UNIQUENESS IN BOUNDARY VALUE PROBLEMS FOR THE SECOND ORDER HYPERBOLIC EQUATION

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Introduction. We study linear normal hyperbolic partial differential equations of the second order, with one dependent variable u, and N independent variables x^i ($i = 1, \ldots, N$). The uniqueness theorem connected with the Cauchy problem for this type of equation is well known and in effect states that if u and its first normal derivatives vanish on a spacelike initial surface S then u vanishes in a certain conical region which contains S (**1**, p. 379). In the present work we also envisage a timelike cylindrical surface T which meets S in a rim C of N-2 dimensions, and we assign a single homogeneous boundary condition, of the type familiar from potential theory, on T. The homogeneous Cauchy conditions are also assumed on that part of S which is inside T. We shall prove that the solution then vanishes identically in the region inside T. If the homogeneous boundary condition is given for a certain "time interval" along T, the proof shows that u vanishes in this same interval of the timelike variable.

The boundary conditions considered are of the Dirichlet, Neumann, and Robin type. Uniqueness with the Dirichlet condition has been proved by Hörmander (**4**) using a different approach by which estimates of the solution in the non-homogeneous case can be found. The method used here is also applied to systems of second order normal hyperbolic equations having similar second order terms.

1. Geometric background. Let the differential equation be written in the form

$$L[u] = a^{ik}\frac{\partial^2 u}{\partial x^i \partial x^k} + \beta^i \frac{\partial u}{\partial x^i} + cu = 0. \tag{1.1}$$

Here the summation convention for repeated indices is understood, i and k ranging from 1 to N. The coefficients in the equation are assumed to be four times continuously differentiable (C^4) functions of the x^i. The signature of the quadratic form $a^{ik}\,\xi_i\xi_k$ is taken to be (1, N-1), so that the equation is normal hyperbolic. We may suppose $a^{ik} = a^{ki}$ without loss of generality.

The geometrical aspect is best treated by means of the Riemannian geometry associated with the coefficients a^{ik}; these latter have under coordinate changes the transformation law of a contravariant tensor. Since (1.1) is hyperbolic, the determinant $|a^{ik}|$ is not zero; hence the matrix (a^{ik}) has an inverse (a_{ik}).

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This definition of the a_{ik}, which in explicit form reads

$$a^{ik}a_{kj} = \delta^i{}_j$$

shows that a_{ik} and a^{ik} are associate tensors, a_{ik} being covariant and symmetric.

The Riemannian metric associated with (1.1) is determined by the line-element

$$ds^2 = a_{ik}\, dx^i\, dx^k,$$

which also has the signature $(1, N - 1)$. The measure of lengths and angles determined by this metric will be used throughout this paper. A vector v^i is timelike if its square length

$$v^2 = a_{ik}v^iv^k = a^{ik}v_iv_k$$

is positive, and spacelike if v^2 is negative. The vector is null if $v^2 = 0$. The normal vector to a surface $\phi = \text{const.}$ has contravariant components $a^{ik}\partial\phi/\partial x^k$, and the surface is spacelike if its normal is timelike and vice versa. The surface is null if its normal vector is null; and is then a characteristic surface of the hyperbolic differential equation. To each point P with coordinates x^i there is associated a null or characteristic conoid with vertex P.

According to the theory of invariant differential operators in Riemannian spaces, the operator L can be written in the form

$$L[u] = \Delta u + b\cdot\nabla u + cu,$$

where ∇u denotes the gradient vector, where c is a scalar invariant and b a vector; and where

$$\Delta u = \frac{1}{\sqrt{a}}\frac{\partial}{\partial x^i}\left(\sqrt{a}\, a^{ik}\frac{\partial u}{\partial x^k}\right)$$

is the Laplacian of u (**2**, §4.2).

The timelike cylindrical hypersurface T upon which the boundary conditions shall be assigned is understood to be topologically equivalent to the product of a compact closed $(N - 2)$-dimensional manifold (spacelike) and a line (timelike). By a spacelike cross-section of T, or surface spanning T, is meant a spacelike surface which divides the region interior to T into two separated regions. With respect to the spanning surface, these correspond to the past and future. We assume that T does not touch or intersect itself and therefore that the interior region is well defined. The equation defining T in terms of the given coordinates shall be four times continuously differentiable.

