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# POSITIVITY NOTIONS FOR HOLOMORPHIC LINE BUNDLES OVER COMPACT RIEMANN SURFACES

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Since the early 1950's, when Kodaira "discovered" positive line bundles, the notion of positivity has undergone a continuous evolution. This paper is intended as an introduction to the study of positivity notions. More specifically, I consider the simplest case - line bundles over compact Riemann surfaces - and compare five positivity notions for such bundles. The results obtained are certainly not new; they are, in fact, known in much greater generality. However, by restricting to the dimension one case, I am able to make use of Riemann surface techniques to significantly simplify the proofs. In fact, this article should be easily understood by anyone familiar with the contents of Gunning's *Lectures on Riemann surfaces*.

## 0.

The notion of a positive line bundle over a compact complex manifold was originally introduced by Kodaira in the early 1950's to obtain his famous characterization of projective manifolds (see [7], [8], [9]). Since that time, the notion of positivity has undergone a continuous evolution. Positive line bundles turned out to have many nice properties each of which had a natural generalization to vector bundles. The resulting definitions, however, were not always equivalent for bundles of fiber dimension greater than one (see [11], [4]). Grauert, in [2], introduced the notion of weak negativity and was thus able to extend Kodaira's results to normal complex

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spaces. Moreover, this notion extended to linear fiber spaces and thus, by duality, implied the first positivity notion for coherent sheaves. More recent work has centered on coherent sheaves and linear fiber spaces ([1], [3], [12], [13], [14]) although there do remain some open questions in the locally-free case.

The object of this paper is to study the various existing positivity notions in the simplest of all cases, line bundles over compact Riemann surfaces. Here the various definitions are, for the most part, equivalent. However, proving these equivalences does require some relatively high-powered machinery. More specifically, use is made of the Serre Duality Theorem, the Riemann-Roch Theorem, results from Kähler geometry, and Remmert's holomorphic reduction theory. Thus, even this simple case provides a serious introduction to the notions and methods of the study of positivity.

We shall give four different definitions for positivity and prove that they are all equivalent. A general outline of the notions and the methods employed in proving equivalence is given in the following diagram (*cf.* [4]):

cohomological positivity
algebro-geometric

Riemann-Roch Theorem $\Downarrow$ $\Uparrow$ Serre Duality

numerical positivity (= positive Chern class)
topological

Kähler Identities $\Downarrow$ $\Uparrow$ Trivial

positive curvature
differential geometric

Trivial $\Downarrow$ $\Uparrow$ Holomorphic Reduction Theory

weak positivity
function theoretic

We also consider the notion of ampleness and prove that

very ample $\Rightarrow$ positive

positive $\Rightarrow$ sufficiently high powers are very ample.

In §1, we recall some basic facts about line bundles over compact

Riemann surfaces. The primary reference here is [5]. In §2, we prove the equivalence of cohomological and numerical positivity and that positivity implies cohomological positivity. Then, in §3, we recall some basic properties of compact Kähler manifolds and proceed to show that numerical positivity implies positivity. §4 is devoted to a discussion of ampleness and its relation to positivity. Finally, in §5, we consider Grauert's notion of weak positivity.

## 1.

Let $M$ be a compact Riemann surface and $\mathcal{O}$ the sheaf of holomorphic functions on $M$. Let $\mathcal{O}^*$ be the sheaf of nowhere vanishing holomorphic functions on $M$. Then (see [5]), $H^1(M, \mathcal{O}^*)$ is the group of equivalence classes of holomorphic line bundles over $M$. If $\xi \in H^1(M, \mathcal{O}^*)$, then $\xi$ is represented by data of nowhere vanishing holomorphic functions $\{\xi_{\alpha\beta}\}$ with $\xi_{\alpha\beta}\xi_{\beta\gamma} = \xi_{\alpha\gamma}$. A section of $\xi$ consists of functions $\{f_\alpha\}$ with $f_\alpha = \xi_{\alpha\beta}f_\beta$; we may speak of continuous $C^\infty$, holomorphic, and meromorphic sections. A metric of $\xi$ is given by data of $C^\infty$-functions $\{r_\alpha\}$ with $r_\alpha > 0$ and $r_\alpha = |\xi_{\beta\alpha}|^2 r_\beta$. ($\{r_\alpha\}$ induces a pointwise inner product on sections by the formula $\langle f_\alpha, g_\alpha\rangle = f_\alpha r_\alpha \overline{g_\alpha}$ which is well-defined since $f_\alpha r_\alpha \overline{g_\alpha} = \xi_{\alpha\beta} f_\beta r_\alpha \overline{\xi_{\alpha\beta}}\,\overline{g_\beta} = f_\beta |\xi_{\alpha\beta}|^2 r_\alpha \overline{g_\beta} = f_\beta r_\beta \overline{g_\beta}$.) The curvature form of the metric $\{r_\alpha\}$ is the globally defined (1, 1)-form

$$\Theta = \bar{\partial}\partial \log(r_\alpha) = -\partial\bar{\partial} \log(r_\alpha)$$

which is well-defined since $\xi_{\alpha\beta}$ is holomorphic.

