ε_1 or ε_2. Similarly,

$$\int_{\Gamma^{(1)}_\rho(\zeta^{(0)};H_0)\cap\{|\zeta''_1|=\varepsilon_2\}} \frac{|\bar{g}_\nu \psi^{(\nu)}_0(\zeta)|}{\|g\|^{2n}} |\mathrm{d}\bar{\zeta}''_2 \wedge \cdots \wedge \mathrm{d}\bar{\zeta}''_n \wedge \mathrm{d}\zeta''_2 \wedge \cdots \wedge \mathrm{d}\zeta''_n| \leqslant C\delta^{2(r-1)}. \tag{6.26}$$

Recalling (6.20) and (6.21), the inequalities (6.25) and (6.26) give

$$\left| \int_{\Gamma^{(1)}_\rho(\zeta^{(0)};H_0)} \Psi(\lambda) \right| = \mathrm{O}(\varepsilon_2)$$

which implies (II) and (II′) for $k = 1$.

6.3. PROOF OF (III): ESTIMATE OF THE INTEGRAL $\Psi_{H_1,\ldots,H_p}(\lambda)$ $(1 \leqslant p \leqslant n-1)$ OVER $\Gamma^{(k)}_\rho(\zeta^{(0)}; H_\infty)$

We again assume $k = 1$. Let $(\zeta'_1, \ldots, \zeta'_n)$ be the coordinates as in Definition 3.3. We denote by L' the subspace $L' = \cap_{H \ni \zeta^{(0)}} H$ which is assumed to have dimension $n - r$. Then $\mathcal{B}$ coincides with $\mathcal{A}_{L'}$. We put $L = L' \cap H_\infty$.

Now $\Gamma^{(1)}_\rho(\zeta^{(0)}; H_\infty)$ is defined by (5.12). By introducing new coordinates $(\zeta''_1, \ldots, \zeta''_n)$ according to

$$\zeta''_1 = \zeta'_1/\zeta'_{r+1} = \zeta_1, \ldots, \zeta''_r = \zeta'_r/\zeta'_{r+1}, \zeta''_{r+1} = \zeta'_{r+1}, \ldots, \zeta''_n = \zeta'_n,$$

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we see that $\Gamma_\rho^{(1)}(\zeta^{(0)}; H_\infty)$ is contained in

$$|\zeta''_{r+1}| = \varepsilon_1, K^{-1}\varepsilon_2 \leqslant |\zeta''_j| \leqslant K\delta/\varepsilon_1 \ (1 \leqslant j \leqslant r), |\zeta''_{r+2}| \leqslant K, \cdots, |\zeta''_n| \leqslant K$$

for a suitable positive constant K. Also, analogous to (6.14) and (6.15), $\Psi_{H_1,\ldots,H_p}(\lambda)$ can be expressed as

$$\Psi_{H_1,\ldots,H_p}(\lambda) \equiv \frac{\psi_{H_1,\ldots,H_p}(\zeta)}{\|g\|^{2n}} \mathrm{d}\bar{\zeta}''_1 \wedge \cdots \wedge \mathrm{d}\bar{\zeta}''_r \wedge \mathrm{d}\bar{\zeta}''_{r+2} \wedge \cdots \wedge \mathrm{d}\bar{\zeta}''_n \wedge \\ \wedge \mathrm{d}\zeta''_1 \wedge \cdots \wedge \mathrm{d}\zeta''_n (\mathrm{mod}\ \mathrm{d}\bar{\zeta}''_{r+1})$$

where $\psi_{H_1,\ldots,H_p}(\zeta)$ denotes $\sum_{\nu=1}^n \bar{g}_\nu \psi^{(\nu)}_{H_1,\ldots,H_p}(\zeta)$ as in (6.15).