Though we may regard T as extending far into the past and future, we shall in practice work with a finite length of it defined in the following way. Let R denote the region interior to T which is covered by a 1-parameter family of spacelike surfaces Σ_t spanning T. If S is a given spacelike initial surface, we may take for R the region covered by surfaces Σ_t geodesically parallel to S. Therefore such a region R always exists, given S, though it may be of finite

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extent. If the space is flat, there exists a family Σ_t of parallel planes such that R extends arbitrarily far into the future.

For simplicity we assume that the initial surface S is Σ_0. It is actually sufficient if the rim $C = S \cap T$ should lie in the closure of R, since we can suppose that the solution function vanishes identically in R "before" or "below" S.

Then the problem to be studied may be formulated as follows. Let

$$L[u] = 0 \qquad \text{in } R; \tag{1.2}$$

and let the Cauchy data for u on S vanish:

$$u = 0, \ \frac{\partial u}{\partial n} = 0 \qquad \text{on } S. \tag{1.3}$$

Let u satisfy on T either

$$u = 0 \quad \text{or} \quad \frac{\partial u}{\partial n} = 0. \tag{1.4}$$

We show that u vanishes in R. We may assume that u vanishes in the interior of T below $S = \Sigma_0$.

From the theory of Cauchy's problem it is known that (1.2) and (1.3) ensure the vanishing of u in the domain of dependence D_s upon S (see **1**, p. 310; **2**, p. 62; **3**, ch. 9 §5). This domain D_s is bounded by two characteristic surfaces each containing the rim $C = T \cap S$. We shall first establish the result for a second family of surfaces S_t constructed in the following section, which may exist only over small timelike intervals. This will demonstrate that if (1.3) hold on Σ_t, then u will vanish on Σ_τ, for $t \leqslant \tau \leqslant t + \epsilon$, where ϵ may depend on t. From the Heine-Borel theorem it will follow that (1.2), (1.3), and (1.4) ensure the vanishing of u throughout R.

At each step of this process we know that u vanishes in the domain of dependence on Σ_t. That is, u vanishes on $\Sigma_{t+\epsilon}$ except for a strip of width of order ϵ adjoining the boundary $\Sigma_{t+\epsilon} \cap T$ of $\Sigma_{t+\epsilon}$. We have therefore to prove that u vanishes in this strip.

2. A coordinate system. Let the initial surface S cut T in the rim $C = C_0$ of $N - 2$ dimensions. We now denote by S_0 the portion of S bounded by T.

Let $T(n)$ denote the timelike surface geodesically parallel to T at inward distance n. In drawing this surface we first select a compact region of T, such as the portion of T intercepted by Σ_{-t} and Σ_t. The parallel surfaces generated by this region of T are then defined for n sufficiently small, say $0 \leqslant n \leqslant n_0$. and are of class C^3 in the x^i. The $T(n)$ are timelike since their normals are spacelike geodesics (**1**, p. 365; **3**, p. 45).

Now let timelike geodesics g_T of the cylinder subspace T be drawn from each point of C_0, orthogonal to C_0 in T. The locus of points at subspace geodesic

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distance t from C_0 is an $N - 2$ dimensional surface C_t, geodesically parallel to C_0 in T. The g_T are also orthogonal to C_t in T, and hence also when both are regarded as loci in the full space-time (**3**, p. 69).

At each point of C_t let the spacelike geodesics, normal to T, in the full space, be drawn inward; these have length n_0. The geodesics drawn from C_t generate a C^3 surface S_t which is topologically the product of C_t by an interval. Let S_t meet $T(n_0)$ in the inner rim $C_t(n_0)$.

Provided that n_0 is chosen sufficiently small, and that a compact t-interval is considered, the surfaces S_t will be spacelike. To verify this, we note that on T the normals to S_t coincide in direction with the timelike sub-geodesics g_T. Since the direction of the normal varies continuously on S_t, there is an n-interval of positive length $n(t)$ whereon S is spacelike. We have therefore to choose $n_0 < \min n(t)$.