Consider the commutative diagram (with exact rows) of sheaves over $M$;

$$\begin{array}{ccccccccc} 0 & \to & Z & \xrightarrow{j} & \mathcal{O} & \xrightarrow{e^{2\pi i f}} & \mathcal{O}^* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Z & \xrightarrow{j} & C & \xrightarrow{e^{2\pi i f}} & C^* & \to & 0 \end{array}$$

where $Z$ is the constant sheaf of integers and $C$ the sheaf of

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$C^\infty$-functions on $M$. This induces a commutative diagram in cohomology

$$\begin{array}{ccc} H^1(M,\mathcal{O}^*) & \xrightarrow{c} & H^2(M,Z) \\ \downarrow & & \downarrow \\ H^1(M,C^*) & \xrightarrow[\cong]{} & H^2(M,Z) \end{array} .$$

For $\xi \in H^1(M, \mathcal{O}^*)$, $c(\xi) \in H^2(M, Z)$ is called the Chern class of $\xi$; as the diagram indicates, $c(\xi)$ depends only on the smooth structure of the line bundle. Let $\mathbb{C}$ be the constant sheaf of complex numbers. Then the sheaf inclusion $Z \to \mathbb{C}$ induces a map $H^2(M, Z) \to H^2(M, \mathbb{C})$. We have:

**PROPOSITION** ([5], p. 100). *Let* $\xi = \{\xi_{\alpha\beta}\} \in H^1(M, \mathcal{O}^*)$. *Let* $\{r_\alpha\}$ *be a metric on* $\xi$ *with curvature form* $\Theta$. *Then, under the de Rham isomorphism,* $(i/2\pi)\Theta$ *represents the image in* $H^2(M, \mathbb{C})$ *of the Chern class of* $\xi$. *That is,*

$$c(\xi) = \iint_M \frac{i}{2\pi}\,\Theta = \frac{i}{2\pi}\iint_M -\,\partial\bar\partial \log(r_\alpha) = \frac{1}{2\pi i}\iint_M \partial\bar\partial \log(r_\alpha) .$$

Note. Since $\dim_R M = 2$, $H^2(M, \mathbb{C}) \cong \mathbb{C}$. In the above proposition, we have used the explicit isomorphism given by integration of forms.

Proof of proposition. The proof is a simple diagram chase; see [5] for details.

Let $f = \{f_\alpha\}$ be a global meromorphic cross-section of $\xi$ and let $p \in M$. By the order of $f$ at $p$, denoted $\nu_p(f)$, we mean the order at $p$ of the meromorphic function $f_\alpha$. Since $f_\alpha = \xi_{\alpha\beta}f_\beta$ with $\xi_{\alpha\beta}$ holomorphic and non-vanishing, this definition is clearly independent of the choice of $\alpha$. Furthermore, since $M$ is compact, $\nu_p(f) = 0$ for all but a finite number of points of $M$ (unless, of course, $f$ is trivial).

**PROPOSITION** ([5], p. 103). *Let* $\xi = \{\xi_{\alpha\beta}\} \in H^1(M, \mathcal{O}^*)$ *and let* $f = \{f_\alpha\}$ *be any non-trivial meromorphic cross-section of* $\xi$. *Then*

$$c(\xi) = \sum_{p \in M} \nu_p(f) .$$

Proof. Let $\{p_i\}$ be the set of points where $f$ has non-zero order. Choose a covering, $\{U_\alpha\}$, of $M$ such that for each $p_i$ there exists an open set $V_i$ with $p_i \in V_i \subset U_{\alpha_i}$ but such that $V_i \cap U_\alpha = \emptyset$ for $\alpha \neq \alpha_i$. Let $f = \{f_\alpha\}$ with $f_\alpha$ meromorphic in $U_\alpha$ and $f_\alpha = \xi_{\alpha\beta} f_\beta$. Then $|f_\alpha|^2$ is $C^\infty$ and non-vanishing in $U_\alpha - \{(\cup p_i) \cap U_\alpha\}$. Thus $\left\{|f_\alpha|^2\right\}$ may be altered in the various $V_i$ to yield data of $C^\infty$ positive functions $\{g_\alpha\}$ such that $g_\alpha = |\xi_{\alpha\beta}|^2 g_\beta$ and $g_\alpha = |f_\alpha|^2$ in $U_\alpha - \{(\cup V_i) \cap U_\alpha\}$. It follows that $g_\alpha^{-1} = |\xi_{\beta\alpha}|^2 g_\beta^{-1}$ so $\left\{g_\alpha^{-1}\right\}$ defines a metric in $\xi$ and hence

$$c(\xi) = \frac{1}{2\pi i} \iint_M \partial\bar{\partial} \log\left(g_\alpha^{-1}\right) = \frac{1}{2\pi i} \bar{\partial}\partial \log\left(g_\alpha\right) .$$

Now outside of $\cup V_i$ we have $\log\left(g_\alpha\right) = \log\left(|f_\alpha|^2\right) = \log\left(f_\alpha\right) + \log\left(\bar{f}_\alpha\right)$ and $f_\alpha$ is holomorphic. Thus

$$\begin{aligned} c(\xi) &= \frac{1}{2\pi i} \iint_{\cup V_i} \bar{\partial}\partial \log\left(g_\alpha\right) = \frac{1}{2\pi i} \sum_i \iint_{V_i} \bar{\partial}\partial \log\left(g_\alpha\right) \\ &= \frac{1}{2\pi i} \sum_i \iint_{V_i} d\partial \log\left(g_\alpha\right) = \frac{1}{2\pi i} \sum_i \int_{\partial V_i} \partial \log\left(f_\alpha\right) \\ &= \frac{1}{2\pi i} \sum_i \int_{\partial V_i} d \log\left(f_\alpha\right) = \sum \nu_{p_i}(f) . \end{aligned}$$

COROLLARY. *If* $c(\xi) < 0$, *then* $\Gamma\left(M, \theta(\xi)\right) = H^0\left(M, \theta(\xi)\right) = 0$.

Proof. The order of a holomorphic cross-section is non-negative.

