Can. J. Math., Vol. XLI, No. 4, 1989, pp. 612–625

WEAKLY CLOSE-TO-CONVEX MEROMORPHIC FUNCTIONS

LAURELLEN LANDAU-TREISNER AND ALBERT E. LIVINGSTON

1. Introduction. Classes of functions, meromorphic and univalent in

$$\Delta = \{z : |z| < 1\}$$

with simple pole at $z = p$, $0 < p < 1$, have been discussed in several places in the literature ([**3**], [**6**], [**8**], [**10**], [**11**], and [**12**]). The purpose of this paper is to discuss a class of Close-to-Convex functions with pole at p analogous to the class of Close-to-Convex functions with pole at zero studied by Libera and Robertson [**9**].

Let $\Sigma(p)$ be the class of functions which are univalent and analytic in $\Delta - \{p\}$, with a simple pole at $z = p$, $0 < p < 1$. A function f in $\Sigma(p)$ with $f(0) = 1$ is said to be in $\Lambda^*(p)$ if f maps Δ onto a domain whose complement is starlike with respect to the origin. The class $\Lambda^*(p)$ has been studied in ([**3**], [**8**], [**10**], [**11**], and [**12**]). Functions $f(z)$ in $\Lambda^*(p)$ are characterized by the fact that there exists F in Σ^*, the class of meromorphic univalent starlike functions with pole at zero of residue one, such that

$$f(z) = \frac{-pz}{(z-p)(1-pz)}F(z). \tag{1.1}$$

We let $I(0)$ be the class studied by Libera and Robertson [**9**]. Thus h is in $I(0)$, if h is analytic in $\Delta - \{0\}$ with a simple pole of residue one at $z = 0$ such that there exists G in Σ^* and α, $|\alpha| \leqq \pi$, so that

$$\operatorname{Re}\left(\frac{zh'(z)}{e^{i\alpha}G(z)}\right) > 0$$

for $0 < |z| < 1$.

Analogously, if $0 < p < 1$, we let $I(p)$ be the class of functions f, analytic in $\Delta - \{p\}$, with a simple pole at $z = p$ and such that there exists g in $\Lambda^*(p)$, an α, $|\alpha| \leqq \pi$, and a $\delta, 0 < \delta < 1$, so that

$$\operatorname{Re}\left(\frac{zf'(z)}{e^{i\alpha}g(z)}\right) > 0 \tag{1.2}$$

for $\delta < |z| < 1$.

Received May 2, 1988.

https://doi.org/10.4153/CJM-1989-027-7 Published online by Cambridge University Press

In what follows we let $\mathcal{P}^*$ be the class of functions $P(z)$, analytic in Δ satisfying $\operatorname{Re} P(z) > 0$ for z in Δ.

THEOREM 1. *If f is in $I(p)$, there exists g in $\Lambda^*(p)$, an $\alpha, |\alpha| \leqq \pi$ and a $P(z)$ in $\mathcal{P}^*$ such that for z in $\Delta - \{p\}$,*

$$(1.3) \qquad f'(z) = \frac{e^{i\alpha}}{(z-p)(1-pz)} g(z)P(z).$$

Proof. There exists g in $\Lambda^*(p)$, an $\alpha, |\alpha| \leqq \pi$, and a $\delta, 0 < \delta < 1$, such that (1.2) holds.

Let $\delta < r < 1$ and $f_r(z) = f(rz)$ and $g_r(z) = g(rz)$, then

$$(1.4) \qquad \operatorname{Re}\left(\frac{zf_r'(z)}{e^{i\alpha}g_r(z)}\right) > 0$$

for $|z| = 1$. The function $zf_r'(z)/e^{i\alpha}g_r(z)$ in analytic in $\overline{\Delta}$ except for a simple pole at $z = p/r$. Let

$$P_r(z) = \frac{(z - \frac{p}{r})(1 - \frac{p}{r}z)}{z}\left(\frac{zf_r'(z)}{e^{i\alpha}g_r(z)}\right).$$