We now assert that for sufficiently small t, the inner rim $C_t(n_0)$ lies in D_S. We note, again presuming n_0 was chosen sufficiently small in the first place, that $C_0(n_0)$ lies in the interior of D_S. Since $C_t(n_0)$ varies continuously with t, the statement is valid for some t-interval, say $0 \leqslant t \leqslant t_0$. Consequently u and its derivatives will vanish identically on $C_t(n_0)$ in this interval.

Coordinates $\xi_1, \ldots, \xi_{N-1}, t$ of the points on S_t will be assigned as follows. Let $\xi_1, \ldots, \xi_{N-2}$ be coordinates parametrizing C_0. Given P on S_t, and $T(n)$, follow the spacelike geodesic of S_t through P back to C_t on T, and then follow the subspace geodesic g_T along T to C_0, meeting C_0 at $\xi_1, \ldots, \xi_{N-2}$, say. We then take $\xi_1, \ldots, \xi_{N-2}, \xi_{N-1} \equiv n$ and t as coordinates of P on S_t.

It is easily shown that the g_T are C^2 curves (with respect to the x^i), and that the geodesics of S_t are C^3. It follows that the (ξ_r, t) coordinates are related to the (x^i) coordinates by a C^2 transformation. Hence also the components a^{ik} of the metric tensor, expressed in the system, are C^1 functions of ξ_r and t.

On the boundary surface T, the parametric lines of the ξ_r, $\alpha = 1, \ldots, N - 1$ are orthogonal to the parametric lines of t. To show this, we recall that the g_T, which are the parametric lines of t on T, are the orthogonal trajectories of the C_t. This shows the statement is true for $\xi_1, \ldots, \xi_{N-2}$. Finally, the parametric lines of $n = \xi_{N-1}$ are the geodesics normal to T, and therefore also orthogonal to the g_T. This shows that the parametric lines of ξ_{N-1} are orthogonal to the t-lines as well. It follows that in the (ξ_r, t) coordinates the components a_{rN} $(r = 1, \ldots, N - 1)$ of the metric tensor vanish on T. Therefore the contravariant components $a^{rN} (r = 1, \ldots, N - 1)$ which appear as coefficients in the differential equation, also vanish there, as a brief computation shows. Thus, on T,

$$a^{rN} = 0, \qquad r \neq N. \tag{2.1}$$

3. The basic inequality. In the (ξ^α, t) coordinate system the differential equation can be written

$$L[u] = a^{NN}u_{tt} + a^{rs}u_{\xi^r\xi^s} + 2a^{Nr}u_{\xi^r} + \beta^r u_{\xi^r} + \beta^N u_t + cu = 0, \tag{3.1}$$

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where summation over r and s runs from 1 to $N - 1$. The quadratic form $Q_0(x) = a^{rs}x_r x_s$ is negative definite on account of the signature of the overall metric. Also $a^{NN} > 0$. Let us therefore define a positive definite metric in the subspace S_t: t = const. by writing

$$g^{rs} = -\frac{a^{rs}}{a^{NN}}. \tag{3.2}$$

We may define the associate tensor g_{rs} in the usual way since $|g^{rs}| \neq 0$. The volume element dS_t is $\sqrt{g}\, d\xi_1 \ldots d\xi_{N-1}$ where $g = |g_{rs}|$.

Let us denote by ∇_t and Δ_t the gradient and Laplacian operators in the subspace S_t. Then the differential equation takes the form

$$u_{tt} = \Delta_t u + b_t \cdot \nabla_t u + c_1 u + \beta_1 u_t + \sum_{r=1}^{N-1} \beta_1^r u_{t\xi^r} \tag{3.3}$$

where $(b_t)^r = -\beta^r/a^{NN}$, $c_1 = -c/a^{NN}$, $\beta_1 = -\beta^N/a_{NN}$, and

$$\beta_1^r = -2a^{Nr}/a^{NN}. \tag{3.4}$$

We note that the vector $\beta_1{}^r$ vanishes on the rim C_t.

