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SOME FURTHER EXTENSIONS OF HARDY'S INEQUALITY

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1. **Introduction.** Let $p>1$, $r\neq 1$, and let $f(x)$ be a non-negative function defined in $[0,\infty)$. The following inequality is due to G. H. Hardy [5, Ch. IX]:

$$\int_0^\infty x^{-r}F^p(x)\,dx \le \left(\frac{p}{|r-1|}\right)^p \int_0^\infty x^{-r}(xf(x))^p\,dx, \tag{1.1}$$

where $F(x)=\int_0^x f(t)\,dt$ or $=\int_x^\infty f(t)\,dt$ according as $r>1$ or $r<1$.

This inequality has important applications in analysis, especially in the study of Fourier series, and has been generalized in various directions by a number of authors (see for example, [1]–[3], [6]–[9]). The case when $p<0$ has also been discussed, for example, in [1].

It is easy to see that (1.1) breaks down when $r=1$, as in this case the left hand side of (1.1) is infinite unless $f(x)$ is almost everywhere zero, while the integral on the right hand side may be finite. Recently, on splitting $[0,\infty)$, the interval of integration, into $[0,1]$ and $[1,\infty)$, the author [3] has proved the following four corresponding inequalities for $r=1$ and $p>1$:

$$\int_1^\infty x^{-1}\left(\int_x^\infty f(t)\,dt\right)^p dx \le p^p \int_1^\infty x^{-1}(x\log x\, f(x))^p\,dx, \tag{1.2}$$

$$\int_0^1 x^{-1}\left(\int_0^x f(t)\,dt\right)^p dx \le p^p \int_0^1 x^{-1}(x(-\log x)f(x))^p\,dx, \tag{1.3}$$

$$\int_1^\infty x^{-1}\left(\int_1^x f(t)\,dt/\log x\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_1^\infty x^{-1}(xf(x))^p\,dx, \tag{1.4}$$

$$\int_0^1 x^{-1}\left(\int_x^1 f(t)\,dt/(-\log x)\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^1 x^{-1}(xf(x))^p\,dx. \tag{1.5}$$

The object of this paper is to obtain four-fold generalizations of (1.2)–(1.5), in which the Lebesgue integral is replaced by Lebesgue–Stieltjes integrals, the factor log is replaced by $\log^r$, the power p on the left hand sides is replaced by q and the range $1<p<\infty$ is extended to $-\infty<p<\infty$ $(p\neq 0)$. Namely, we shall

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prove inequalities such as the following:

$$(1.6)\quad \int_1^\infty g^{-1}(x)(\log g(x))^{-r}F^q(x)\,dg(x) \le A\left\{\int_1^\infty g^{p-1}(x)(\log g(x))^{[(q-r+1)p/q]-1}f^p(x)\,dg(x)\right\}^{q/p},$$

and

$$(1.7)\quad \int_0^1 g^{-1}(x)(|\log g(x)|)^{-r}F^q(x)\,dg(x) \le A\left\{\int_0^1 g^{p-1}(x)(|\log g(x)|)^{[(q-r+1)p/q]-1}f^p(x)\,dg(x)\right\}^{q/p},$$

where $g^{-1}(x)$ denotes $(g(x))^{-1}$, $F(x)$ is an integral of $f(x)$ and $A>0$ depends on p, q and r only. In fact, as in [3], we shall prove more precise inequalities in which the ranges of integration are sub-intervals of $[1,\infty)$ and $[0,1]$, which reduce to (1.6) and (1.7) on passing to the limits.

2. **Main results.** Throughout this paper we let p, q, and r be real numbers, $A=(|q/(r-1)|)^{(p-1)q/p}(|p/(r-1)|)$ and $\delta=[(q-r+1)p/q]-1$ (provided that these quantities are finite). We let $f(x)$ be a non-negative measurable function defined on $[0,\infty)$, and let $g(x)$ be a continuous non-decreasing function defined in $[0,\infty)$, such that $g(0)=0$, $g(x)\ne 0$ when $x\ne 0$, $g(1)=1$, $g(x)\ne 1$ when $x\ne 1$ and $g(\infty)=\infty$. We shall also let, provided that the integrals in question exist,

$$F_i(x)=\int_{E_i} f(t)\,dg(t),\qquad \theta_i(x)=\int_{E_i} g^{p-1}(t)f^p(t)(|\log g(t)|)^{\delta+[(r-1)/q]}\,dg(t),$$

where $i=1,2,3,4$, and E_i's are intervals in $[0,\infty)$ defined as follows:

$$E_1=[x,\infty)(1\le x<\infty),\qquad E_2=[1,x](1<x<\infty),$$
$$E_3=[0,x](0<x\le 1)\quad\text{and}\quad E_4=[x,1](0\le x<1).$$

