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INCLUSIONS FOR CLASSES OF LACUNARY SETS

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1. Introduction. A sequence, $a_1 < a_2 < a_3 < \dots$, of positive integers is called *lacunary* if the difference sequence $d_n = a_{n+1} - a_n$ tends to infinity as $n \to \infty$.

In several recent papers we have made use of these sequences in analysis and combinatorics. In [**6**] we show that the class $\mathscr{L}$ of all sets which are either finite or the range of a lacunary sequence is "full" in the sense that if (t_k) is a real sequence and $\sum_{k \in L} |t_k| < \infty$ for each $L \in \mathscr{L}$, then (t_k) is an l_1 sequence, that is,

$$\sum_{k=1}^{\infty} |t_k| < \infty.$$

In [**3**] the class $\mathscr{Z}$ of all finite unions of sets of $\mathscr{L}$ is shown to consist of exactly those sets of integers, A, whose characteristic sequence, χ_A, is in the well known summability space $bs + c_0$. More recently, in [**1**], we study lacunary sequences in connection with the conjecture of P. Erdös that, if a set A of integers satisfies $\sum_{a \in A} 1/a = \infty$, then A contains arbitrarily long arithmetic progressions. It turns out that Erdös' conjecture is true if, and only if, it is true for all sets in $\mathscr{L}$, and that the conjecture is indeed true for all sets in $\mathscr{L}_1$, a certain full subclass of $\mathscr{L}$ to be defined below.

In this paper we introduce some natural subclasses of $\mathscr{L}$ and prove inclusions among them and among their closures with respect to finite unions and subsets. These subclasses were suggested by the combinatorial and analytical work done in [**1**] and [**3**]. Furthermore, the use of lacunary sets goes back as far as the classical contribution of G. G. Lorentz [**5**]. These statements notwithstanding, the proofs of these inclusions became so demanding that the results seem to generate an interest in themselves aside from any possible applications.

For a class $\mathscr{S}$ of subsets of the natural numbers I we define $\mathscr{S}^*$ and $[\mathscr{S}]$ to be the "hereditary closure" and closure under finite unions of $\mathscr{S}$ respectively, that is,

$$\mathscr{S}^* = \{A : A \subset S \quad \text{for some } S \in \mathscr{S}\},$$

$$[\mathscr{S}] = \{A : A = S_1 \cup S_2 \cup \dots \cup S_k \quad \text{for some } S_i \in \mathscr{S} \text{ and } k \geqq 0\}.$$

It is easy to see that $[\mathscr{S}^*] = [\mathscr{S}]^*$. Moreover, a class of sets $\mathscr{A}$ is of the form $[\mathscr{S}^*]$ if and only if $\mathscr{A} = 2^I$ or $\mathscr{A}$ is a "zero-class", that is, the class

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of sets of zero upper density with respect to some density on I (see [**2**] and [**4**]).

We now define the subclasses of $\mathscr{L}$ in which we are interested. For an integer $j \geqq 0$ define $\mathscr{L}_{M_j}$ to be the class of all lacunary sequences for which $s \leqq t$ implies that $d_s \leqq d_t + j$. Further, we define $\mathscr{L}_1$ to be the "monotone" lacunary sequences $\mathscr{L}_{M_0}$. Finally, define two subclasses of $\mathscr{L}_1$ thus:

$$\mathscr{L}_2 = \{A \in \mathscr{L}_1 : \textstyle\sum_{a \in A} 1/a = \infty\}, \quad \mathscr{L}_3 = \mathscr{L}_1 - \mathscr{L}_2.$$

2. Inclusions. The remainder of this paper will be devoted to proving the following diagrams. In every case the inclusion itself is a trivial consequence of the definitions. It is in proving the two classes to be equal or unequal, as the case may be, that the real difficulties arise.