Since $(z - p/r)(1 - pz/r)$ is real and positive for $|z| = 1$, it follows from (1.4) that $\operatorname{Re} P_r(z) > 0$ for $|z| = 1$. Since $P_r(z)$ is analytic for $|z| \leqq 1$, it follows that $\operatorname{Re} P_r(z) > 0$ for $|z| \leqq 1$. Since $P_r(0) = -pf'(0)e^{-i\alpha}$ is independent of r, there exists a sequence r_n tending to 1 such that P_{r_n} converges uniformly on compact subsets of Δ to $P(z)$ in $\mathcal{P}^*$. Since $f_{r_n}(z)$ and $g_{r_n}(z)$ converge uniformly on compact subsets of $\Delta - \{p\}$ to $f(z)$ and $g(z)$ respectively, it follows that

$$P(z) = \frac{(z-p)(1-pz)}{z}\left(\frac{zf'(z)}{e^{i\alpha}g(z)}\right)$$

from which we obtain (1.3).

COROLLARY 1. *If $f(z)$ is in $I(p)$, then $f'(z) \neq 0$ for $z \neq p$.*

Because of the corollary, there is no loss in generality in assuming that $f'(0) = 1$ for $f(z)$ in $I(p), 0 < p < 1$. In the sequel we therefore make the added assumption that $f'(0) = 1$ for f in $I(p), 0 < p < 1$.

Because of Theorem 1, we also define another class of functions $I^*(p)$. We will say that $f(z)$ is in $I^*(p), 0 < p < 1$, if it is analytic in $\Delta - \{p\}$ with a simple pole at $z = p$ and $f'(0) = 1$, and there exists $g(z)$ in $\Lambda^*(p)$, a $P(z)$ in $\mathcal{P}^*$ and an $\alpha, |\alpha| \leqq \pi$, so that

$$(1.5) \qquad f'(z) = \frac{e^{i\alpha}}{(z-p)(1-pz)} g(z)P(z).$$

A function f is said to be in $I^*(0)$ if it is analytic in $\Delta - \{0\}$ with a simple pole at $z = 0$ and there exists $g(z)$ in Σ^*, $P(z)$ in $\mathcal{P}^*$ and $\alpha, |\alpha| \leqq \pi$, such that

$$(1.6) \qquad f'(z) = e^{i\alpha} g(z) P(z).$$

Thus, by Theorem 1, $I(p) \subset I^*(p)$. Also, $I(0) = I^*(0)$.

We widen the class $I^*(p)$ to allow for logarithmic singularities at $z = p$. In the sequel the statement "$f'(z)$ is analytic in $\Delta - \{p\}$" will refer to a function $f(z)$ which is analytic in $\Delta - \{z : p \leqq z < 1\}$ and such that $f'(z)$ can be analytically continued in $\Delta - \{p\}$. In what follows $\Lambda^*(0) = \Sigma^*$.

We will say that f is in $J^*(p), 0 \leqq p < 1$, if $f'(z)$ is analytic in $\Delta - \{p\}$ and there exists g in $\Lambda^*(p)$, and $\alpha, |\alpha| \leqq \pi$ and $P(z)$ in $\mathcal{P}^*$ such that

$$(1.7) \qquad f'(z) = \frac{e^{i\alpha}}{(z-p)(1-pz)} g(z) P(z).$$

The essential difference between $J^*(p)$ and $I^*(p)$ is that in $J^*(p)$ we are allowing the function to possibly have a logarithmic type singularity at $z = p$. That is, for z in $\{z : |z - p| < 1 - p\} - \{z : p \leqq z < 1\}$

$$f(z) = \frac{\alpha}{z-p} + \beta \log(z-p) + \sum_{n=0}^{\infty} c_n (z-p)^n.$$

If f satisfies (1.7) with $0 < p < 1$, then it is easily seen that $f'(z)/f'(0)$ has the form (1.7). We will thus assume, without loss of generality, that $f'(0) = 1$ for all f in $J^*(p), 0 < p < 1$. Similarly we may assume with loss of generality that

$$\operatorname{Res}(zf'; 0) = -1 \quad \text{for all } f \text{ in } J^*(0).$$

The following two functions will be important in the sequel. Let F_1 and F_2 be defined by

$$(1.8) \qquad F_1'(z) = \frac{p^2(1-z)^3}{(z-p)^2(1-pz)^2(1+z)}$$

and

$$(1.9) \qquad F_2'(z) = \frac{p^2(1+z)^3}{(z-p)^2(1-pz)^2(1-z)}.$$

Both functions $F_1(z)$ and $F_2(z)$ are members of $J^*(p)$ but not of $I^*(p)$.