We consider the positive definite integral

$$E(t) = \int_{S_t} \{u_t^2 + (\nabla_t u)^2\}\, dS_t, \tag{3.5}$$

which vanishes only if all first derivatives of u with respect to ξ^r and t vanish on S_t. Now, differentiating with respect to t, we find

$$\frac{dE(t)}{dt} = 2\int_{S_t} \Big\{u_t u_{tt} + \nabla_t u \cdot \nabla_t u_t + C^{rs} u_{\xi^r} u_{\xi^s} + [u_t^2 + (\nabla_t u)^2] \frac{\partial \log |g|^{\frac{1}{2}}}{\partial t}\Big\}\, dS_t. \tag{3.6}$$

Here the C^{rs} contain partial derivatives of the g^{rs} with respect to t. Replacing u_{tt} by its expression in (3.2), we have

$$\frac{dE(t)}{dt} = 2\int_{S_t} \{u_t \Delta_t u + \nabla_t u \cdot \nabla_t u_t + u_t b_t \cdot \nabla_t u + c_1 u u_t + \beta_2 u_t^2 + \beta^r u_{t\xi^r} u_t + C_2{}^{rs} u_{\xi^r} u_{\xi^s}\}\, dS_t. \tag{3.7}$$

Here summation over r, s from 1 to $N - 1$ is understood, while β_2 and the $C_2{}^{rs}$ are new coefficients, of the same type as β_1 and C^{rs}, which include the terms arising from the derivative of $g(t)$.

The terms in the integrand of (3.7) will be separated into three sets. To the first two terms we apply the Gauss theorem for the domain S_t with boundary $C_t - C_t(n_0)$, finding

$$\int_{S_t} \{u_t \Delta_t u + \nabla_t u_t \cdot \nabla_t u\}\, dS_t = \int_{S_t} \nabla_t \cdot \{u_t \nabla_t u\}\, dS_t = \int_{C_t - C_t(n_0)} u_t \frac{\partial u}{\partial n}\, ds_t. \tag{3.8}$$

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Here ds_t is the "surface element" on C_t and $C_t(n_0)$, while $\partial/\partial n$ denotes the normal derivative in S_t across C_t or $C_t(n_0)$. However on C_t the direction of this differentiation is also normal to T in the full space, and is consequently the normal to T in the full space. That is, the derivative $\partial u/\partial n$ in (3.8) is the normal derivative in the sense of the given boundary value problems. We also note that $C_t(n_0)$ lies in D_S for $0 \leqslant t \leqslant t_0$, and since u vanishes identically in D_S the integral over $C_t(n_0)$ drops out, provided $t < t_0$. Thus the integrals of the first two terms on the right are together equal to

$$\int_{C_t} u_t \frac{\partial u}{\partial n} ds_t.$$

The term containing the mixed second derivatives may be transformed to an integral containing only first derivatives. We have, applying the Gauss theorem on S_t to the vector $u_t^2\beta_1^r$,

$$\int_{C_t - C_t(n_0)} \beta_{1n} u_t^2 ds_t = \int_{S_t} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^r}(\sqrt{g}\, u_t^2 \beta_1^r)\, dS_t$$
$$= \int_{S_t} u_t^2 \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^r}(\sqrt{g}\, \beta_1^r)\, dS_t + 2\int_{S_t} \beta_1^r u_t u_{t\xi^r} dS_t.$$

Here the integral over $C_t(n_0)$ vanishes as before, and now the integral over C_t drops out since all the components of the "vector" β_1^r vanish on C_t. We then obtain

$$\int_{S_t} \beta_1^r u_t u_{t\xi^r} dS_t = -\tfrac{1}{2} \int_{S_t} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^r}(\beta_1^r \sqrt{g})\, u_t^2 dS_t. \tag{3.9}$$

From (3.7), (3.8) and (3.9) we find

$$\frac{dE(t)}{dt} = 2\int_{C_t} u_t u_n ds_t$$
$$+ 2\int_{S_t} \{u_t b^r u_{\xi^r} + cuu_t + \beta_2 u_t^2 + C_1^{rs} u_{\xi^r} u_{\xi^s}\}\, dS_t,$$

where β_2 incorporates the term (3.9). Here the integral over S_t is a quadratic functional of u and the first derivatives of u. We integrate this equation from 0 to t, noting that $E(0) = 0$ owing to the homogeneous Cauchy conditions on S_0. Thus

$$E(t) = 2\int_0^t d\tau \int_{C_\tau} u_\tau u_n ds_\tau \tag{3.10}$$
$$+ 2\int_0^t d\tau \int_{S_\tau} \{u_\tau b^r u_{\xi^r} + cuu_\tau + \beta_2 u_\tau^2 + C_1^{rs} u_{\xi^r} u_{\xi^s}\}\, dS_\tau.$$