We denote by

$$|\Psi(\lambda)|_{H_\infty,\zeta^{(0)}} = \max_{\zeta\in\Gamma_\rho^{(1)}(\zeta^{(0)};H_\infty)} \frac{|\psi(\zeta)|}{\|g\|^{2n}},$$

$$|\Psi_{H_1,\ldots,H_p}(\lambda)|_{H_\infty,\zeta^{(0)}} = \max_{\zeta\in\Gamma_\rho^{(1)}(\zeta^{(0)};H_\infty)} \frac{|\psi_{H_1,\ldots,H_p}(\zeta)|}{\|g\|^{2n}},$$

and seek bounds for these quantities. First, from (2.7) and (2.9), we deduce the inequalities

$$\|g\|^2 \geqslant C_0\left(\sum_{H\in\mathcal{B}} \frac{1}{|\sum_{\nu=1}^r \nu_{H,\nu}\zeta''_\nu|^2} + \varepsilon_1^{-2}\right), \tag{6.27}$$

$$|g_\nu| \leqslant C_1\left(\sum_{H\in\mathcal{B}} \frac{1}{|\sum_{\nu=1}^r \nu_{H,\nu}\zeta''_\nu|} + \varepsilon_1^{-1}\right). \tag{6.28}$$

Next, we deduce the analogue of Lemma 6.2.

LEMMA 6.4. *Assume that* $H_1, \ldots, H_q \in \mathcal{B}$ *and that* $H_{q+1}, \ldots, H_p \notin \mathcal{B}$ $(0 \leqslant q \leqslant r)$. *Then*

$$\max_{\zeta\in\Gamma_\rho^{(1)}(\zeta^{(0)},H_0)} |\psi^{(\nu)}_{H_1,\ldots,H_p}(\zeta)| \leqslant C\varepsilon_1^{2(p-q)-n+r+1} \prod_{j=1}^q \left|\sum_{\nu=1}^r \nu_{H_j,\nu}\zeta''_\nu\right|^{-2}. \tag{6.29}$$

Proof. Since $\zeta''_{r+1} = \zeta_1$, we have

$$\mathrm{d}\left(\frac{1}{\bar{\zeta}_1}\right) \equiv 0 (\mathrm{mod}\ \mathrm{d}\bar{\zeta}''_{r+1}),$$

$$\mathrm{d}\left(\frac{\bar{\zeta}_\nu}{\bar{\zeta}_1}\right) \equiv \sum_{j=1}^r \tilde{e}_{j,\nu}\mathrm{d}\bar{\zeta}''_j + \sum_{\mu=r+2}^n \tilde{e}_{\mu,\nu}\mathrm{d}\bar{\zeta}''_\mu/\bar{\zeta}''_{r+1} (\mathrm{mod}\ \mathrm{d}\bar{\zeta}''_{r+1}).$$

On the right-hand side the coefficients with respect to $\mathrm{d}\bar{\zeta}''_1, \ldots, \mathrm{d}\bar{\zeta}''_r$ are bounded, while the one with respect to $\mathrm{d}\bar{\zeta}''_{r+2}, \ldots, \mathrm{d}\bar{\zeta}''_n$ are $\mathrm{O}(1/\varepsilon_1)$. Moreover, if $H \in \mathcal{B}$, we have in view of (6.12)

$$\mathrm{d}\overline{\left(\frac{\zeta_1}{\tilde{f}_H(\zeta)}\right)} \equiv -\frac{\overline{\zeta}''_{r+1}}{\overline{\tilde{f}_H(\zeta)}^2}\left[\sum_{\nu=1}^{r} v_{H,\nu}\overline{\zeta''_{r+1}\mathrm{d}\zeta''_{\nu}}\right] (\mathrm{mod}\ \mathrm{d}\overline{\zeta''_{r+1}}),$$

whence the coefficients w.r.t. $\mathrm{d}\overline{\zeta''_1}, \ldots, \mathrm{d}\overline{\zeta''_r}$ are of order at most $\mathrm{O}(|\sum_{\nu=1}^{r} v_{H,\nu}\zeta''_\nu|^{-2})$. On the other hand, if $H \notin \mathcal{B}$, we have in view of (6.11)