2. An alternate definition of $J^*(p)$. In [**8**], functions of $\Lambda^*(p)$ were defined by their relationship with functions of Σ^*. A somewhat different relationship can be found between functions of $J^*(p), 0 < p < 1$, and $J^*(0)$. We first need the following lemma.

LEMMA 1. *If g is in $\Lambda^*(p)$ and $\beta_p = \operatorname{Res}(g;p)$ then*

$$G(z) = \frac{1-p^2}{\beta_p} g\left(\frac{z+p}{1+pz}\right)$$

is in Σ^* and $\beta_p = (1-p^2)/G(-p)$. Conversely, if G is in Σ^* and $\beta_p = (1-p^2)/G(-p)$, then

$$g(z) = \frac{\beta_p}{1-p^2} G\left(\frac{z-p}{1-pz}\right)$$

is in $\Lambda^*(p)$ and $\beta_p = \operatorname{Res}(g;p)$.

Proof. The proof follows from the fact that a properly normalized function is a member of Σ^* or $\Lambda^*(p)$ if and only if it maps Δ onto a domain whose complement is starlike with respect to the origin.

THEOREM 2. *If f is in $J^*(p)[I^*(p)]$ and $\alpha_p = -\operatorname{Res}((z-p)f';p)[\alpha_p = \operatorname{Res}(f;p)]$ then there exists h in $J^*(0)[I^*(0)]$ such that*

$$f(z) = \frac{\alpha_p}{1-p^2} h\left(\frac{z-p}{1-pz}\right). \tag{2.1}$$

Conversely, if h is in $J^*(0)[I^*(0)]$ and $\alpha_p = 1/h'(-p)$, then $f(z)$, defined by (2.1) is in $J^*(p)[I^*(p)]$.

Proof. We will prove the statement about $J^*(p)$. The proof concerning $I^*(p)$ is similar. If f is in $J^*(p)$ then there exists g in $\Lambda^*(p)$, P in $\mathcal{P}^*$ and an α, $|\alpha| \leqq \pi$, such that

$$P(z) = \frac{(z-p)(1-pz)f'(z)}{e^{i\alpha}g(z)}.$$

Thus

$$P_1(z) = \frac{1}{1-p^2} P\left(\frac{z+p}{1+pz}\right)$$

is also in $\mathcal{P}^*$. Now consider the functions

$$h(z) = \frac{1-p^2}{\alpha_p} f\left(\frac{z+p}{1+pz}\right)$$

where $\alpha_p = -\operatorname{Res}((z-p)f';p)$ and

$$G(z) = \frac{1-p^2}{\beta_p} g\left(\frac{z+p}{1+pz}\right)$$

where $\beta_p = \text{Res}(g;p)$. Then,

$$(2.2) \qquad P_1(z) = \frac{\alpha_p}{\beta_p}\frac{zh'(z)}{e^{i\alpha}G(z)} = \left|\frac{\alpha_p}{\beta_p}\right|\frac{zh'(z)}{e^{i\gamma}G(z)}$$

for a suitably chosen γ.

Since g is in $\Lambda^*(p)$, G is in Σ^* by Lemma 1. Also h' is analytic in $\Delta - \{0\}$ with a pole of order 2 at $z = 0$ and $\text{Res}(zh';0) = -1$. From (2.2) we have

$$h'(z) = \frac{e^{i\gamma}G(z)Q(z)}{z}$$

where $Q(z) = |\beta_p/\alpha_p|P_1(z)$ is in $\mathcal{P}^*$. Thus h is in $J^*(0)$ and (2.1) holds. The proof of the converse is similar.

Remark. Since functions in $I^*(0)$ need not be univalent [**9**], it follows from Theorem 2, that functions in $I^*(p)$ need not be univalent.