The second term on the right is again a quadratic functional of u and its first derivatives. Now if, as we shall assume, these vanish for $t = 0$, we can follow the method (**1**, p. 310) used in the uniqueness theorem for the Cauchy problem, and find an estimate for this quantity in terms of $E(t)$. The calculation

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is straightforward, and shows that the second term on the right is less than

$$K\int_0^t E(\tau)\,d\tau,$$

for some constant K.

Let us now assume that the surface integral, the first term on the right of (3.10), is non-positive. Then we have

$$E(t) \leqslant K\int_0^t E(\tau)\,d\tau,$$

and on integrating from 0 to α, say, we find

$$\int_0^\alpha E(t)\,dt \leqslant \alpha \max E(t) \leqslant K\alpha \int_0^\alpha E(\tau)\,d\tau.$$

Thus either $K\alpha \geqslant 1$, or else $E(t) \equiv 0$ for $0 \leqslant t \leqslant \alpha$. The first alternative is surely false for $\alpha < K^{-1}$, hence we conclude that $E(t) \equiv 0$, $0 \leqslant t < K^{-1}$. That is, all first derivatives of u vanish in the region bounded by T, S, and S_{K-1}. Since $u = 0$ on S, $u \equiv 0$ in the region.
This establishes the following

LEMMA. *If*

$$J(t) = \int_0^t d\tau \int_{C\tau} u_\tau u_n ds_\tau \leqslant 0,$$

for $0 \leqslant t \leqslant t_1$, *then* $u \equiv 0$ *in the region covered by* S_t, $0 \leqslant t \leqslant t_1$.

4. Uniqueness for the Dirichlet and Neumann problems. The uniqueness in the large follows from this Lemma by an application of a "Heine-Borel" argument. If Σ_t is a surface of the family covering R, and if u has vanishing Cauchy data on Σ_t, then we may take Σ_t as the S_0 of the Lemma. Assuming that $J(t) \leqslant 0$, we see that u vanishes in a t-neighbourhood of Σ_t. Now u has zero Cauchy data on Σ_0; and so vanishes in a t-neighbourhood of Σ_0. If now

$$t_0 = \underset{u\neq 0}{\text{g.l.b.}}\ t,$$

we see that $t_0 > t$ for every t^1 of R. Hence $u \equiv 0$ in R.

Any boundary condition which ensures that the integrand $u_n u_\tau$ of $J(t)$ in the Lemma is non-positive will lead to a uniqueness theorem. For the Dirichlet boundary $u = 0$ on T we have by differentiation, tangential to T, $u_\tau \equiv 0$, and hence $J(t) \equiv 0$. Similarly for the Neumann condition $u_n = 0$. Indeed it is sufficient if $u = 0$ in a certain open subset of T and if $u_n = 0$ on the complementary part of T. Thus we have

THEOREM I. *Let R be a region covered by spacelike surfaces Σ_t and bounded by $S = \Sigma_0$, Σ_{t_1}, and the timelike surface T. Let $L[u] = 0$ in R, let $u \in C^2$ in R and let u have vanishing Cauchy data on S. If then either*

$$u = 0 \quad \text{or} \quad u_n = 0 \qquad \text{on } T$$

then

$$u \equiv 0 \qquad \text{in } R.$$

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5. Uniqueness for the Robin problem. The boundary condition here is

$$\frac{\partial u(p)}{\partial n} + h(p)\,u(p) = 0, \qquad p \in T. \tag{5.1}$$

We shall assume that $h(p)$ is C^2 in the original coordinates, and show that a function $k(P)$ can be constructed, so that

$$v(P) = k(P)\,u(P) \tag{5.2}$$

satisfies a similar hyperbolic equation

$$L_1[v] = \Delta v + b_1\cdot\nabla v + c_1 v = 0, \tag{5.3}$$

together with the Neumann boundary condition

$$\frac{\partial v}{\partial n} = 0. \tag{5.4}$$

From the theorem just proved will then follow the vanishing of v, and hence u, in R.