$$\mathrm{d}\overline{\left(\frac{\zeta_1}{\tilde{f}_H(\zeta)}\right)} \equiv -\frac{\overline{\zeta}''_{r+1}}{\overline{\tilde{f}_H(\zeta)}^2}\left[\sum_{\nu=1}^{r} v_{H,\nu}\overline{\zeta''_{r+1}\mathrm{d}\zeta''_{\nu}} + \sum_{\nu=r+2}^{n} v_{H,\nu}\mathrm{d}\overline{\zeta''_{\nu}}\right] (\mathrm{mod}\ \mathrm{d}\overline{\zeta''_{r+1}}).$$

In the right-hand side the coefficients w.r.t. $\mathrm{d}\overline{\zeta}''_1, \ldots, \mathrm{d}\overline{\zeta}''_r$ are of order at most $\mathrm{O}(\varepsilon_1^2)$, while the ones w.r.t. $\mathrm{d}\overline{\zeta''_{r+2}}, \ldots, \mathrm{d}\overline{\zeta''_n}$ are of order $\mathrm{O}(\varepsilon_1)$. □

Now, from (6.27) and (6.28) we have

$$\frac{|g_\nu|}{\|g\|^{2n}} \leqslant C_2\left(\sum_{H\in\mathcal{B}}\frac{1}{|\sum_{\nu=1}^{r} v_{H,\nu}\zeta''_\nu|^2} + \frac{1}{\varepsilon_1^2}\right)^{-n+\frac{1}{2}}.$$

Hence, due to Lemma 6.4,

$$|\Psi_{H_1,\ldots,H_p}(\lambda)|_{H_\infty,\zeta^{(0)}} \leqslant C_2\varepsilon_1{}^{2(r-q)+2(p-q)-1}\prod_{j=1}^{q}\left(1+\frac{|\sum_{\nu=1}^{r} v_{H_j,\nu}\zeta''_\nu|^2}{\varepsilon_1^2}\right)^{-1}. \tag{6.30}$$

The final step is to use (6.30) to estimate the integral

$$\int_{\Gamma_p^{(1)}(\zeta^{(0)};H_\infty)} \Psi_{H_1,\ldots,H_p}(\lambda).$$

The analogue of Lemma 6.3 is required.

LEMMA 6.5

$$\int_{K^{-1}\varepsilon_2 \leqslant |\tau| \leqslant K\delta/\varepsilon_1} \frac{|\mathrm{d}\tau\wedge\mathrm{d}\overline{\tau}|}{\frac{|\tau|^2}{\varepsilon_1^2}+1} \leqslant \pi\varepsilon_1^2\log(1+K^2\delta^2/\varepsilon_1^4),$$

$$\int_{K^{-1}\varepsilon_2 \leqslant |\tau| \leqslant K\delta/\varepsilon_1} |\mathrm{d}\overline{\tau}\wedge\mathrm{d}\tau| \leqslant \pi K^2(\delta/\varepsilon_1)^2$$

In (6.30) we may assume that $\sum_{\nu=1}^{r} v_{H_1,\nu}\zeta'_\nu, \ldots, \sum_{\nu=1}^{r} v_{H_q,\nu}\zeta'_\nu$ as well as some $n-q$ elements among $\zeta''_1, \ldots, \zeta''_n$ are linearly independent. Fixing ζ''_{r+1} such that $|\zeta''_{r+1}| = \varepsilon_1$, the preceding Lemma can now be applied to estimate the following integral with respect to $\mathrm{d}\zeta''_1, \ldots, \mathrm{d}\zeta''_r, \mathrm{d}\overline{\zeta}''_1, \ldots, \mathrm{d}\overline{\zeta}''_r$, and then carry it out with respect to