THEOREM 1. *For $1\le p\le q<\infty$ or $-\infty<q\le p<0$, $r\ne 1$, we have*

$$(2.1)\quad \int_1^\infty g^{-1}(x)(\log g(x))^{-r}F_i^q(x)\,dg(x)\le A\left\{\int_1^\infty g^{p-1}(x)(\log g(x))^{\delta}f^p(x)\,dg(x)\right\}^{q/p},$$

where $i=1$ when $(r-1)/q<0$ and $i=2$ when $(r-1)/q>0$.

More precisely, if the integral on the right hand side of (2.1) is finite, then $\theta_i(x)$ ($i=1$ when $(r-1)/q<0$ and $i=2$ when $(r-1)/q>0$) is finite for every $x\in(1,\infty)$, $(\log g(x))^{(1-r)/q}\theta_i(x)\to 0$ as $x\to 1+$ and as $x\to\infty$; in this case, for

$1 \le c \le \infty$ we have

$$\int_1^c g^{-1}(x)(\log g(x))^{-r} F_1^q(x)\, dg(x) \le A\left\{\int_1^c g^{p-1}(x)(\log g(x))^{\delta} f^p(x)\, dg(x) + (\log g(c))^{(1-r)/q}\theta_1(c)\right\}^{q/p} \tag{2.2}$$

when $(r-1)/q < 0$, and

$$\int_c^\infty g^{-1}(x)(\log g(x))^{-r} F_2^q(x)\, dg(x) \le A\left\{\int_c^\infty g^{p-1}(x)(\log g(x))^{\delta} f^p(x)\, dg(x) + (\log g(c))^{(1-r)/q}\theta_2(c)\right\}^{q/p} \tag{2.3}$$

when $(r-1)/q > 0$, where $(\log g(c))^{(1-r)/q}\theta_i(c)$ ($i = 1$ when $(r-1)/q < 0$ and $i = 2$ when $(r-1)/q > 0$) at $c = 1$ and at $c = \infty$ are interpreted respectively as their limits as $c \to 1+$ and as $c \to \infty$.

THEOREM 2. *For* $1 \le p \le q < \infty$ *or* $-\infty < q \le p < 0$, $r \ne 1$, *we have*

$$\int_0^1 g^{-1}(x)(-\log g(x))^{-r} F_i^q(x)\, dg(x) \le A\left\{\int_0^1 g^{p-1}(x)(-\log g(x))^{\delta} f^p(x)\, dg(x)\right\}^{q/p}, \tag{2.4}$$

where $i = 3$ *when* $(r-1)/q < 0$ *and* $i = 4$ *when* $(r-1)/q > 0$.

More precisely, if the integral on the right hand side of (2.4) is finite, then $\theta_i(x)$ ($i = 3$ when $(r-1)/q < 0$ and $i = 4$ when $(r-1)/q > 0$) is finite for every $x \in (0, 1)$, $(-\log g(x))^{(1-r)/q}\theta_i(x) \to 0$ as $x \to 0+$ and as $x \to 1-$; in this case, for $0 \le c \le 1$ we have

$$\int_c^1 g^{-1}(x)(-\log g(x))^{-r} F_3^q(x)\, dg(x) \le A\left\{\int_c^1 g^{p-1}(x)(-\log g(x))^{\delta} f^p(x)\, dg(x) + (-\log g(c))^{(1-r)/q}\theta_3(c)\right\}^{q/p} \tag{2.5}$$

when $(r-1)/q < 0$, and

$$\int_0^c g^{-1}(x)(-\log g(x))^{-r} F_4^q(x)\, dg(x) \le A\left\{\int_0^c g^{p-1}(x)(-\log g(x))^{\delta} f^p(x)\, dg(x) + (-\log g(c))^{(1-r)/q}\theta_4(c)\right\}^{q/p} \tag{2.6}$$

when $(r-1)/q > 0$, where $(-\log g(c))^{(1-r)/q}\theta_i(c)$ ($i = 3$ when $(r-1)/q < 0$ and $i = 4$ when $(r-1)/q > 0$) at $c = 0$ and at $c = 1$ are interpreted respectively as their limits as $c \to 0+$ and as $c \to 1-$.