$$(1) \qquad \begin{matrix} [\mathscr{L}_2] \subsetneq & & & & \\ & [\mathscr{L}_1] \subsetneq [\mathscr{L}_{M_i}] \subsetneq [\mathscr{L}_{M_j}] \subsetneq [\mathscr{L}] \\ [\mathscr{L}_3] \subsetneq & & & & \end{matrix}$$

where $1 \leqq i < j$ and $[\mathscr{L}_2]$ and $[\mathscr{L}_3]$ are incomparable. If we remove the closure under finite unions from each of the above classes the same inclusions hold by definition. However we get the following for hereditary closure * of these classes.

$$(2) \qquad \begin{matrix} \mathscr{L}_2^* \subsetneq & \\ & \mathscr{L}_1^* \subsetneq \mathscr{L}_{M_i}^* = \mathscr{L}^* = \mathscr{L} \\ \mathscr{L}_3^* \subsetneq & \end{matrix}$$

for all $i \geqq 1$. $\mathscr{L}_2^*$ and $\mathscr{L}_3^*$ remain incomparable. Finally, taking both closures we get

$$(3) \qquad [\mathscr{L}_3^*] \subsetneq [\mathscr{L}_2^*] = [\mathscr{L}_1^*] \subsetneq [\mathscr{L}].$$

We omit the simple proof of our first proposition.

PROPOSITION 1. *If* $\mathscr{A} \subset \mathscr{B} \subset 2^I$ *and* $\mathscr{A}$ *is full, so is* $\mathscr{B}$.

PROPOSITION 2. $\mathscr{A} \subset 2^I$ *is full if and only if* $[\mathscr{A}]$ *is full if and only if* $\mathscr{A}^*$ *is full.*

Proof. If $\mathscr{A}$ is full then by Proposition 1, $[\mathscr{A}]$ and $\mathscr{A}^*$ are full.

Suppose $[\mathscr{A}]$ is full, and (t_k) is a real sequence such that

$$\textstyle\sum_{k=1}^{\infty} |t_k| = \infty.$$

Then there exists $A \in [\mathscr{A}]$ such that

$$\textstyle\sum_{k \in A} |t_k| = \infty.$$

Let $A = A_1 \cup A_2 \cup \ldots \cup A_n$ where $A_i \in \mathscr{A}$ for $i = 1, 2, \ldots, n$. Then there exist i such that

$$\sum_{k \in A_i} |t_k| = \infty.$$

Hence $\mathscr{A}$ is full.

Suppose that $\mathscr{A}^*$ is full. If

$$\sum_{k=1}^{\infty} |t_k| = \infty,$$

there exists $A \in \mathscr{A}^*$ such that

$$\sum_{k \in A} |t_k| = \infty.$$

Let $A \subset B$ where $B \in \mathscr{A}$. Then obviously

$$\sum_{k \in B} |t_k| = \infty$$

and $B \in \mathscr{A}$. Therefore $\mathscr{A}$ is full.

PROPOSITION 3. *$\mathscr{L}$, $\mathscr{L}_1$, $\mathscr{L}_2$ are full.*

Proof. Since $\mathscr{L}_2 \subset \mathscr{L}_1 \subset \mathscr{L}$, we only need to show that $\mathscr{L}_2$ is full. Let (t_k) be a real sequence such that

$$\sum_{k=1}^{\infty} |t_k| = \infty.$$

For each n, there exists $b_n \in I$ such that

$$\sum_{k=1}^{\infty} |t_{(b_n + k2^n)}| = \infty.$$

We construct two sequences $(M_n)_{n=2}^{\infty}$, and $(N_n)_{n=1}^{\infty}$ in I with the following properties:

(4) $\quad N_n < M_{n+1} < N_{n+1} \quad (n \geqq 1)$

(5) $\quad N_n \equiv M_n \equiv b_n \bmod 2^n \quad (n \geqq 2)$

(6) $\quad M_{n+1} \equiv N_n \bmod (2^n + 1) \quad (n \geqq 1)$

(7) $\quad M_n > b_n \quad (n \geqq 2)$

(8) $\quad \sum_{a \in B[2^n, M_n, N_n]} |t_a| > 1 \quad (n \geqq 2)$

(9) $\quad \sum_{a \in B[2^n, M_n, N_n]} 1/a > 1 \quad (n \geqq 2)$

where $B[s, a, b] = \{a, a + s, a + 2s, \ldots, a + [(b - a)/s]s\}$.