$d\zeta''_{r+1}, \dots, d\zeta''_n, d\bar{\zeta}''_{r+2}, \dots, d\bar{\zeta}''_n$. We have

$$\begin{aligned}
&\left|\int_{\Gamma^{(1)}_\rho(\zeta^{(0)};H_\infty)} \frac{|\psi_{H_1,\dots,H_p}(\lambda)|}{\|g\|^{2n}} \overline{|d\zeta''_1 \wedge \cdots d\zeta''_r \wedge d\zeta''_{r+2} \wedge \cdots d\zeta''_n} \wedge d\zeta''_1 \wedge \cdots \wedge d\zeta''_n\right| \\
&\quad \leqslant C_2 \varepsilon_1^{2(p-q)-1} \{\varepsilon_1^2 \log(1 + K^2\delta^2/\varepsilon_1^4)\}^q \times \\
&\qquad \times \int_{\substack{|\zeta''_{r+1}|=\varepsilon_1, \\ |\zeta''_j| \leqslant K, (j=r+2,\dots,n)}} \left|\overline{d\zeta''_{r+2} \wedge \cdots \wedge d\zeta''_n} \wedge d\zeta''_{r+1} \wedge \cdots \wedge d\zeta''_n\right| \\
&\quad \leqslant C_3 \varepsilon_1^{2(p-q)} \{\varepsilon_1^2 \log(1 + K^2\delta^2/\varepsilon_1^4)\}^q. \qquad (6.31)
\end{aligned}$$

Hence for $p \geqslant 1$

$$\lim_{\varepsilon_2 \downarrow 0} \lim_{\varepsilon_1 \downarrow 0} \int_{\Gamma^{(1)}_\rho(\zeta^{(0)};H_\infty)} \Psi_{H_1,\dots,H_p}(\lambda) = 0.$$

As a consequence,

$$\lim_{\varepsilon_2 \downarrow 0} \lim_{\varepsilon_1 \downarrow 0} \int_{\Gamma^{(1)}_\rho(\zeta^{(0)};H_\infty)} (\Psi(\lambda) - \Psi_0(\lambda)) = 0,$$

which is the identity (III).

6.4. PROOF OF (IV)

To make explicit the dependence of g_ν on λ, here we write $g_\nu(\cdot|\lambda)$ in place of g_ν. On $\Gamma^{(1)}_\rho(\zeta^{(0)}; H_\infty)$, we can write the difference $\Psi_0(\lambda) - \Psi_0(0)$ in the form

$$\Psi_0(\lambda) - \Psi_0(0) = \chi(\zeta)\overline{d\zeta_2 \wedge \cdots \wedge d\zeta_n} \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n,$$

where $\chi(\zeta) = (\|g(\cdot|\lambda)\|^{-2n} - \|g(\cdot|0)\|^{-2n})\bar{\zeta}_1^{-n}\zeta_1^{-n-1}$. We put

$$\|\Psi_0(\lambda) - \Psi_0(0)\|_{\zeta^{(0)},H_\infty} = \mathrm{Max}_{\zeta \in \Gamma^{(1)}_\rho(\zeta^{(0)};H_\infty)} |\chi(\zeta)|.$$

Since

$$g_\nu(\cdot|\lambda) - g_\nu(\cdot|0) = -\sum_{H \in \mathcal{A}} \frac{\lambda_H u_{H,\nu}\zeta_1}{\tilde{f}_H(\zeta)},$$

we have

$$\begin{aligned}
\|g(\cdot|\lambda) - g(\cdot|0)\| &\leqslant C_4 \left(\sum_{\zeta^{(0)} \in H} \frac{|\zeta_1|}{|\tilde{f}_H(\zeta)|} + |\zeta_1| \right) \\
&= C_4 \varepsilon_1 \left(\sum_{\zeta^{(0)} \in H} \frac{1}{|\tilde{f}_H(\zeta)|} + 1 \right).
\end{aligned}$$

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On the other hand, from (2.8),

$$\|g(\cdot|\lambda)\|^2 \geqslant C_0\frac{1}{|\zeta_1|^2}, \qquad \|g(\cdot|0)\|^2 \geqslant C_0\frac{1}{|\zeta_1|^2}.$$