THEOREM 3. *When* $0<q\le p\le 1$, *Theorems* 1 *and* 2 *hold with the inequality signs in* (2.1)–(2.6) *reversed.*

If, in particular, $g(t)=t$ and $1<p=q<\infty$, then (2.1) and (2.4) reduce to (1.2) and (1.3) when $r=0$, and reduce to (1.4) and (1.5) when $r=p$.

Theorems 1–3 break down when $r=1$. Take Theorem 1 as an example. When $r=1$, the left hand side of (2.1) is always infinite (unless $f(x)$ is almost everywhere zero, in the case when $q>0$), while the integral on the right hand side may be finite. Nevertheless, if we decompose $[1,\infty)$, the interval of integration in (2.1), into $[1,c_0]$ and $[c_0,\infty)$, where $1<c_0<\infty$, $g(c_0)=e$ and $g(x)\ne e$ when $x\ne c_0$, then for $1\lneq p\le q<\infty$ or $-\infty<q\le p<0$ we have

$$(2.7)\quad \int_{c_0}^{\infty} g^{-1}(x)(\log g(x))^{-1}(\log\log g(x))^{-r}F_5^q(x)\,dg(x) \le A\left\{\int_{c_0}^{\infty} g^{p-1}(x)(\log g(x))^{p-1}(\log\log g(x))^{\delta}f^p(x)\,dg(x)\right\}^{q/p},$$

and

$$(2.8)\quad \int_{1}^{c_0} g^{-1}(x)(\log g(x))^{-1}(-\log\log g(x))^{-r}F_6^q(x)\,dg(x) \le A\left\{\int_{1}^{c_0} g^{p-1}(x)(\log g(x))^{p-1}(-\log\log g(x))^{\delta}f^p(x)\,dg(x)\right\}^{q/p},$$

where $r\ne 1$, $F_5(x)=\int_x^\infty f(t)\,dg(t)$ or $=\int_{c_0}^x f(t)\,dg(t)$ according as $(r-1)/q<0$ or $(r-1)/q>0$, and $F_6(x)=\int_1^x f(t)\,dg(t)$ or $=\int_x^{c_0} f(t)\,dg(t)$ according as $(r-1)/q<0$ or $(r-1)/q>0$. If $0<q\le p\le 1$, then (2.7) and (2.8) hold with the inequality signs reversed.

Again, both (2.7) and (2.8) break down when $r=1$; and for this case additional inequalities involving $(\log\log\log g(x))^{-r}$, $(\log(-\log\log g(x)))^{-r}$ $(r\ne 1)$, etc., can be obtained by further decomposing the intervals $[1,c_0]$ and $[c_0,\infty)$ into $[1,c_1]$, $[c_1,c_0]$, $[c_0,c_2]$ and $[c_2,\infty)$, where $1<c_1<c_0<c_2<\infty$, $g(c_1)=e^{1/e}$, $g(x)\ne e^{1/e}$ when $x\ne c_1$, $g(c_2)=e^e$ and $g(x)\ne e^e$ when $x\ne c_2$. To avoid too much complication, however, we shall not go further in this direction, except that in the next section we shall state how (2.7) and (2.8) can be proved.

3. **Proofs of theorems.**

LEMMA. *Let* $-\infty\le a\le b\le\infty$ *and* $-\infty\le\alpha\le\beta\le\infty$. *Suppose that* $\xi(x)$ *and* $\eta(x)$ *are continuous and non-decreasing,* $\alpha\le\xi(x)\le\eta(x)\le\beta$ *for* $a\le x\le b$, *and that* $\lambda(t)$ *is non-decreasing for* $\alpha\le t\le\beta$. *Suppose also that* $h(x,t)$ *is non-negative and measureable for* $a\le x\le b$ *and* $\alpha\le t\le\beta$. *Let* $\chi(x,t)$ *be defined by*

$$\chi(x,t)=\begin{cases}1, & \text{when } a\le x\le b \text{ and } \xi(x)\le t\le\eta(x),\\ 0, & \text{otherwise.}\end{cases}$$

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Then:

(i) *when* $1 \le p \le q < \infty$ *or* $-\infty < q \le p < 0$ *we have*

$$\text{(3.1)}\quad \int_a^b g^{-1}(x)\left(\int_{\xi(x)}^{\eta(x)} h^{1/p}(x,t)\,d\lambda(t)\right)^q\left(\int_{\xi(x)}^{\eta(x)} d\lambda(t)\right)^{(1-p)q/p} dg(x)$$
$$\le\left\{\int_\alpha^\beta\left(\int_a^b \chi(x,t)\,g^{-1}(x)h^{q/p}(x,t)\,dg(x)\right)^{p/q} d\lambda(t)\right\}^{q/p};$$