## 2.

We have seen that for $\xi \in H^1(M, \theta^*)$, $c(\xi) = \iint_M \frac{i}{2\pi} \Theta$ where $\Theta$ is the curvature form of any metric on $\xi$. It follows that a sufficient condition for $c(\xi) > 0$ is that $\xi$ carry on a metric whose curvature form $\Theta$ has the property that $i\Theta$ is positive at every point of $M$. This

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suggests the following definitions.

**DEFINITION.** A $(1, 1)$-form $\omega$ on a compact Riemann surface $M$ is *positive* if, for every point $p \in M$, there exists a coordinate neighbourhood $(U, z)$ of $p$ such that $\omega|U = i\varphi dz \wedge \overline{dz}$ with $\varphi > 0$ throughout $U$.

**DEFINITION.** A line bundle $\xi$ over a compact Riemann surface $M$ is called *positive* if there exists a metric on $\xi$ with curvature form $\Theta$ such that $i\Theta$ is a positive differential form.

**DEFINITION.** A line bundle $\xi$ over a compact Riemann surface $M$ is called *numerically positive* if $c(\xi) \in H^2(M, Z) \cong Z$ is positive.

**THEOREM.** *Let $M$ be a compact Riemann surface and $\xi \in H^1(M, \theta^*)$. If $\xi$ is positive then $c(\xi) > 0$; that is to say, positivity implies numerical positivity.*

Proof. Let $\{(U_\alpha, z_\alpha)\}$ be an open covering of $U$ such that $i\Theta|U_\alpha = i\varphi_\alpha dz_\alpha \wedge \overline{dz}_\alpha$. Let $\rho_\alpha$ be a partition of unity subordinate to $\{U_\alpha\}$. Then $i\Theta = i \sum_\alpha \rho_\alpha\varphi_\alpha dz_\alpha \wedge \overline{dz}_\alpha$.

Thus

$$c(\xi) = \iint_M \frac{i}{2\pi}\Theta = \frac{1}{2\pi}\sum_\alpha \iint_{U_\alpha} i\rho_\alpha\varphi_\alpha dz_\alpha \wedge \overline{dz}_\alpha$$

$$= \frac{1}{2\pi}\sum_\alpha \iint_{U_\alpha} i\rho_\alpha\varphi_\alpha (dx_\alpha + idy_\alpha) \wedge (dx_\alpha - idy_\alpha)$$

$$= \frac{1}{\pi}\sum_\alpha \iint_{U_\alpha} \rho_\alpha\varphi_\alpha dx_\alpha dy_\alpha > 0 .$$

We note that it is not at all obvious that numerical positivity implies positivity.

**DEFINITION.** A line bundle $\xi$ over a compact Riemann surface $M$ is called *cohomologically positive* if for any line bundle $\eta$ over $M$ there exists an integer $m = m(c(\eta))$ such that

$$H^k(M, \theta(\xi^n\eta)) = 0 \quad \text{for all } n \geq m \text{ and all } k > 0 .$$

Note. The usual definition of cohomological positivity (see, for example [4]) is stronger in that $\eta$ is allowed to be any coherent sheaf and weaker in that $m$ is allowed to depend on $\eta$ itself rather than its Chern class (even when $\eta$ is invertible). The two notions are equivalent.

**THEOREM.** *Let $M$ be a compact Riemann surface and $\xi \in H^1(M, \theta^*)$. Then $\xi$ is cohomologically positive if and only if $\xi$ is numerically positive.*

**COROLLARY.** *Positivity implies cohomological positivity.*

Proof of theorem. If $\mu$ is any line bundle over $M$, then there is a fine resolution of $\theta(\mu)$ :

$$0 \to \theta(\mu) \to E^{0,0}(\mu) \xrightarrow{\bar\partial} E^{0,1}(\mu) \xrightarrow{\bar\partial} 0$$

where $E^{p,q}(\mu)$ is the sheaf of smooth $(p, q)$-forms on $M$ with coefficients in $\mu$. It follows that for any $\mu$, $H^k(M, \theta(\mu)) = 0$ for all $k \geq 2$. Furthermore, by the Serre Duality Theorem,

$$H^1(M, \theta(\mu)) \cong H^0(M, \theta(K\mu^{-1}))$$

where $K$ is the canonical bundle over $M$. ($\theta(K) = \theta^{1,0}$ equals the sheaf of abelian differentials.)

Suppose now that $\xi$ is numerically positive. By the above remarks, it suffices to show that for any line bundle $\eta$ over $M$ there is an integer $m$ such that $H^0(M, \theta(K\xi^{-n}\eta^{-1})) = 0$ for all $n \geq m$. Now $c(K\xi^{-n}\eta^{-1}) = c(K) - nc(\xi) - c(\eta)$. Since $c(\xi) > 0$, it is clear that, for sufficiently large $n$, $c(K\xi^{-n}\eta^{-1}) < 0$ and hence $H^0(M, \theta(K\xi^{-n}\eta^{-1})) = 0$. (Actually we can compute the smallest possible value for $m$ since we know by the Riemann-Roch Theorem that $c(K) = 2g - 2$ where $g$ is the genus of $M$.)