Therefore

$$\|\Psi_0(\lambda)-\Psi_0(0)\|_{H_\infty,\zeta^{(0)}} \leqslant C_5\varepsilon_1\left(\sum_{\zeta^{(0)}\in H}\frac{1}{|\tilde{f}_H(\zeta)|}+1\right).$$

Hence

$$\left|\int_{\Gamma_\rho^{(1)}(\zeta^{(0)};H_\infty)}(\Psi_0(\lambda)-\Psi_0(0))\right| \leqslant C_5\varepsilon_1^2$$

because, ζ_1 being fixed such that $|\zeta_1|=\varepsilon_1$, the function $\sum_{\zeta^{(0)}\in H}\frac{1}{|\tilde{f}_H(\zeta)|}+1$ is summable with respect to the positive measure on $\Gamma_\rho^{(1)}(\zeta^{(0)};H_\infty)\cap\{\zeta_1;\,|\zeta_1|=\varepsilon_1\}$ attached to the form $\overline{\mathrm{d}\zeta_2\wedge\cdots\wedge\mathrm{d}\zeta_n}\wedge\mathrm{d}\zeta_2\wedge\cdots\wedge\mathrm{d}\zeta_n$. The identity (IV) follows.

Since each of the identities (I)–(IV) have now been established, (5.17) follows and Theorem 1.3 is thus now proved.

7. Application

We fix a chamber $\triangle_j$ in $M(\mathcal{A})\cap\mathbf{R}^n$. Given an arbitrary point $w=(w_1,\ldots,w_n)\in\mathbf{R}^n$, there exists the unique $x=(x_1,\ldots,x_n)\in\triangle_j$ such that $g_1(x)=\cdots=g_n(x)=0$, which is to say the mapping

$$T_j : w_\nu = x_\nu - \sum_{H\in\mathcal{A}}\frac{\lambda_H u_{H,\nu}}{f_H(x)} \quad 1\leqslant\nu\leqslant n \tag{7.1}$$

from $\triangle_j$ onto $\mathbf{R}^n$ is a diffeomorphism. Theorem 1.3 can be used prove a integration formula for this change of variables.

PROPOSITION 7.1. *Assume that $f(x)$ is a summable function on $\mathbf{R}^n$ and that $g(w)=f(T_j^{-1}(w))$ does not depend on j. Then*

$$\int_{\mathbf{R}^n} f(x)\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n = \int_{\mathbf{R}^n} g(w)\mathrm{d}w_1\wedge\cdots\wedge\mathrm{d}w_n. \tag{7.2}$$

Proof. According to (1.8) we have

$$\sum_{j=1}^{\kappa}\left[\det\left(\frac{\partial w_\nu}{\partial x_\mu}\right)_{\mu,\nu=1}^{n}\right]^{-1}_{x=T_j^{-1}(w)} = 1. \tag{7.3}$$

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Hence

$$\begin{aligned}\int_{\mathbf{R}^n} f(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n &= \sum_{j=1}^{\kappa} \int_{\triangle_j} f(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\ &= \sum_{j=1}^{\kappa} \int_{\mathbf{R}^n} \frac{f(T_j^{-1}(w))}{[\det(\frac{\partial w_\nu}{\partial x_\mu})^n_{\mu,\nu=1}]_{x=T_j^{-1}(w)}} \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_n \\ &= \int_{\mathbf{R}^n} g(w)\mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_n\end{aligned}$$

as required. □

In the case $n = 1$, setting $f(x) = h(T(x))$ in (7.2) gives

$$\int_{-\infty}^{\infty} h\left(x - \sum_{j=1}^{p} \frac{\lambda_j}{x - a_j}\right) \mathrm{d}x = \int_{-\infty}^{\infty} h(w)\,\mathrm{d}w \quad (\lambda_j > 0, a_j \in \mathbf{R}).$$

This formula was first obtained by G. Boole [5] in the nineteenth century, and has also been considered in more recent times [9]. For general n some special cases of (7.2) have been conjectured in [6], and some explicit examples given assuming the validity of the conjecture.