(ii) *when* $0 < q \le p \le 1$; (3.1) *holds with the inequality sign reversed.*

Proof. First let $1 \le p \le q < \infty$. We have by Hölder's inequality

$$\int_{\xi(x)}^{\eta(x)} h^{1/p}(x,t)\,d\lambda(t) \le \left\{\int_{\xi(x)}^{\eta(x)} h(x,t)\,d\lambda(t)\right\}^{1/p}\left\{\int_{\xi(x)}^{\eta(x)} d\lambda(t)\right\}^{1-1/p}.$$

Hence

$$\int_a^b g^{-1}(x)\left(\int_{\xi(x)}^{\eta(x)} h^{1/p}(x,t)\,d\lambda(t)\right)^q\left(\int_{\xi(x)}^{\eta(x)} d\lambda(t)\right)^{(1-p)q/p} dg(x)$$
$$\le \int_a^b g^{-1}(x)\left(\int_{\xi(x)}^{\eta(x)} h(x,t)\,d\lambda(t)\right)^{q/p} dg(x)$$
$$= \int_a^b\left(\int_\alpha^\beta \chi(x,t)g^{-p/q}(x)\,h(x,t)\,d\lambda(t)\right)^{q/p} dg(x)$$
$$\le\left\{\int_\alpha^\beta\left(\int_a^b \chi(x,t)g^{-1}(x)\,h^{q/p}(x,t)\,dg(x)\right)^{p/q} d\lambda(t)\right\}^{q/p},$$

where the last inequality follows from the generalized form of Minkowski's inequality ([10, p. 19]). This proves the Lemma for the case $1 \le p \le q < \infty$.

For other cases we only have to observe that here Hölder's inequality is reversed when $-\infty < p \le 1$ (cf. [5, §2.8, §9.13]) and the generalized form of Minkowski's inequality is reversed when $0 < q/p \le 1$.

Proof of Theorem 1. Let $\alpha = 1$, $\beta = \infty$, $\lambda(t) = [q/(r-1)](\log g(t))^{(r-1)/q}$ and $h(x,t) = g^p(t)f^p(t)(\log g(t))^{\delta+1}(\log g(x))^{[((1-r)/p)-1]p/q}$.

We shall prove the theorem for the case $(r-1)/q < 0$ only, as the proof for $(r-1)/q > 0$ follows almost exactly the same lines.

Suppose that $(r-1)/q < 0$, and that the integral on the right hand side of (2.1) is finite. As $(\log g(x))^{(r-1)/q}$ is non-increasing, when $x > 1$ we have

$$\theta_1(x) \le (\log g(x))^{(r-1)/q}\int_x^\infty g^{p-1}(t)f^p(t)(\log g(t))^\delta\,dg(t)$$
$$\le (\log g(x))^{(r-1)/q}\int_1^\infty g^{p-1}(t)f^p(t)(\log g(t)^\delta\,dg(t) < \infty.$$

Hence $\theta_1(x)$ is finite for every $x \in (1,\infty)$, and $(\log g(x))^{(1-r)/q}\theta_1(x) \to 0$ as $x \to \infty$.

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Let $a=1$, $1<b<\infty$, $\xi(x)=x$ and $\eta(x)\equiv\infty$. We have

$$\chi(x,t)=\begin{cases}1, & \text{when } 1\le x\le b \text{ and } x\le t\le\infty,\\ 0, & \text{otherwise.}\end{cases}$$

Straightforward calculation shows that

$$\int_a^b g^{-1}(x)\left(\int_{\xi(x)}^{\eta(x)} h^{1/p}(x,t)\,d\lambda(t)\right)^q\left(\int_{\xi(x)}^{\eta(x)} d\lambda(t)\right)^{(1-p)q/p} dg(x)$$
$$=(q/(1-r))^{(1-p)q/p}\int_1^b g^{-1}(x)(\log g(x))^{-r}\left(\int_x^\infty f(t)\,dg(t)\right)^q dg(x).$$