Take $N_1 = b_1$ and suppose that we have constructed two sequences $(M_n)_{n=2}^{m-1}$ and $(N_n)_{n=1}^{m-1}$ such that (4) and (6) are true for $n = 1, 2, \ldots, m - 2$ and (5), (7), (8) and (9) are true for $n = 2, 3, \ldots, m - 1$. Since 2^m and $2^{m-1} + 1$ are relatively prime, we can find $M_m \in I$ such that

$$M_m \equiv b_m \bmod 2^m,$$

$$M_m \equiv N_{m-1} \bmod (2^{m-1} + 1),$$

$$M_m > b_m \quad \text{and} \quad M_m > N_{m-1}.$$

Since

$$\sum_{k=1}^{\infty} |t_{(b_m+k2^m)}| = \infty \quad \text{and} \quad M_m \equiv b_m \bmod 2^m$$

we have

$$\sum_{k=1}^{\infty} |t_{(M_m+k2^m)}| = \infty.$$

Clearly

$$\sum_{k=1}^{\infty} 1/(M_m + k2^m) = \infty.$$

Now we can take N_m large enough such that

$$N_m \equiv M_m \bmod 2^m,$$

$$\sum_{a \in B[2^m, M_m, N_m]} |t_a| > 1,$$

and

$$\sum_{a \in B[2^m, M_m, N_m]} 1/a > 1.$$

Let

$$A = \cup_{k=1}^{\infty} (B[2^k + 1, N_k, M_{k+1}] \cup B[2^{k+1}, M_{k+1}, N_{k+1}]).$$

Clearly $A \in \mathscr{L}_2$ and $\sum_{a \in A} |t_a| = \infty$.

PROPOSITION 4. *The class $\mathscr{L}_3$ is not full. Thus $[\mathscr{L}_3^*] \subsetneq [\mathscr{L}_1^*]$.*

Proof. The sequence $1/k$ satisfies

$$\sum_{k=1}^{\infty} 1/k = \infty.$$

But, for any infinite set A in $\mathscr{L}_3$,

$$\sum_{a \in A} 1/a < \infty.$$

The last statement follows since $[\mathscr{L}_1^*]$ is full.

Proposition 4 also establishes the corresponding inclusion in diagrams (1) and (2).

PROPOSITION 5. $[\mathscr{L}_2^*] = [\mathscr{L}_1^*]$.

Proof. Obviously $[\mathscr{L}_2^*] \subset [\mathscr{L}_1^*]$. For $[\mathscr{L}_1^*] \subset [\mathscr{L}_2^*]$, we only need to show $\mathscr{L}_1 \subset [\mathscr{L}_2^*]$. In fact we show that, for any infinite set $A = \{a_n\} \in \mathscr{L}_1$, $A \subset B_1 \cup B_2$, where B_1, B_2 are members of $\mathscr{L}_2$. For $n \geqq 1$, let $d_n = a_{n+1} - a_n$. We know $d_n \leqq d_{n+1}$ for each n and $\lim d_n = \infty$. Thus we can find s_0, $t_1 \in I$ such that

$$d_1 \leqq a_{t_1} - (a_1 + s_0 d_1) < 2d_1 < d_{t_1} \quad \text{and}$$

$$\sum_{j=1}^{s_0} 1/(a_1 + jd_1) > 1.$$

Suppose that we have thus constructed $s_0 < s_1 < \ldots < s_{m-1}$, $t_0 = 1 < t_1 < \ldots < t_m$ such that

$$d_{t_{k-1}} \leqq a_{t_k} - (a_{t_{k-1}} + s_{k-1}d_{t_{k-1}}) < 2d_{t_{k-1}} < d_{t_k}$$

and

$$\sum_{j=1}^{s_{k-1}} 1/(a_{t_{k-1}} + jd_{t_{k-1}}) > 1$$

for $k = 1, 2, \ldots, m$. Again, since $d_n \leqq d_{n+1}$ for each n and $\lim d_n = \infty$, we can find s_m and t_{m+1} such that