Conversely, suppose that $c(\xi) \leq 0$. We will show that, in this case, $\xi$ is not cohomologically positive by exhibiting a line bundle $\eta$ such that $H^1(M, \theta(\xi^n\eta))$ has positive dimension for arbitrarily large values of $n$. Recall that if $\mu \in H^1(M, \theta^*)$ then

$$\dim_{\mathbb{C}}[H^0(M, \theta(\mu))] - \dim_{\mathbb{C}}[H^1(M, \theta(\mu))] = c(\mu) + 1 - g ;$$

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this is the Riemann-Roch Theorem. By Serre Duality, this becomes

$$\dim_{\mathbb{C}}[H^0(M, \theta(\mu))] - \dim_{\mathbb{C}}[H^0(M, \theta(K\mu^{-1}))] = c(\mu) + 1 - g .$$

Introducing the notation $\gamma(\sigma) = \dim_{\mathbb{C}}[H^0(M, \theta(\sigma))]$ , we may rewrite this as

$$\gamma(\mu) - \gamma(K\mu^{-1}) = c(\mu) + 1 - g .$$

In particular, letting $\mu = \xi^n\eta$ , we have

$$\begin{aligned}\gamma(\xi^n\eta) - \gamma(K\xi^{-n}\eta^{-1}) &= c(\xi^n\eta) + 1 - g \\ &= nc(\xi) + c(\eta) + 1 - g \\ &\leq c(\eta) + 1 - g\end{aligned}$$

since $c(\xi) \leq 0$ . Thus

$$\gamma(K\xi^{-n}\eta^{-1}) \geq g - 1 - c(\eta) + \gamma(\xi^n\eta) \geq g - 1 - c(\eta) .$$

It follows that if $\eta$ is chosen so that $c(\eta) < g - 1$ then

$$\gamma(K\xi^{-n}\eta^{-1}) = \dim_{\mathbb{C}}[H^0(M, \theta(K\xi^{-n}\eta^{-1}))] = \dim_{\mathbb{C}}[H^1(M, \theta(\xi^n\eta))] > 0$$

for all $n > 0$ . (We note that line bundles of all Chern classes exist. In fact if $d \in H^0(M, \mathcal{D})$ is any holomorphic divisor on $M$ and $\eta$ is the corresponding line bundle then $c(\eta) = -(\text{order of } d)$ .)

Note. We have shown that if $c(K\mu^{-1}) < 0$ then $H^k(M, \theta(\mu)) = 0$ for all $k \geq 1$ . Thus, if $\mu K^{-1}$ has positive Chern class then $H^k(M, \theta(\mu)) = 0$ for all $k \geq 1$ . Kodaira's Vanishing Theorem (see, for example [16]) says that if $\mu K^{-1}$ is positive then $H^k(M, \theta(\mu)) = 0$ for all $k \geq 1$ . Thus, we have proven the Kodaira Theorem for Riemann surfaces under the (*a priori*) weaker condition that only the Chern class be positive.

## 3.

We now proceed to show that $c(\xi) > 0$ implies that $\xi$ is positive. This, in conjunction with our previous results, will establish the equivalence of positivity, numerical positivity, and cohomological positivity. We shall make use of the fact that every Riemann surface is, in fact, a Kähler manifold (see, for example, [16], p. 212). We recall the following important property of Kähler manifolds:

PROPOSITION ([15], p. 72; [10], p. 130). *Let* $M$ *be a compact Kähler manifold and let* $\varphi$ *be a purely imaginary* $(\overline{\varphi} = -\varphi)$ *exact* $(1, 1)$*-form on* $M$ *. Then there is a smooth real-valued function* $g$ *on* $M$ *such that* $\varphi = \partial\overline{\partial}g$ .

We shall need the following lemma.

LEMMA (*cf.* [5], p. 102). *Let* $M$ *be a compact Riemann surface. Then there exists a positive differential form on* $M$ . (*Note that any such form is necessarily closed since* $\dim_R M = 2$ .)

Proof. Let $R$ be the sheaf of smooth real-valued functions on $M$ , and $R^*$ the sheaf of smooth positive functions. Clearly $\exp : R \to R^*$ is a sheaf isomorphism; since $R$ is fine, this implies

$$H^1(M, R^*) = H^1(M, R) = 0 .$$

Let $\{(U_\alpha, z_\alpha)\}$ be a system of coordinate charts for $M$ . Let $\eta_{\alpha\beta} = |\partial z_\alpha/\partial z_\beta|^2$ ; then $\eta_{\alpha\beta}\eta_{\beta\gamma} = \eta_{\alpha\gamma}$ so $\eta_{\alpha\beta}$ is a cocycle and, by the above, a coboundary. Thus there exist positive functions $\{r_\alpha\}$ such that $\eta_{\alpha\beta} = r_\beta/r_\alpha$ , that is, $r_\beta = \eta_{\alpha\beta}r_\alpha = |\partial z_\alpha/\partial z_\beta|^2 r_\alpha$ . Let $\Psi_\alpha = ir_\alpha dz_\alpha \wedge \overline{dz}_\alpha$ . Then

$$\begin{aligned}\Psi_\alpha = ir_\alpha dz_\alpha \wedge \overline{dz}_\alpha &= i|\partial z_\beta/\partial z_\alpha|^2 r_\beta(\partial z_\alpha/\partial z_\beta)dz_\beta \wedge (\overline{\partial z}_\alpha/\partial z_\beta)\overline{dz}_\beta \\ &= ir_\beta dz_\beta \wedge \overline{dz}_\beta = \Psi_\beta\end{aligned}$$

on $U_\alpha \cap U_\beta$ . Thus $\{\Psi_\alpha\}$ defines a global $(1, 1)$-form $\Psi$ and $\Psi$ is positive by construction.

Note. $\Psi$ is just the fundamental form of an Hermitian metric on the holomorphic tangent bundle of $M$ .