the last quantity is by Lemma (i) not exceeding

$$\left\{\int_1^\infty\left(\int_1^b \chi(x,t)\,g^{-1}(x)\,h^{q/p}(x,t)\,dg(x)\right)^{p/q} d\lambda(t)\right\}^{q/p}$$
$$=\left\{\int_1^b\left(\int_1^t g^{-1}(x)(\log g(x))^{[(1-r)/p]-1}\,dg(x)\right)^{p/q}\right.$$
$$\times g^{p-1}(t)\,f^p(t)(\log g(t))^{\delta+[(r-1)/q]}\,dg(t)$$
$$+\int_b^\infty\left(\int_1^b g^{-1}(x)(\log g(x))^{[(1-r)/p]-1}\,dg(x)\right)^{p/q}$$
$$\left.\times g^{p-1}(t)f^p(t)\,(\log g(t))^{\delta+[(r-1)/q]}\,dg(t)\right\}^{q/p}$$
$$=[p/(1-r)]\left\{\int_1^b g^{p-1}(t)f^p(t)\,(\log g(t))^{\delta}\,dg(t)+(\log g(b))^{(1-r)/q}\theta_1(b)\right\}^{q/p}$$

We have therefore proved (2.2) for $1<c<\infty$.

Now consider the case when $c=1$ or $c=\infty$. We have already proved that $(\log g(x))^{(1-r)/q}\theta_1(x)\to 0$ as $x\to\infty$, so that the case $c=\infty$ of (2.2) is proved. In order to prove that $(\log g(x))^{(1-r)/q}\theta_1(x)\to 0$ as $x\to 1+$, we suppose that $\varepsilon>0$ is arbitrarily fixed. For $1<x<x'<\infty$ we have

$$(\log g(x))^{(1-r)/q}\theta_1(x)$$
$$=(\log g(x))^{(1-r)/q}\left(\int_x^{x'}+\int_{x'}^\infty\right)g^{p-1}(t)f^p(t)(\log g(t))^{\delta+[(r-1)/q]}\,dg(t)$$
$$=J_1+J_2,\text{ say.}$$

We recall that $g(x)$ is continuous and non-decreasing, $g(x)\to 1$ as $x\to 1+$, so that $(\log g(x))^{(1-r)/q}$ is non-decreasing and $\to 0$ as $x\to 1+$. since $(\log g(x))^{(1-r)/q}$ is non-decreasing,

$$J_1\le\int_x^{x'} g^{p-1}(t)f^p(t)(\log g(t))^{\delta}\,dg(t)\le\int_1^{x'} g^{p-1}(t)f^p(t)(\log g(t))^{\delta}\,dg(t).$$

Hence $J_1<\varepsilon$ when x' is sufficiently closed to 1. Having fixed x', as

$(\log g(x))^{(1-r)/q} \to 0$ as $x \to 1+$, we have

$$J_2 = (\log g(x))^{(1-r)/q} \int_{x'}^{\infty} g^{p-1}(t) f^p(t) (\log g(t))^{\delta+[(r-1)/q]} \, dg(t) < \varepsilon$$

when x is sufficiently closed to 1. Hence $J_1 + J_2 < 2\varepsilon$ when x is sufficiently closed to 1, or $(\log g(x))^{(1-r)/q}\theta_1(x) \to 0$ as $x \to 1+$. We have therefore also proved (2.2) for $c = 1$, hence for $1 \le c \le \infty$. (2.1) is the special case of (2.2) in which $c = \infty$.

In order to prove Theorem 1 for $(r-1)/q > 0$, we apply Lemma (i) with $1 < a < \infty$, $b = \infty$, $\xi(x) \equiv 1$, $\eta(x) = x$, the same α, β, $h(x, t)$, and $\lambda(t)$ as for $(r-1)/q < 0$.

In order to prove Theorem 2 for the case $(r-1)/q < 0$, we apply Lemma (i) with $\alpha = 0$, $\beta = 1$, $\lambda(t) = [-q/(r-1)](-\log g(t))^{(r-1)/q}$, $h(x, t) = g^p(t) f^p(t)(-\log g(t))^{\delta+1}(-\log g(x))^{[((1-r)/p)-1]p/q}$, $b = 1$, $0 < a < 1$, $\xi(x) \equiv 0$, and $\eta(x) = x$. For the case $(r-1)/q > 0$ we apply Lemma (i) with $a = 0$, $0 < b < 1$, $\xi(x) = x$, $\eta(x) \equiv 1$, the same α, β, $\lambda(t)$, and $h(x, t)$ as for $(r-1)/q < 0$

The proof of Theorem 3 is also omitted, as it is exactly the same as those of Theorems 1 and 2, except that part (ii) of the Lemma is applied instead of part (i).