$$s_{m-1} < s_m \quad \text{and} \quad t_m < t_{m+1}$$

$$d_{t_m} \leqq a_{t_{m+1}} - (a_{t_m} + s_m d_{t_m}) < d_{t_{m+1}} \quad \text{and}$$

$$\sum_{j=1}^{s_m} 1/(a_{t_m} + jd_{t_m}) > 1.$$

For $n = 1, 2, 3, \ldots$, let

$$P_n = \{a_{t_n}, a_{t_n} + d_{t_n}, \ldots, a_{t_n} + s_n d_{t_n}\}$$

$$W_n = \{a_{t_n}, a_{(t_n+1)}, \ldots, a_{t_{n+1}}\}.$$

Then we have,

$$\sum_{a \in P_n} 1/a > 1 \quad \text{and} \quad A = \cup_{n=1}^{\infty} W_n.$$

Let

$$B_1 = P_1 \cup W_2 \cup P_3 \cup W_4 \cup \ldots \cup P_{2n-1} \cup W_{2n} \cup \ldots.$$

$$B_2 = W_1 \cup P_2 \cup W_3 \cup P_4 \cup \ldots \cup W_{2n-1} \cup P_{2n} \cup \ldots.$$

Clearly $B_i \in \mathscr{L}_2$ for $i = 1, 2$ and $A \subset B_1 \cup B_2$.

We have shown that $[\mathscr{L}_2^*] = [\mathscr{L}_1^*]$. We proceed to show that $\mathscr{L}_2^* \subsetneqq \mathscr{L}_1^*$.

Definition 1. (1) Let $a, x_1, x_2, \ldots, x_n$ be positive integers with $a = x_1 + x_2 + \ldots + x_n$ and $x_1 \leqq x_2 \leqq \ldots \leqq x_n$. Then $(x_1, x_2, \ldots, x_n)$ is called a *partition* of a of length n.

(2) let $(a_1, a_2, \ldots, a_n)$ be any finite sequence of positive integers and let

(10) $(y_{11}, y_{12}, \ldots, y_{1k_1}, y_{21}, y_{22}, \ldots, y_{2k_2}, \ldots, y_{n1}, \ldots, y_{nk_n})$

be a nondecreasing sequence such that $(y_{i1}, y_{i2}, \ldots, y_{ik_i})$ is a partition of a_i. Then the block (10) is called a *partition* of the sequence $(a_1, a_2, \ldots, a_n)$.

Definition 2. Let (x_n) be a sequence and $(t(n))$ a strictly increasing sequence of positive integers with $t(1) = 1$. Then

$$(x_{t(n)}, x_{t(n)+1}, \ldots, x_{t(n+1)-1})$$

is called the n-th *part* of $(x_n)_{n=1}^{\infty}$ with respect to $(t(n))$.

LEMMA 6. *Let $p > 2$ be a prime number and let $(a_1, a_2, \ldots, a_p)$ be the sequence with $a_i = p$, for all $i = 1, 2, \ldots, p$. Let*

$$(y_{11}, y_{12}, \ldots, y_{1k_1}, y_{21}, \ldots, y_{2k_2}, \ldots, y_{p1}, y_{p2}, \ldots, y_{pk_p})$$

be a partition of $(a_1, a_2, \ldots, a_p)$ with $y_{11} > 1$. Then $k_p = 1$ and $y_{p1} = p$.

Proof. Suppose that $k_p > 1$. Then $y_{pk_p} < p$ and since p is a prime,

$$y_{p1} < y_{pk_p}.$$

It follows that $k_i > 1$ for all $i < p$ since if $k_i = 1$, then

$$y_{i1} = p > y_{pk_p},$$

which is a contradiction. Furthermore,

$$y_{i1} < y_{ik_i}$$

since $a_i = p$ is a prime. Therefore $1 < y_{11} < y_{21} < \ldots < y_{p1} < p$ which is impossible.

PROPOSITION 7. $\mathscr{L}_2^* \subsetneq \mathscr{L}_1^*$.