3. Integral means. In this section we use a technique of Baernstein [**1**] as employed by Leung [**7**] to obtain bounds on the integral means of $|f'|$. For this purpose we first mention some results that will be used.

For $g(x)$, a real valued integrable function on $[-\pi,\pi]$, the Baernstein $*$-function is defined by

$$g^*(\theta) = \sup_{|E|=2\theta}\int_E g,$$

for $0 < \theta \leqq \pi$, where $|E|$ is the Lebesque measure of the set E in $[-\pi,\pi]$.

Statements A, B and C of the following Lemma were proven by Baernstein [**1**] and statement D was proven by Leung [**7**].

LEMMA 2. (A) *For g, h in $L^1[-\pi,\pi]$, the following are equivalent:*

(i) *For every convex non-decreasing function Φ on $(-\infty,\infty)$*

$$\int_{-\pi}^{\pi}\Phi(g(x))dx \leqq \int_{-\pi}^{\pi}\Phi(h(x))dx.$$

(ii) *For every t in $(-\infty,\infty)$*

$$\int_{-\pi}^{\pi}[g(x)-t]^+dx \leqq \int_{-\pi}^{\pi}[h(x)-t]^+dx.$$

(iii) $g^*(\theta) \leqq h^*(\theta)$ *for* $0 \leqq \theta \leqq \pi$.

(B) *If f is in $S = \{f : f$ is analytic and univalent in Δ with $f(0) = 0$ and $f'(0) = 1\}$, then for each r, $0 < r < 1$,*

$$(\pm\log|f(re^{i\theta})|)^* \leqq (\pm\log|K(re^{i\theta})|)^*$$

for any Koebe function $K(z) = z/(1 - e^{i\alpha}z)^2$.

(C) *For* g *in* $L^1[-\pi, \pi]$, *if* $\overline{g}(x)$ *is the symmetric non-increasing rearrangement of* g [**4**], *then*

$$g^*(\theta) = \int_{-\theta}^{\theta} \overline{g}(x)dx = (\overline{g})^*(\theta).$$

(D) *For* g, h *in* $L^1[-\pi, \pi]$,

$$[g(\theta) + h(\theta)]^* \leqq g^*(\theta) + h^*(\theta)$$

with equality if and only if g *and* h *are both symmetric and non-increasing on* $[0, \pi]$.

THEOREM 3. *For any* f *in* $J^*(p)$ *and every convex non-decreasing function* Φ *on* $(-\infty, \infty)$,

$$\int_{-\pi}^{\pi} \Phi(\pm \log |f'(re^{i\theta})|)d\theta \leqq \int_{-\pi}^{\pi} \Phi(\pm \log |F_2'(re^{i\theta})|)d\theta$$

where F_2 *is in* $J^*(p)$ *and is defined by* (1.9).

Proof. If f is in $J^*(p)$ then combining (1.1) and (1.7), there exists G in Σ^* an α, $|\alpha| \leqq \pi$, and P in $\mathcal{P}^*$ such that

$$f'(z) = \frac{-pe^{i\alpha}z}{(z-p)^2(1-pz)^2}G(z)P(z).$$

Thus

$$(3.1) \qquad \log |f'(re^{i\theta})| = \log \frac{pr}{|re^{i\theta} - p|^2|1 - pre^{i\theta}|^2} + \log |G(re^{i\theta})| + \log |P(re^{i\theta})|.$$

The first term on the right side of (3.1) is symmetric and non-increasing on $[0, \pi]$. For the second term, since G is in Σ^*, $1/G$ is in S and by Lemma 2 (B)

$$[\log |G(re^{i\theta})|]^* = \left[-\log \frac{1}{|G(re^{i\theta})|}\right]^* \leqq [-\log |K(re^{i\theta})|]^*.$$

Choose $K(z) = z/(1+z)^2$, then

$$(3.2) \qquad [\log |G(re^{i\theta})|]^* \leqq \left[\log \frac{|1 + re^{i\theta}|^2}{r}\right]^*$$