THEOREM. *Let* $M$ *be a compact Riemann surface and* $\xi \in H^1(M, \theta^*)$ *. If* $c(\xi) > 0$ *, then* $\xi$ *is positive.*

Proof. Let $\Psi$ be a positive differential form on $M$ . Then certainly $\iint_M \Psi > 0$ . Let $\alpha = \left[\iint_M \Psi\right]^{-1} c(\xi)$ . Then $\alpha > 0$ so $\alpha\Psi$ is a positive form and, by construction, $\iint_M \alpha\Psi = c(\xi)$ . Let $\{r_\alpha\}$ be any metric on $\xi$ and $\Theta$ its curvature form. Then $\iint_M (i/2\pi)\Theta - \alpha\Psi = 0$ ; that is, $(i/2\pi)\Theta - \alpha\Psi$ represents the 0 cohomology class in $H^2(M, \mathbb{C})$ and is therefore exact. Let $\Omega = -2\pi i\alpha\Psi$ so that $i\Omega$ is positive and $(i/2\pi)\Omega = \alpha\Psi$ . Then $\Omega - \Theta$ is exact and thus, by the proposition, there is a real-valued function $g$ such that $\Omega - \Theta = \partial\bar{\partial}g$ . Let $\sigma_\alpha = e^{-g}r_\alpha$ . Then $\log(\sigma_\alpha) = -g + \log(r_\alpha)$ and

$$-\partial\bar{\partial} \log(r_\alpha) = \partial\bar{\partial}g - \partial\bar{\partial} \log(r_\alpha) = \partial\bar{\partial}g + \Theta = \Omega .$$

Thus $\{\sigma_\alpha\}$ is a metric on $\xi$ whose curvature form $\Omega$ has the property that $i\Omega$ is positive; $\xi$ is positive.

## 4.

We have seen that if $c(\xi) < 0$ then $\gamma(\xi) = \dim_{\mathbb{C}}\left[H^0(M, \theta(\xi))\right] = 0$ . However $\gamma(\xi) = 0$ is not a satisfactory notion of negativity. Although $c(\xi) < 0$ does imply $\gamma(\xi) = 0$ , it is possible for $\gamma(\xi)$ to be zero even if $c(\xi) \geq 0$ . As an example, consider any line bundle $\xi$ which is trivial as a differentiable line bundle but not as a holomorphic line bundle. Then $c(\xi) = 0$ . If $\xi$ had a holomorphic section, then it would have to be nowhere vanishing and that would imply that $\xi$ is holomorphically trivial. Thus $\gamma(\xi) = 0$ even though $c(\xi) = 0$ . It is true, however, that if $\xi$ is "sufficiently" positive then $\xi$ must have non-trivial sections. Indeed, by the Riemann-Rich Theorem, $\gamma(\xi) \geq c(\xi) + 1 - g$ so if $c(\xi) \geq g$ then $\gamma(\xi) \geq 1$ .

The above remarks lead us to the consideration of line bundles with "many" sections. For $\xi \in H^1(M, \theta^*)$ and $p \in M$ , we denote by $\xi|_p$ the fiber at $p$ of $\xi$ . $\Gamma(\xi)$ will, as usual, denote the complex vector space of global holomorphic cross-sections of $\xi$ and $\Gamma_p(\xi)$ the subspace of

$\Gamma(\xi)$ consisting of those sections that vanish at $p$. Let $\{(U_\alpha, z_\alpha)\}$ be a system of coordinate charts for $M$. Then the canonical bundle, $K$, has transition functions $\{k_{\alpha\beta}\}$ where $k_{\alpha\beta} = dz_\beta/dz_\alpha$; $K$ is just the holomorphic cotangent bundle. For each $p \in M$, we introduce a map $\lambda_p : \Gamma_p(\xi) \to (\xi K)|_p$ as follows: $\gamma \in \Gamma_p(\xi)$ is given by data of holomorphic functions $\{f_\alpha\}$ with $f_\alpha(p) = 0$ and $f_\alpha = \xi_{\alpha\beta} f_\beta$. Consider the data $\{\partial f_\alpha/\partial z_\alpha|_p\}$. We have

$$\begin{aligned}\partial f_\alpha/\partial z_\alpha|_p &= \partial f_\alpha/\partial z_\beta|_p(\partial z_\beta/\partial z_\alpha|_p) = \partial(\xi_{\alpha\beta}f_\beta)/\partial z_\beta|_p(\partial z_\beta/\partial z_\alpha|_p)\\ &= \{(\partial\xi_{\alpha\beta}/\partial z_\beta)f_\beta|_p+\xi_{\alpha\beta}(\partial f_\beta/\partial z_\beta|_p)\}(\partial z_\beta/\partial z_\alpha|_p)\\ &= \xi_{\alpha\beta}(\partial z_\beta/\partial z_\alpha|_p)(\partial f_\beta/\partial z_\beta|_p)\end{aligned}$$

since $f_\beta(p) = 0$. Thus $\lambda_p(\{f_\alpha\}) = \{\partial f_\alpha/\partial z_\alpha|_p\}$ is a well-defined element of $(\xi K)|_p$. (An element of $(\xi K)|_p$ is given by data of complex constants $\{a_\alpha\}$ with $a_\alpha = \xi_{\alpha\beta}(p)(\partial z_\beta/\partial z_\alpha|_p)a_\beta$.)

**DEFINITION.** A line bundle $\xi$ over a compact Riemann surface $M$ is called *very ample* if

(i) the global sections of $\xi$ generate all its fibers; that is, for each $p \in M$ there is an exact sequence $0 \to \Gamma_p(\xi) \to \Gamma(\xi) \to \xi|_p \to 0$, and

(ii) that map $\lambda_p : \Gamma_p(\xi) \to (\xi K)|_p$ is surjective for each $p \in M$.

**THEOREM.** *Let $M$ be a compact Riemann surface and $\xi \in H^1(M, \theta^*)$:*

(1) *if $\xi$ is very ample then $\xi$ is positive;*

(2) *if $\xi$ is positive then there is an integer $n$ such that $\xi^m$ is very ample for all $m \ge n$.*

Proof. (1) If $\xi$ is ample and $p \in M$, then, by condition (ii), there is a section of $\xi$ vanishing at $p$. It follows that $c(\xi) \ge 1$ so that $\xi$ is numerically positive and hence positive.