Proof. We construct $A \in \mathscr{L}_1^* - \mathscr{L}_2^*$. Let p_m be the m-th prime number. Let $D_m = (p_m, p_m, \ldots, p_m)$ be p_m repetitions of p_m. Let

$$\{d_n\} = (D_1, D_2, \ldots, D_m, \ldots)$$

and finally let the sequence $A = (a_n)$ be defined such that $a_1 = 1$ and $a_{n+1} = a_n + d_n$.

Clearly $A \in \mathscr{L}_1 \subset \mathscr{L}_1^*$. Suppose that $A \in \mathscr{L}_2^*$ and so $A \subset B = \{b_u\}$, where $B \in \mathscr{L}_2$. Let $e_u = b_{u+1} - b_u$ for $u \geqq 1$. Since B is lacunary there exists N such that, for any $k \geqq N$, $e_k > 1$. If

$$t(m) = 1 + \textstyle\sum_{i=1}^{m-1} p_i,$$

then $\{a_{t(m)}, a_{t(m)+1}, \ldots, a_{t(m+1)-1}\}$ is the m-th part of A corresponding to the m-th part D_m of $\{d_n\}$. Take m such that $b_N \leqq a_{t(m)}$. For each i, since $A \subset B$, some part of $\{e_u\}$ is a partition of D_i. Then $b_N \leqq a_{t(m)} = b_s$, for some s, and thus $N \leqq s$ and $e_s > 1$. By Lemma 6, if

$$a_{t(m+1)} \leqq b_u < b_{u+1} \leqq a_{t(m+2)-1},$$

then

$$p_m \leqq e_u \leqq p_{m+1}.$$

By Bertrand's postulate (i.e., $p_{j+1} < 2p_j$) we get

$$(1/2)p_{m+1} < p_m \leqq e_u.$$

Hence $e_u + e_{u+1} > p_{m+1}$. This implies that $e_u = p_{m+1}$. Thus A and B are asymptotically equal. Hence $B \in \mathscr{L}_2$ implies $A \in \mathscr{L}_2$. But the following computation shows that $A \notin \mathscr{L}_2$. For each n,

$$\begin{aligned}\textstyle\sum_{k=t(m)+1}^{t(m+1)} 1/a_k &= \textstyle\sum_{k=1}^{p_m} 1/\{a_{t(m)} + (k-1)p_m\}\\
&< \int_0^{p_m} 1/\{a_{t(m)} + xp_m\}dx\\
&= \frac{1}{p_m}\log\frac{a_{t(m+1)}}{a_{t(m)}}\\
&= \frac{1}{p_m}\log\frac{1 + p_1^2 + \ldots + p_m^2}{1 + p_1^2 + \ldots + p_{m-1}^2}\\
&= \frac{1}{p_m}\log\left\{1 + \frac{p_m^2}{1 + p_1^2 + \ldots + p_{m-1}^2}\right\}\\
&\leqq \frac{1}{p_m}\cdot\frac{p_m^2}{1 + p_1^2 + \ldots + p_{m-1}^2}\\
&\leqq \frac{p_m}{1 + p_1^2 + \ldots + p_{m-1}^2}\\
&< \frac{p_m}{1 + \sum_{k=1}^{m-1} k^2}.\end{aligned}$$

Thus, using the Prime Number Theorem,

$$\begin{aligned}\textstyle\sum_{a\in A} 1/a &= 1 + \textstyle\sum_{m=1}^{\infty} \left(\sum_{k=t(m)+1}^{k=t(m+1)} 1/a_k\right)\\
&< 1 + \sum_{m=1}^{\infty}\frac{p_m}{1 + \sum_{k=1}^{m-1} k^2}\\
&< r + s\sum_{m=1}^{\infty}\frac{m\log m}{m^3}\\
&< r + s\sum_{m=1}^{\infty}\frac{\log m}{m^2} < \infty\end{aligned}$$

where r and s are positive constants.

PROPOSITION 8. *$\mathscr{L}_2^* \not\subset \mathscr{L}_3^*$ and $\mathscr{L}_3^* \not\subset \mathscr{L}_2^*$.*

Proof. Suppose that $\mathscr{L}_2^* \subset \mathscr{L}_3^*$. Since $\mathscr{L}_2^*$ is full, $\mathscr{L}_3^*$ would also be full. This contradicts Proposition 4.