We now construct the function $k(P)$, requiring $k(P) > 0$, $k(P) \in C^2$. Let n denote inward normal distance from T, and let p_0 be the foot of the normal geodesic to T which passes through a point P in the interior of T, sufficiently close to T. For $0 \leqslant n \leqslant \frac{1}{2}n_0$ we set

$$k(P) = k_0(P) = e^{h(p_0)n} > 0. \tag{5.5}$$

For $n \geqslant n_0$ we set $k(P) \equiv 1$ and for $\frac{1}{2}n_0 \leqslant n \leqslant n_0$ we construct the function

$$k(P) = k_0(P)\,\rho\left(\frac{n}{n_0}\right) + 1 - \rho\left(\frac{n}{n_0}\right). \tag{5.6}$$

Here $\rho(x) \in c\infty$ shall be non-negative, equal to 1 for $x \leqslant \frac{1}{2}$, and equal to zero for $x \geqslant 1$.

In the neighbourhood of T we have, then,

$$v(P) = e^{h(p_0)n}u(P),$$

so

$$\frac{\partial v(p_0)}{\partial n} = e^{h(p_0)n}\left\{\frac{\partial u(p_0)}{\partial n} + h(p_0)\,u(p_0)\right\} = 0.$$

That is, $v(P)$ satisfies a homogeneous Neumann condition on T. If u has vanishing Cauchy data on an initial surface S then $u \equiv 0$ in the domain of dependence D_s. Thus $v \equiv 0$ in D_s and so v has vanishing Cauchy data on S as well.

The differential equation satisfied by v is found by straightforward calculation to be (5.3), where

$$b_1 = -2\nabla \log k(P) + b,$$

and

$$c_1 = k(P)\,L[1/k(P)].$$

Since $k(P) \geqslant \delta > 0$ these coefficients are bounded, and continuous. The proof of Theorem I requires only these properties of these coefficients.

This establishes the uniqueness theorem for the boundary condition of the third kind.

THEOREM II. *Let S and T satisfy the conditions of Theorem I, and let R be defined as above. Let* $u \in C^2$, $L[u] = 0$ *in R, and let*

$$\frac{\partial u(p)}{\partial n} + h(p)\, u(p) = 0, \qquad p \in T$$

where $h(p) \in C^2$. *Then* $u(P) \equiv 0$ *in R.*

This result is also seen to be valid if $u \equiv 0$ on an open subset of T while the above Robin boundary condition holds on the complementary portion of T. It is seen that no condition of positivity for $h(p)$ is required.

6. Systems with similar second order terms. The preceding theorems hold also for systems of linear normal hyperbolic second order equations in which the principal parts of all equations are similar. Consider the system

$$\Delta u_{(m)} + \sum_n b_{mn} \cdot \nabla u_{(n)} + \sum_n c_{mn} u_{(n)} = 0, \tag{6.1}$$

wherein m and n are indices of enumeration for the M dependent variables $u_{(m)}$. Also b_{mn} are an array of vectors, and c_{mn} of scalars, which we assume to be C^1 and C, respectively. Here Δ has its previous meaning. For this system, the uniqueness for the Cauchy problem has in effect been proved in **(2**, pp. 62–64**)**, since the argument given there applies to any system of the form (6.1).

The coordinate system of §2 being taken as before, we may repeat the calculations of §3 leading to the Lemma. We write the differential equations in the form

$$u_{(m)\,tt} = \Delta_t u_{(m)} + \sum_n B_{mn} \cdot \nabla_t u_{(n)} + \sum_n c_{mn} u_{(n)} + \sum_n \beta_{mn} u_{(n)\,t} + \beta_1{}^r u_{(m)\,t\xi^r}. \tag{6.2}$$