(2) Suppose $\xi$ is positive and hence cohomologically positive. Let

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$p \in M$ and choose a system $\{(U_\alpha, z_\alpha)\}$ of coordinate charts for $M$ such that $p \in U_{\alpha_0}$ but $p \notin U_\alpha$ for $\alpha \neq \alpha_0$. Let $g_{\alpha_0}$ be a meromorphic function in $U_{\alpha_0}$ with a single simple pole at $p$ and, for $\alpha \neq \alpha_0$, let $g_\alpha$ be any nowhere vanishing holomorphic function in $U_\alpha$. Let $\eta_p$ be the line bundle with transition functions $\{\eta_{\alpha\beta}\}$ where $\eta_{\alpha\beta} = g_\alpha/g_\beta$. The data $\{g_\alpha\}$ satisfy $g_\alpha = \eta_{\alpha\beta}g_\beta$ and thus define a global meromorphic cross-section of $\eta_p$. Thus $c(\eta_p) = -1$ and $\eta_p$ has a global meromorphic section with one simple pole at $p$. For any $m$, we have exact sequences:

$$0 \to \xi^m\eta_p \xrightarrow{\varphi_p} \xi^m \to \xi^m|_{(p)} \to 0$$

(I)

$$0 \to \xi^m\eta_p^2 \xrightarrow{\Psi_p} \xi^m\eta_p \to \xi^m\eta_p\Big|_{(p)} \to 0 .$$

The map $\varphi_p : \xi^m\eta_p \to \xi^m$ is defined as follows. A local section of $\xi^m\eta_p$ consists of data of holomorphic functions $\{f_\alpha\}$ with

$$f_\alpha = \xi^m_{\alpha\beta}\eta_{\alpha\beta}f_\beta = \xi^m_{\alpha\beta}g_\alpha/g_\beta f_\beta .$$

Then $\varphi_p(\{f_\alpha\})$ is given by the data $\{f_\alpha/g_\alpha\}$. (We are actually contracting with the point bundle $\eta_p^{-1} = \eta^*$.) $\Psi_p$ is defined analogously. The last sheaf in each sequence is defined by the sequence as the quotient of the first two. Now these sequences lead to exact sequences in cohomology:

$$0 \to \Gamma\left(\xi^m\eta_p\right) \to \Gamma(\xi^m) \to \Gamma\left(\xi^m|_{(p)}\right) \to H^1\left(M, \theta\left(\xi^m\eta_p\right)\right)$$

(II)

$$\Gamma\left(\xi^m\eta_p\right) \to \Gamma\left(\xi^m\eta_p\Big|_{(p)}\right) \to H^1\left(M, \theta\left(\xi^m\eta_p^2\right)\right) .$$

LEMMA. *(a)* $\Gamma\left(\xi^m\eta_p\right) = \Gamma_p(\xi^m)$.

*(b)* $\Gamma\left(\xi^m|_{(p)}\right) = \xi^m|_p$.

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*(c)* $\Gamma\left(\xi^m\eta_p\big|_{(p)}\right) = (\xi^m k)\big|_p$ .

Proof of lemma. *(a)* $f \in \Gamma\left(\xi^m\eta_p\right)$ is given by data $\{f_\alpha\}$ with $f_\alpha = \xi^m_{\alpha\beta}\eta_{\alpha\beta}f_\beta$ . $\varphi(f)$ is given by the data $\{f_\alpha/g_\alpha\}$ and thus $\varphi(f)(p) = 0$ . On the other hand, if $f_\alpha = \xi^m_{\alpha\beta}f_\beta$ with $f_\alpha(p) = 0$ , then $f_\alpha g_\alpha$ is holomorphic for all $\alpha$ and thus defines a section of $\xi^m\eta_p$ .

*(b)* $h \in \Gamma\left(\xi^m\big|_{(p)}\right)$ is given by data $\{h_\alpha\}$, $\left\{f^\beta_\alpha\right\}$ such that $f^\beta_\alpha(p) = 0$ , $f^\beta_\alpha = \xi^m_{\beta\gamma}\eta_{\beta\gamma}f^\gamma_\alpha$ , and $h_\alpha - f^\beta_\alpha/g_\beta = \xi^m_{\alpha\beta}h_\beta$ . (The data $\left\{f^\beta_\alpha\right\}$ define a section of $\xi^m\eta_p$ over $U_\alpha$ such that $h_\alpha - \varphi\left(\left\{f^\beta_\alpha\right\}\right) = \xi^m_{\alpha\beta}h_\beta$ .) Since $f^\beta_\alpha(p) = 0$ , we have $h_\alpha(p) = \xi^m_{\alpha\beta}(p)h_\beta(p)$ so that $\{h_\alpha(p)\}$ defines a section of $\xi^m\big|_p$ .

*(c)* Proceed as in *(b)*, considering $\left\{\partial h_\alpha/\partial z_\alpha\big|_p\right\}$ .

Incorporating the lemma into the exact sequences (II), we get exact sequences

$$0 \to \Gamma_p(\xi^m) \to \Gamma(\xi^m) \to \xi^m\big|_p \to H^1\left(M, \theta\left(\xi^m\eta_p\right)\right)$$

$$\Gamma_p(\xi^m) \to (\xi^m K)\big|_p \to H^1\left(M, \theta\left(\xi^m\eta_p^2\right)\right)$$

for each $p$ . By the cohomological positivity of $\xi$ , there is an integer $n = n\left(c(\eta_p), c\left(\eta_p^2\right)\right)$ such that

$$H^1\left(M, \theta\left(\xi^m\eta_p\right)\right) = H^1\left(M, \theta\left(\xi^m\eta_p^2\right)\right) = 0$$

for all $m \geq n$ . But $c(\eta_p) = -1$ , $c\left(\eta_p^2\right) = -2$ for all $p$ ; thus, the integer $n$ may be chosen simultaneously for all $p$ . This completes the proof.