Suppose that $\mathscr{L}_3^* \subset \mathscr{L}_2^*$, then $\mathscr{L}_2^* = \mathscr{L}_3^* \cup \mathscr{L}_2^* = \mathscr{L}_1^*$ which contradicts Proposition 7.

Next we show that $[\mathscr{L}_2^*] \subsetneq [\mathscr{L}]$. This will establish the corresponding inclusions in diagram (3) and (after Proposition 12 below) in diagram (2). First we present two lemmas.

LEMMA 9. *Let x, u and v be positive integers. Suppose that*

$$x + (x-1) + \ldots + (x-u+1)$$

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$$= d_1 + d_2 + \dots + d_\alpha,$$
$$(x - u) + (x - u - 1) + \dots + (x - u - v + 1)$$
$$= d_{\alpha+1} + \dots + d_{\alpha+\beta},$$
$$d_1 \leqq d_2 \leqq \dots \leqq d_{\alpha+\beta} \quad \textit{and} \quad d_1 > (1/2)uv(u + v).$$

Then we have $d_1 < d_{\alpha+\beta}$.

Proof. Suppose that $d_1 = d_2 = \dots = d_{\alpha+\beta}$. Then

$$ux - (1/2)u(u - 1) = \alpha d_1$$
$$vx - (1/2)v(2u + v - 1) = \beta d_1.$$

It follows that

$$uvx - (1/2)uv(u - 1) = \alpha v d_1$$
$$uvx - (1/2)uv(2u + v - 1) = \beta u d_1.$$

Subtracting, we get

$$(1/2)uv(u + v) = (\alpha v - \beta u)d_1.$$

Thus d_1 divides $(1/2)uv(u + v)$, which contradicts the hypothesis.

We omit the proof of the second lemma:

LEMMA 10. *Let* M_t, H_t $(t = 1, 2, \dots, r)$, G *and* B *be given reals which satisfy*

$$H_{t+1} = H_t + M_t \quad \textit{for } t = 1, 2, \dots, r - 1 \quad \textit{and}$$
$$M_t = (1 + G)^{t-1}M_1 \quad \textit{for } t = 1, 2, \dots, r.$$

Then $M_t = G(H_t + B)$ *for* $t = 1, 2, \dots, r$.

PROPOSITION 11. $[\mathscr{L}_1^*] \subsetneq [\mathscr{L}]$.

Proof. Containment is clear since $\mathscr{L}_1^* \subset \mathscr{L}^* = \mathscr{L}$. For $m \geqq 1$, let

$$D_m = (m^2 + m - 1, m^2 + m - 2, \dots, m).$$

The sequence $(d_n) = (D_1, D_2, D_3, \dots)$ will be the difference sequence for a set $A = \{a_n\}$ with $a_1 = 1$. It is clear that $A \in \mathscr{L}$. We will prove that $A \notin [\mathscr{L}_1^*]$. Let us assume, otherwise, that $A \subset A_1 \cup A_2 \cup \dots \cup A_r$ where each $A_i \in \mathscr{L}_1$. For each i, $1 \leqq i \leqq r$, we write

$$A_i = \{a_n^i\} \quad \text{and} \quad d_n^i = a_{n+1}^i - a_n^i.$$

Since the A_i are lacunary sets there is an N such that $n \geqq N$ implies

$$d_n^i \geqq (3r)^3 \quad \text{for all } i.$$

Take

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$$a^* = \max\{d_N^i : 1 \leqq i \leqq r\}.$$