As an example of an explicit integration formula which follows from (7.2), let $\mathcal{A}$ be the central hyperplane arrangement attached to the A type root system. $\mathcal{A}$ is invariant under the permutation group of nth degree, and κ is equal to $n!$. We assume that λ_H *are all equal to the same* λ_0 *(*$\lambda_0 > 0$*)*. The mapping T_j is given by

$$w_\nu = x_\nu - \lambda_0 \sum_{\mu \neq \nu, \mu=1}^{n} \frac{1}{x_\nu - x_\mu}.$$

We take as

$$g(w) = \varphi(w_1^2 + \cdots + w_n^2) \tag{7.4}$$

for a one variable function φ. Since

$$\sum_{\nu=1}^{n} w_\nu^2 = \sum_{\nu=1}^{n} x_\nu^2 + \lambda_0^2 \sum_{\nu=\mu=1, \nu\neq\mu}^{n} \frac{1}{(x_\nu - x_\mu)^2} - n(n-1)\lambda$$

we have from (7.2)

$$\begin{aligned}&\int_{\mathbf{R}^n} \varphi\left(\sum_{\nu=1}^{n} x_\nu^2 + \lambda_0^2 \sum_{\nu=\mu=1, \nu\neq\mu}^{n} \frac{1}{(x_\nu - x_\mu)^2} - n(n-1)\lambda_0\right) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\ &\quad = \int_{\mathbf{R}^n} \varphi(w_1^2 + \cdots + w_n^2)\mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_n.\end{aligned} \tag{7.5}$$

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In particular,

$$\int_{\mathbf{R}^n} \exp\left(-\sum_{\nu=1}^{n} x_\nu^2 - \lambda_0^2 \sum_{\nu=\mu=1,\nu\neq\mu}^{n} \frac{1}{(x_\nu - x_\mu)^2}\right) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = \mathrm{e}^{-n(n-1)\lambda_0} \pi^{n/2}. \tag{7.6}$$

This was first obtained by G. Gallavotti and C. Marchioro [8] using the semi-classical limit formula for the Schrödinger equation corresponding to the Calogero-Sutherland model and also by Francoise [7] using geometric argument on integrable Hamiltonian flows of the same model. The latter author has also extended the formula (7.6) to the B type root system, which is also a special case of (7.3). Indeed with the mapping T_j given by

$$w_\nu = x_\nu - \lambda_0 \sum_{\mu\neq\nu,\mu=1}^{n} \left(\frac{1}{x_\nu - x_\mu} + \frac{1}{x_\nu + x_\mu}\right) - \frac{\lambda_1}{x_\nu}$$

$(\lambda_0, \lambda_1 > 0)$ we have

$$\sum_{\nu=1}^{n} w_\nu^2 = \sum_{\nu=1}^{n} \left(x_\nu^2 + \frac{\lambda_1^2}{x_\nu^2}\right) - 2\lambda_1 N - 2\lambda_0 N(N-1) + \\ + \lambda_0^2 \sum_{\nu=\mu=1,\nu\neq\mu}^{n} \left(\frac{1}{(x_\nu - x_\mu)^2} + \frac{1}{(x_\nu + x_\mu)^2}\right),$$

and so from (7.3) with g given by (7.4) we obtain

$$\int_{\mathbf{R}^n} \varphi\left(\sum_{\nu=1}^{n}\left(x_\nu^2 + \frac{\lambda_1^2}{x_\nu^2}\right) + \lambda_0^2 \sum_{\nu=\mu=1,\nu\neq\mu}^{n} \left(\frac{1}{(x_\nu - x_\mu)^2} + \frac{1}{(x_\nu + x_\mu)^2}\right) - \\ - 2\lambda_1 N - 2\lambda_0 N(N-1)\right) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\ = \int_{\mathbf{R}^n} \varphi(w_1^2 + \cdots + w_n^2) \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_n.$$

In the special case $\varphi(x) = \mathrm{e}^{-x}$ this reduces to the formula presented in [7].

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