Defining

$$E_{(m)}(t) = \int_{S_t} \{u_{(m)\,t}^2 + (\nabla_t u_{(m)})^2\}\, dS_t, \tag{6.3}$$

we find on differentiating with respect to t that

$$\frac{1}{2}\frac{dE_{(m)}(t)}{dt} = \int_{S_t} \{u_{(m)\,t}\Delta_t u_{(m)} + \nabla_t u_{(m)} \nabla_t u_{(m)\,t} + \sum_n \bar{B}_{mn} \cdot \nabla_t u_{(n)} u_{(m)\,t} + \sum_n \bar{C}_{mn} u_{(n)} u_{(m)\,t} + \sum_n \bar{\beta}_{mn} u_{(n)\,t}^2 + \beta_1^r u_{(m)\,t} u_{(m)\,t\xi^r}\}\, dS_t. \tag{6.4}$$

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The first two terms on the right again lead to the surface integral

$$\int_{C_t - C_t(n_0)} u_{(m)t} \frac{\partial u_{(m)}}{\partial n} dS_t, \tag{6.5}$$

while the last term, containing the $u_{(m)\tau\xi^r}$, yields an integral containing only first derivatives, namely

$$-\frac{1}{2}\int_{S_t} u_{(m)t}^2 \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^\tau}(\sqrt{g}\,\beta_1^\tau)\,dS_t.$$

We therefore find

$$\frac{dE_{(m)}}{dt} = Q_{(m)}(u_{(n)},\ u_{(n)\xi^r},\ u_{(n)t}), \tag{6.6}$$

where $Q_{(m)}$ denotes a quadratic integral expression over S_t. Again, integrating, we have

$$E_{(m)}(t) = \int_0^t Q_{(m)} d\tau \tag{6.7}$$

and it follows easily that the integral on the right is dominated by the expression

$$K\int_0^t \sum_{(m)} E_{(m)}(\tau)\,d\tau.$$

We therefore find

$$E(t) \equiv \sum_m E_{(m)}(t) \leqslant MK\int_0^t E(\tau)\,d\tau, \tag{6.8}$$

provided only that for the sum of the surface integrals we have

$$J(t) = \sum_m \int_0^t d\tau \int_{C_\tau} u_{(m)\tau} \frac{\partial u_{(m)}}{\partial n}\,dS_\tau \leqslant 0. \tag{6.9}$$

From (6.8) follows $E(t) \equiv 0$ for $0 \leqslant t \leqslant t_1$, say, and the uniqueness in the large follows as before. The condition (6.9) is satisfied if either

$$u_{(m)} = 0 \text{ or } \frac{\partial u_{(m)}}{\partial n} = 0 \qquad \text{on } T.$$

Indeed $u_{(m)}$ may vanish in an open set $T_{(m)}$ and $\partial_{u(m)}/\partial n$ in the complementary set $T - T_{(m)}$; and the result holds.

Again, we may replace the Neumann condition by the Robin condition

$$\frac{\partial u_{(m)}}{\partial n} + h_{(m)}u_{(m)} = 0, \qquad h_{(m)} \in C^2,$$

by setting

$$v_{(m)}(P) = k_{(m)}(P)\,u_{(m)}(P),$$

as in (5.2). We verify at once that the $v_m(P)$ also satisfy a system of the form (6.1), and that

$$\frac{\partial v_{(m)}(p_0)}{\partial n} = e^{h_{(m)}(p_0)n}\left(\frac{\partial u_{(m)}}{\partial n} + h_{(m)}u_{(m)}\right)$$
$$= 0.$$

Thus the $v_{(m)}$ vanish identically under the boundary conditions discussed above.

We state these results as follows:

THEOREM III. *Let S, T and R satisfy the conditions of Theorem I, and let $u_{(m)}$ be solutions of the system* (6.1) *with zero Cauchy data on Σ_0. Let*

$$u_{(m)} = 0 \qquad (m = 1, \ldots, M),$$

on an open subset $T_{(m)}$ of T, and let

$$\frac{\partial u_{(m)}}{\partial n} + h_{(m)}u_{(m)} = 0$$

on the complementary subset $T - T_{(m)}$. Then

$$u_{(m)} = 0$$

in R.

An important special case of systems such as (6.1) is the equation $\Delta\phi = 0$ of generalized potential theory. The above result applies to the Dirichlet problem for these differential forms. However, the Neumann problem for this equation involves tangential derivatives in spacelike directions on T, and so is not amenable to this method. In fact, uniqueness does not hold for this Neumann problem.

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