## 5.

In this final section, we consider Grauert's notion of weak positivity and his proof of the equivalence of positivity and weak positivity for line bundles. (It is not known whether weak positivity implies positivity for bundles of fiber dimension greater than one.) In previous sections, we have seen that restricting attention to Riemann surfaces led to simplified proofs of the equivalence of various positivity notions. In this case, however, we have no such simplification to offer. We content ourselves, therefore, with a brief synopsis of Grauert's argument.

Let $M$ be a complex manifold, $D \subset M$ an open subset, and $\varphi$ a twice-differentiable real-valued function in $D$. We denote by $H_\varphi$ the complex Hessian of $\varphi$; $H_\varphi$ is an Hermitian form on $TM|D$ where $TM$ is the holomorphic tangent bundle of $M$. In terms of local coordinates $(z_1, \ldots, z_n)$ on $M$, $H_\varphi = \sum (\partial^2\varphi/\partial z_i \partial \bar{z}_j) dz_i \otimes d\bar{z}_j$. For the remainder of this section, it is useful to think of a line bundle over $M$ as a particular geometric object $L$ lying over $M$ rather than an element of $H^1(M, \theta^*)$.

**DEFINITION.** Let $M$ be a complex manifold and $D$ a relatively compact subdomain of $M$ with $C^2$-boundary. $D$ is called *strongly pseudoconvex* if there is an $M$-neighbourhood $W$ of $\partial D$ and a real-valued $C^2$-function $\varphi$ on $W$ such that:

(i) $W \cap D = \{p \in W \mid \varphi(p) < 0\}$;

(ii) $(d\varphi)_p \neq 0$ for all $p \in W$;

(iii) $H_\varphi$ is positive definite on all $T_pM$ for all $p \in W$.

**PROPOSITION.** *Let $M$ be a compact Riemann surface, $L \to M$ a holomorphic line bundle, and $\{r_\alpha\}$ a metric in $L$. Let $p$ be the square-norm function on $L$ induced by $\{r_\alpha\}$; that is*

$$\rho(\xi) = \langle \xi, \xi \rangle = \xi_\alpha r_\alpha \bar{\xi}_\alpha = |\xi_\alpha|^2 r_\alpha .$$

*Then $\{r_\alpha\}$ has negative curvature if and only if the unit-disc bundle*

$U = \{\xi \in L \mid \rho(\xi) < 1\}$ *is strongly pseudoconvex.*

Proof. Suppose first that $\{r_\alpha\}$ has negative curvature, that is $-\partial\bar\partial \log(r_\alpha) < 0$ and hence $\partial\bar\partial \log(r_\alpha) > 0$. Since $U = \{\xi \in L \mid \rho(\xi) < 1\}$ it suffices to show that $\partial\bar\partial\rho$ is positive definite. Now

$$\partial\bar\partial\rho = \partial\bar\partial e^{\log\rho} = \partial\left(e^{\log\rho}\bar\partial \log \rho\right) = \rho\left(\partial \log \rho \wedge \bar\partial \log \rho\right) + \rho\partial\bar\partial \log \rho .$$

Thus

$$\begin{aligned}
&\rho^{-1}\partial\bar\partial\rho \\
&= \partial \log \rho \wedge \bar\partial \log \rho + \partial\bar\partial \log \rho \\
&= \left(\partial \log \xi_\alpha + \partial \log r_\alpha\right) \wedge \left(\bar\partial \log \bar\xi_\alpha + \bar\partial \log r_\alpha\right) + \partial\bar\partial \log r_\alpha \\
&= \left(\xi_\alpha^{-1}\partial\xi\alpha + r_\alpha^{-1}\partial r_\alpha\right) \wedge \left(\bar\xi_\alpha^{-1}\overline{\partial\xi}_\alpha + r_\alpha^{-1}\overline{\partial r}_\alpha\right) + \partial\bar\partial \log r_\alpha \\
&= \left\{\left(1/|\xi_\alpha|^2\right)\partial\xi_\alpha \wedge \overline{\partial\xi}_\alpha + \left(1/r_\alpha\bar\xi_\alpha\right)\partial r_\alpha \wedge \overline{\partial\xi}_\alpha + \left(1/\xi_\alpha r_\alpha\right)\partial\xi_\alpha \wedge \overline{\partial r}_\alpha + \left(1/r_\alpha^2\right)\partial r_\alpha \wedge \overline{\partial r}_\alpha\right\} \\
&\qquad + \partial\bar\partial \log r_\alpha .
\end{aligned}$$

Now the term in brackets is easily seen to be positive semidefinite. Since $\partial\bar\partial \log r_\alpha$ is, by assumption, positive definite, it follows that $\rho^{-1}\partial\bar\partial\rho$ and hence $\partial\bar\partial\rho$ is positive definite.

Conversely, suppose that

$$U = \{\xi \in L \mid \rho(\xi) < 1\} = \{\xi \in L \mid \log \rho(\xi) < 0\}$$

is strongly pseudoconvex. Then $\log \rho(\xi)$ is a "defining function" for a strongly pseudoconvex domain; it follows (see [*6*], p. 262) that $\partial\bar\partial \log \rho = \partial\bar\partial \log r_\alpha$ is positive definite.