For the third term, since $p^{-1}P(z)$ is a function of positive real part and $|p^{-1}P(0)| = 1$, by Corollary 1 of [7]

$$(3.3) \qquad [\log |P(re^{i\theta})|]^* \leqq \left[\log p \left|\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right|\right]^*.$$

Using Lemma 2 (D), we have from (3.1)

$$[\log|f'(re^{i\theta})|]^* \leqq \left[\log\frac{pr}{|re^{i\theta}-p|^2|1-pre^{i\theta}|^2}\right]^* + [\log|G(re^{i\theta})|]^* + [\log|P(re^{i\theta})|]^*.$$

Now using (3.2) and (3.3), we obtain

$$(3.4)\qquad [\log|f'(re^{i\theta})|]^* \leqq \left[\log\frac{pr}{|re^{i\theta}-p|^2|1-pre^{i\theta}|^2}\right]^* + \left[\log\frac{|1+re^{i\theta}|^2}{r}\right]^* + \left[\log p\left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|\right]^*.$$

Since all three functions on the right of (3.4) are symmetric and non-increasing on $[0,\pi]$, by Lemma 2 (D)

$$[\log|f'(re^{i\theta})|]^* \leqq \left[\log\frac{p^2|1+re^{i\theta}|^3}{|re^{i\theta}-p|^2|1-pre^{i\theta}|^2|1-re^{i\theta}|}\right]^* = [\log|F_2'(re^{i\theta})|]^*.$$

By Lemma 2, Part A

$$\int_{-\pi}^{\pi}\Phi(\log|f'(re^{i\theta})|)d\theta \leqq \int_{-\pi}^{\pi}\Phi(\log|F_2'(re^{i\theta})|)d\theta.$$

The proof of the inequality

$$\int_{-\pi}^{\pi}\Phi(-\log|f'(re^{i\theta})|)d\theta \leqq \int_{-\pi}^{\pi}\Phi(-\log|F_2'(re^{i\theta})|)d\theta$$

is similar.

COROLLARY 2. *For any f in $J^*(p)$ and any real λ,*

$$(3.5)\qquad \int_{-\pi}^{\pi}|f'(re^{i\theta})|^\lambda d\theta \leqq \int_{-\pi}^{\pi}|F_2'(re^{i\theta})|^\lambda d\theta.$$

Proof. Apply Theorem 3 with $\Phi(x) = e^{tx}$, $t > 0$.

COROLLARY 3. *If f is in $J^*(p)$, then for $r \neq p$*

$$(3.6)\qquad F_2'(-r) \leqq |f'(re^{i\theta})| \leqq F_2'(r)$$

where $F_2'(z)$ is defined by (1.9).

The bounds in (3.6) are of course sharp in $J^*(p)$ and the upper bound gives

$$\max_{|z|=r} |f'(z)| = O\left(\frac{1}{1-r}\right)$$

for f in $J^*(p)$ and hence for f in $I^*(p)$. The order estimate $O(1/(1-r))$ can not be improved in $I^*(p)$.

To see this we will construct a function f_ϵ for each ϵ, $0 < \epsilon < 1$, such that f_ϵ is in $I^*(p)$ and

$$\lim_{r \to 1} (1-r)^\epsilon |f_\epsilon'(r)| = \infty.$$

Given any ϵ, $0 < \epsilon < 1$, choose ϵ' with $\epsilon < \epsilon' < 1$, then

$$G_\epsilon(z) = \frac{(1-z)^{1-\epsilon'}(1+z)^{1+\epsilon'}}{z}$$

is in Σ^*, and

$$g_\epsilon(z) = \frac{-p(1-z)^{1-\epsilon'}(1+z)^{1+\epsilon'}}{(z-p)(1-pz)} = -\frac{pz}{(z-p)(1-pz)} G_\epsilon(z)$$

is in $\Lambda^*(p)$. Let

$$f_\epsilon(z) = \frac{1}{g_\epsilon'(0)} g_\epsilon(z),$$

then f_ϵ is in $I^*(p)$ and

$$\lim_{r \to 1} (1-r)^\epsilon |f_\epsilon'(r)| = \infty.$$

4. Coefficient estimates.

LEMMA 3. *If g is in $\Lambda^*(p)$ and*

$$g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

for $|z| < p$, then for $n \geqq 1$

$$(4.1) \qquad |b_n| \leqq \frac{(1+p)(1-p^{2n})}{(1-p)p^n}.$$

Equality is attained in (4.1) *by the function*

$$g(z) = \frac{-p(1+z)^2}{(z-p)(1-pz)}.$$

Proof. Using a result of Jenkins [**5**] it was pointed out in [**10**] that (4.1) would be true for all n for which the Bieberbach conjecture is true. Thus, due to the proof of the conjecture by L. deBranges [**2**], we can say that (4.1) is valid for all n.