We now come to the proofs of (2.7) and (2.8). Set $G_i(x) = g^{-1}(x)(\log g(x))^{-1}(|\log \log g(x)|)^{-r} F_i^q(x)$ $(i = 5, 6)$,

$$H(x) = g^{p-1}(x)(\log g(x))^{p-1}(|\log \log g(x)|)^{\delta} f^p(x).$$

In order to prove (2.7) for the case $(r-1)/q < 0$, we apply Lemma (i) with $\alpha = c_0$, $\beta = \infty$, $\lambda(t) = [q/(r-1)](\log \log g(t))^{(r-1)/q}$, $h(x, t) = g^p(t) f^p(t)(\log g(t))^p \times (\log \log g(t))^{\delta+1}(\log \log g(x))^{[((1-r)/p)-1]p/q}(\log g(x))^{-p/q}$, $a = c_o$, $c_0 < b < \infty$, $\xi(x) = x$ and $\eta(x) \equiv \infty$. For the case $(r-1)/q > 0$, we apply Lemma (i) with $c_0 < a < \infty$, $b = \infty$, $\xi(x) \equiv c_0$, $\eta(x) = x$, the same α, β, $\lambda(t)$, and $h(x, t)$ as for $(r-1)/q < 0$. In fact the results obtained are as follows:

(2.7a) $$\int_{c_0}^{b} G_5(x)\, dg(x) \le A\left\{\int_{c_0}^{b} H(x)\, dg(x) + [\log \log g(b)]^{(1-r)/q}\theta_5(b)\right\}^{q/p}$$

if $(r-1)/q < 0$, $c_0 < b < \infty$, where $\theta_5(b)$

$$= \int_{b}^{\infty} H(x)[\log \log g(x)]^{(r-1)/q}\, dg(x),$$

while

(2.7b) $$\int_{a}^{\infty} G_5(x)\, dg(x) \le A\left\{\int_{a}^{\infty} H(x)\, dg(x) + [\log \log g(a)]^{(1-r)/q}\, \bar{\theta}_5(a)\right\}^{q/p}$$

if $(r-1)/q > 0$, $c_0 < a < \infty$, where $\bar{\theta}_5(a)$

$$= \int_{c_0}^{a} H(x)[\log \log g(x)]^{(r-1)/q}\, dg(x).$$

In order to prove (2.8) for the case $(r-1)/q<0$, we apply Lemma (i) with $\alpha=1$, $\beta=c_0$, $\lambda(t)=[-q/(r-1)](-\log\log g(t))^{(r-1)/q}$, $h(x,t)=g^p(t)f^p(t) \times(\log g(t))^p(-\log\log g(t))^{\delta+1}(-\log\log g(x))^{[((1-r)/p)-1]p/q}(\log g(x))^{-p/q}$, $1<a<c_0$, $b=c_0$, $\xi(x)\equiv 1$ and $\eta(x)=x$. For the case $(r-1)/q>0$, we apply Lemma (i) with $1<b<c_0$, $a=1$, $\xi(x)=x$, $\eta(x)\equiv c_0$, the same α, β, $\lambda(t)$, and $h(x,t)$ as for $(r-1)/q<0$. The results obtained here are:

$$\text{(2.8a)}\quad \int_a^{c_0} G_6(x)\,dg(x)\le A\left\{\int_a^{c_0} H(x)\,dg(x)+[-\log\log g(a)]^{(1-r)/q}\theta_6(a)\right\}^{q/p}$$

if $(r-1)/q<0$, $1<a<c_0$, where $\theta_6(a)$

$$=\int_1^a H(x)[-\log\log g(x)]^{(r-1)/q}\,dg(x),$$

and

$$\text{(2.8b)}\quad \int_1^{b} G_6(x)\,dg(x)\le A\left\{\int_1^{b} H(x)\,dg(x)+[-\log\log g(b)]^{(1-r)/q}\,\bar\theta_6(b)\right\}^{q/p}$$

if $(r-1)/q>0$, $1<b<c_0$, where $\bar\theta_6(b)$

$$=\int_b^{c_0} H(x)[-\log\log g(x)]^{(r-1)/q}\,dg(x).$$

The inequality (2.7) follows from (2.7a) by letting $b\to\infty$, or from (2.7b) by letting $a\to c_0+$. Similarly (2.8) follows from (2.8a) by letting a $\to 1+$, or from (2.8b) by letting $b\to c_0-$.

As before the reverse inequalities to (2.7a)–(2.8b) for the case $0<q\le p\le 1$ follow by using Lemma (ii) instead of Lemma (i) at the appropriate place.

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