**DEFINITION.** A line bundle $L$ over a compact Riemann surface $M$ is called *weakly negative* if its zero-section has a strongly pseudoconvex neighbourhood. $L$ is called *weakly positive* if $L^* = L^{-1}$ is weakly negative.

**THEOREM.** *Let $M$ be a compact Riemann surface and $L \to M$ a holomorphic line bundle. If $L$ is positive then $L$ is weakly positive.*

Proof. If $L$ is positive then $L^*$ is negative and hence, by the proposition, the unit disc bundle in $L^*$ is strongly pseudoconvex. Thus $L^*$ is weakly negative and $L$ is weakly positive.

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**DEFINITION** (see [2]). Let $X$ be a reduced complex space and $A$ a compact analytic set in $X$. $A$ is called *exceptional* if there is a reduced complex space $Y$, a point $p \in Y$, and a surjective holomorphic map $\pi : X \to Y$ such that $\pi(A) = p$ and $\pi : X-A \to Y-\{p\}$ is a biholomorphism.

We quote the following basic result of Grauert without proof (this is where the holomorphic reduction theory comes in):

**PROPOSITION** (Grauert, [2]). *Let $M$ be a compact complex manifold and $L \to M$ a weakly negative line bundle. Then the zero-section of $L$ is an exceptional analytic set in $L$.*

**THEOREM** (Grauert, [2], p. 34). *Let $M$ be a compact Riemann surface (or, more generally, a compact complex manifold) and $L \to M$ a holomorphic line bundle. If $L$ is weakly positive then $L$ is positive.*

Proof. Let $Q$ be the zero-section of $L^*$ and let $\pi : L^* \to Y$ be such that $\pi : Q \to p \in Y$ and $\pi|L^* - Q$ is a biholomorphism. Let $V$ be a neighbourhood of $p = \pi(Q)$ in $Y$ and let $\varphi : V \to \mathbb{C}^n$ be an embedding of $V$ into $\mathbb{C}^n$ with $\varphi(p) = 0 \in \mathbb{C}^n$. Finally, let $p(z) = \sum z_i \bar{z}_i$ be the usual norm in $\mathbb{C}^n$ and let $\tilde{p} = p \circ \varphi \circ \pi : \pi^{-1}(V) \to R$. Then $\partial\bar{\partial}p$ is positive definite off of $Q$ and $Q = \{\xi \in \pi^{-1}(V) \mid \tilde{p}(\xi) = 0\}$. Now define

$$\hat{p}(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{p}(e^{i\sigma}\xi)\,d\sigma .$$

This is well-defined in $T = \bigcap_{\sigma} e^{i\sigma}[\pi^{-1}(V)]$. By differentiating under the integral, we see that $\partial\bar{\partial}\hat{p}$ is positive definite off of $Q$. Furthermore, $\hat{p}$ is invariant under the action $e^{i\sigma}$. Choosing $\eta$ sufficiently small we can arrange that $\{\xi \in T \mid \hat{p}(\xi) < \eta\} \subset\subset T$ and defines a disc bundle in $L$ which is strongly pseudoconvex. It follows that the metric whose unit disc bundle is this disc bundle has negative curvature; the theorem is proven.

## References

[1] Klaus Fritzsche, "Pseudoconvexity properties of complements of analytic subvarieties", *Math. Ann.* **230** (1977), 107-122.

[2] Hans Grauert, "Über Modifikationen und exzeptionelle analytische Mengen", *Math. Ann.* **146** (1962), 331-368.

[3] Hans Grauert und Oswald Riemenschneider, "Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen", *Invent. Math.* **11** (1970), 263-292.

[4] Phillip A. Griffiths, "Hermitian differential geometry, Chern classes, and positive vector bundles", *Global analysis*, 185-251 (University of Tokyo Press, Tokyo, 1969).

[5] R.C. Gunning, *Lectures on Riemann surfaces* (Princeton University Press, Princeton, New Jersey, 1966).

[6] Robert C. Gunning, Hugo Rossi, *Analytic functions of several complex variables* (Prentice-Hall, Englewood Cliffs, New Jersey, 1965).

[7] K. Kodaira, "On cohomology groups of compact analytic varieties with coefficients in some analytic faisceaux", *Proc. Nat. Acad. Sci. USA* **39** (1953), 865-868.

[8] K. Kodaira, "On a differential-geometric method in the theory of analytic stacks", *Proc. Nat. Acad. Sci. USA* **39** (1953), 1268-1273.

[9] K. Kodaira, "On Kähler varieties of restricted type (An intrinsic characterization of algebraic varieties)", *Ann. of Math.* (2) **60** (1954), 28-48.

[10] James Morrow, Kunihiko Kodaira, *Complex manifolds* (Holt, Rinehart and Winston, New York, London, Sydney, 1971).

[11] Shigeo Nakano, "On complex analytic vector bundles", *J. Math. Soc. Japan* **7** (1955), 1-12.

[12] Joshua H. Rabinowitz, "Moišezon spaces and positive coherent sheaves", *Proc. Amer. Math. Soc.* **71** (1978), 237-242.

[13] Joshua H. Rabinowitz, "Positivity notions for coherent sheaves over compact complex spaces", *Invent. Math.* (to appear).

[14] Oswald Riemenschneider, "Characterizing Moišezon spaces by almost positive coherent analytic sheaves", *Math. Z.* 123 (1971), 263-284.

[15] Andre Weil, *Introduction ā lētude des variētēs kählēriennes*, nouvelle edition corrigee (Hermann, Paris, 1971).

[16] R.O. Wells, Jr., *Differential analysis on complex manifolds* (Prentice-Hall, Englewood Cliffs, New Jersey, 1973).

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