Using a comparison of coefficients, some detailed computations and Lemma 3, we can prove the following theorem. We omit the proof, choosing instead to prove a slightly different type of inequality in Theorem 5, which relates the coefficients of a function in $J^*(p)$ to those of a function in $I^*(p)$.

THEOREM 4. *If f is in $J^*(p)$ and*

$$f(z) = a_0 + z + \sum_{n=2}^{\infty} a_n z^n$$

for $|z| < p$, then for $n \geqq 2$

$$\begin{aligned} |a_n| \leqq |a_n^{(2)}| &= \frac{1-p^n}{p^{n-1}(1-p)} \sum_{j=0}^{n-1} \left[(2j+1) - \frac{2j(j+1)}{n} \right] p^j \\ &= \frac{1-p^{2n}}{p^{n-1}} \frac{(1+p)}{(1-p)^3} \left[1 - \frac{4p(1-p^n)}{n(1+p^n)(1-p^2)} \right] \end{aligned}$$

where $a_n^{(2)}$ is the n-th coefficient of F_2 in $J^*(p)$ defined by (1.9).

In what follows we let $\alpha_p = -\text{Res}((z-p)f'; p)$. Note that for f in $I^*(p)$, $\alpha_p = \text{Res}(f; p)$.

LEMMA 4. *For f in $J^*(p)$*

$$\text{(4.2)} \qquad \frac{p^2(1-p)}{(1+p)^3} \leqq |\alpha_p| \leqq \frac{p^2(1+p)}{(1-p)^3}.$$

These bounds are sharp, being attained by F_1 and F_2 given by (1.8) *and* (1.9).

Proof. For f in $J^*(p)$, there exists G in Σ^*, an $\alpha, |\alpha| \leqq \pi$, and P in $\mathcal{P}^*$ so that

$$f'(z) = \frac{-pe^{i\alpha}z}{(z-p)^2(1-pz)^2} G(z)P(z).$$

We then have

$$\alpha_p = \lim_{z \to p} (z-p)^2 f'(z) = [-p^2 e^{i\alpha}/(1-p^2)^2] G(p)P(p).$$

Using well known bounds on $|G(p)|$ and $|P(p)|$, we obtain (4.2).

LEMMA 5. *If h is in $J^*(0)$ and*

$$h(z) = \frac{1}{z} + d\log z + \sum_{n=0}^{\infty} c_n z^n$$

for $0 < |z| < 1$, then

$$|d| \leqq 4.$$

Equality is attained by h in $J^*(0)$, defined by

$$h'(z) = \frac{-(1+z)^3}{z^2(1-z)}.$$

Proof. There exists $G(z) = 1/z + B_0 + B_1 z + \cdots$ in Σ^*, and $P(z) = p_0 + p_1 z + \cdots$ in $\mathcal{P}^*$ so that

$$zh' = e^{i\alpha} G(z) P(z).$$

It follows that

$$d = e^{i\alpha}(B_0 p_0 + p_1). \tag{4.3}$$

It is well known that $|B_0| \leqq 2$ and $|p_1| \leqq 2|p_0| = 2$. We thus obtain $|d| \leqq 4$, from (4.3).

LEMMA 6. *If f is in $J^*(p)$ and*

$$f(z) = \frac{\alpha_p}{(z-p)} + d\log(z-p) + \sum_{n=0}^{\infty} c_n (z-p)^n$$

for $\{z : |z-p| < 1-p\} - \{z : p \leqq z < 1\}$, then

$$\left|\frac{d}{\alpha_p}\right| \leqq \frac{4}{1-p^2}$$

and the bound is sharp.

Proof. By Theorem 2 there exists h in $J^*(0)$ so that

$$\frac{f(z)}{\alpha_p} = \frac{1}{1-p^2} h\left(\frac{z-p}{1-pz}\right). \tag{4.4}$$

Let

$$h(z) = \frac{1}{z} + c\log z + \sum_{n=0}^{\infty} d_n z^n \tag{4.5}$$

https://doi.org/10.4153/CJM-1989-027-7 Published online by Cambridge University Press

for $\{z : |z| < 1\} - \{z : 0 \leqq z < 1\}$. Substituting (4.5) in (4.4) and equating coefficients, we obtain

$$\frac{d}{\alpha_p} = \frac{c}{1-p^2}.$$

Lemma 6 now follows by applying Lemma 5.

For sharpness, consider the function

$$f(z) = \frac{\alpha_p}{1-p^2} h\left(\frac{z-p}{1-pz}\right)$$

where h in $J^*(0)$ is the function given in Lemma 5 and $\alpha_p = 1/h'(-p)$. For this function we have

$$d/\alpha_p = -4/(1-p^2).$$

THEOREM 5. *If f is in $J^*(p)$ and*

$$f(z) = a_0 + z + \sum_{n=2}^{\infty} a_n z^n$$

for $|z| < p$, then

$$\lim_{n\to\infty} \left[\frac{\sup_{f\in J^*(p)} \left|\frac{a_n}{\alpha_p}\right|}{\left|\frac{a_n^{(1)}}{\alpha_p^{(1)}}\right|} \right] = 1$$

where $a_n^{(1)}$ and $\alpha_p^{(1)}$ are the coefficients in the expansion about $z = 0$ and the residue at $z = p$ of

$$f_1(z) = \frac{-p^2(1-z)^2}{(1-p)^2(z-p)(1-pz)},$$

which is in $I^*(p)$.

Proof. For $\frac{(1+p)}{2} < R < 1$ and sufficiently small δ,

$$na_n = \frac{1}{2\pi i}\int_{|w|=R} \frac{f'(w)}{w^n}dw - \frac{1}{2\pi i}\int_{|w-p|=\delta} \frac{f'(w)}{w^n}dw. \tag{4.6}$$

The first integral on the right side of (4.6) can be bounded by using Corollary 2 and the inequality

$$\int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|} \leqq C \log\frac{1}{1-r},$$

where C is a constant independent of r [**13**].

$$(4.7)\quad \left|\frac{1}{2\pi i}\int_{|w|=R}\frac{f'(w)}{w^n}dw\right|$$
$$\leqq \frac{1}{2\pi R^{n-1}}\int_{-\pi}^{\pi}|F_2'(Re^{i\theta})|d\theta$$
$$= \frac{1}{2\pi R^{n-1}}\int_{-\pi}^{\pi}\frac{p^2|1+Re^{i\theta}|^3 d\theta}{|Re^{i\theta}-p|^2|1-pRe^{i\theta}|^2|1-Re^{i\theta}|}$$
$$\leqq \frac{p^2(1+R)^3}{2\pi R^{n-1}(R-p)^2(1-pR)^2}\int_{-\pi}^{\pi}\frac{d\theta}{|1-Re^{i\theta}|}$$
$$\leqq \frac{C}{R^n}\log\left(\frac{1}{1-R}\right)$$

where C is a constant independent of f and R.

For the second integral, we note that

$$\frac{1}{2\pi i}\int_{|w-p|=\delta}\frac{f'(w)}{w^n}dw = \operatorname{Res}\left(\frac{f'(z)}{z^n};p\right).$$

Let

$$f'(z) = -\frac{\alpha_p}{(z-p)^2}+\frac{d}{(z-p)}+\sum_{n=0}^{\infty}c_n(z-p)^n$$

for $|z-p|<1-p$, then

$$\frac{f'(z)}{z^n} = \frac{-\alpha_p}{p^n(z-p)^2}+\left(\frac{d}{p^n}+\frac{n\alpha_p}{p^{n+1}}\right)\frac{1}{z-p}+\cdots$$

for $|z-p|<1-p$. Thus

$$\frac{1}{2\pi i}\int_{|w-p|=\delta}\frac{f'(w)}{w^n}dw = \frac{\alpha_p}{p^{n+1}}\left[\frac{pd}{\alpha_p}+n\right].$$

Using Lemma 6, we obtain

$$(4.8)\quad \left|\frac{1}{2\pi i}\int_{|w-p|=\delta}\frac{f'(w)}{w^n}dw\right| \leqq \frac{|\alpha_p|}{p^{n+1}}\left[\frac{4p}{1-p^2}+n\right].$$

Combining (4.6), (4.7) and (4.8) we have,

$$(4.9)\quad n|a_n| \leqq \frac{-C\log(1-R)}{R^n}+\frac{|\alpha_p|}{p^{n+1}}\left[\frac{4p}{1-p^2}+n\right].$$

Using the fact that $1/|\alpha_p| \leqq (1+p)^3/p^2(1-p)$ from Lemma 4, we obtain for $R > (1+p)/2$.

$$\left|\frac{a_n}{\alpha_p}\right| \leqq \frac{1}{p^{n+1}}\left[\frac{-Cp(p/R)^n \log(1-R)}{n|\alpha_p|} + \frac{4p}{n(1-p^2)} + 1\right]$$
$$\leqq \frac{1}{p^{n+1}}\left[\frac{C_{R,p}}{n} + 1\right]$$

where $C_{R,p}$ is a constant independent of n and f.

Since

$$\left|\frac{a_n^{(1)}}{\alpha_p^{(1)}}\right| = \frac{1-p^{2n}}{p^{n+1}}$$

we obtain from (4.9)

$$1 \leqq \frac{\sup_{f\in J^*(p)}\left|\frac{a_n}{\alpha_n}\right|}{\left|\frac{a_n^{(1)}}{\alpha_p^{(1)}}\right|} \leqq \frac{1}{1-p^{2n}}\left[\frac{C_{R,p}}{n} + 1\right].$$

We thus obtain

$$\lim_{n\to\infty} \frac{\sup_{f\in J^*(p)}\left|\frac{a_n}{\alpha_p}\right|}{\left|\frac{a_n^{(1)}}{\alpha_p^{(1)}}\right|} = 1.$$

References

1. A. Baernstein, *Integral means, univalent functions and circular symmetrization*, Acta Math. *133* (1974), 139–169.
2. L. deBranges, *A proof of the Bieberbach conjecture*, Acta. Math. *154* (1985), 137–152.
3. P. J. Eenigenberg and A. E. Livingston, *Meromorphic starlike functions*, Rocky Mountain J. *11* (1981), 441–457.
4. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities* (Cambridge Univ. Press, New York, 1959).
5. J. A. Jenkins, *On a conjecture of Goodman concerning meromorphic univalent functions*, Michigan Math. J. *9* (1962), 25–27.
6. W. E. Kirwan and G. Schober, *Extremal problems for meromorphic univalent functions*, J. D'Analyse Math. *30* (1976), 330–348.
7. Y. J. Leung, *Integral means of the derivatives of some univalent functions*, Bull. London Math. Soc. *11* (1979), 289–294.
8. R. J. Libera and A. E. Livingston, *Weakly starlike meromorphic univalent functions*, Trans. Amer. Math. Soc. *202* (1975), 181–191.
9. R. J. Libera and M. S. Robertson, *Meromorphic close-to-convex functions*, Michigan Math. J. *8* (1961), 167–175.
10. A. E. Livingston, *Weakly starlike meromorphic univalent functions II*, Proc. Amer. Math. Soc. *62* (1977), 47–53.
11. ——— *A class of meromorphic starlike functions*, Rocky Mountain J. *13*, (1983), 259–269.

12. J. Miller, *Convex and starlike meromorphic functions*, Proc. Amer. Math. Soc. *80* (1980), 607–613.

13. Ch. Pommerenke, *On the coefficients of close-to-convex functions,* Michigan Math. J. *9* (1962), 259–269.

University of Delaware,
Newark, Delaware