Consider the part P_m of A corresponding to D_m. That is

$$P_m = \{a_{\alpha(m)}, a_{\alpha(m)+1}, \ldots, a_{\alpha(m+1)}\}$$

where

$$\alpha(t) = 1 + \textstyle\sum_{i=1}^{t-1} i^2 = (1/6)(t - 1)t(2t - 1) + 1 \quad \text{and}$$

$$(d_{\alpha(m)}, d_{\alpha(m)+1}, \ldots, d_{\alpha(m+1)-1}) = D_m.$$

We consider m large enough so that $a^* \leqq a_{\alpha(m)}$. Let

$$M_0 = 3r,\ B = (1/2)(3r - 1),$$

$$G = 9r^3,\ M_1 = G(m + 3r + B) \quad \text{and}$$

$$M_t = (1 + G)^{t-1}M_1 \quad \text{for } t = 1, 2, \ldots, r.$$

Then we have

$$\begin{aligned} &M_r + M_{r-1} + \ldots + M_1 + M_0 \\ &= (1/G)\{(1 + G)^r - 1\}M_1 + M_0 \\ &= \{(1 + G)^r - 1\}(m + 3r + B) + 3r. \end{aligned}$$

Since $M_r + \ldots + M_0$ is thus a polynomial in m of degree 1, we can further choose m such that $m^2 > M_r + \ldots + M_0$.

We will partition some of P_m into $r + 1$ blocks $L_r, L_{r-1}, \ldots, L_1, L_0$ thus:

$$L_t = \{a_j : \alpha(m + 1) - (M_0 + M_1 + \ldots + M_t) \leqq j \leqq \alpha(m + 1) - (M_0 + M_1 + \ldots + M_{t-1})\}.$$

Hence L_{t+1} is to the left of L_t with the rightmost point of L_{t+1} and the leftmost point of L_t equal. Furthermore, each L_t has $M_t + 1$ points in it and thus represents M_t differences of A. Finally, since

$$M_r + M_{r-1} + \ldots + M_0 + 1 \leqq m^2 = \alpha(m + 1) - \alpha(m),$$

it follows that $\cup L_t \subset P_m$. Also, the rightmost point of L_0 is $a_{\alpha(m+1)}$.

Let H_t be the smallest difference d_n represented in the block L_t (it occurs at the right hand end of L_t). Since, within P_m, the differences decrease by one at each point we clearly get

$$H_{t+1} = H_t + M_t \quad \text{for } 0 \leqq t < r.$$

Note that $H_0 = m$ so that $H_1 = m + 3r$. We can apply Lemma 10 and obtain

$$M_t = G(H_t + B) \quad \text{for } t = 1, 2, \ldots, r.$$

Note that M_t is divisible by $3r$. We now partition L_t into $M_t/3r$ blocks $I_1^t, I_2^t, \ldots, I_{M_t/3r}^t$ thus:

$$I_k^t = \{a_j : \alpha(m + 1) - (M_0 + \dots + M_t) + (k - 1)3r \leqq j \leqq \alpha(m + 1) - (M_0 + \dots + M_t) + k \cdot 3r\}.$$

Here I_k^t is to the left of I_{k+1}^t with one point in common. The number of elements of A in I_k^t is $3r + 1$. Since

$$I_k^t \subset A_1 \cup A_2 \cup \dots \cup A_r,$$

we get that for some i

$$|I_k^t \cap A_i| > 3.$$

Let

$$a_p = a_\delta^i, \quad a_{p'} = a_{\delta+\alpha}^i \quad \text{and} \quad a_{p''} = a_{\delta+\alpha+\beta}^i$$

be three elements of $I_k^t \cap A_i$. The following equations result:

$$x + (x - 1) + \dots + (x - u)$$
$$= d_\delta^i + d_{\delta+1}^i + \dots + d_{\delta+\alpha-1}^i$$
$$(x - u - 1) + (x - u - 2) + \dots + (x - u - v)$$
$$= d_{\delta+\alpha}^i + d_{\delta+\alpha+1}^i + \dots + d_{\delta+\alpha+\beta-1}^i$$

where $x = d_p$, $u = p' - p$, $v = p'' - p'$. Recall

$$d_j^i \leqq d_{j+1}^i \quad \text{and} \quad d_\delta^i > (3r)^3 > (1/2)uv(u + v)$$

(since $u + v \leqq 3r$). We can apply Lemma 9 and get

$$d_\delta^i < d_{\delta+\alpha+\beta-1}^i.$$

Thus we conclude that, for any I_k^t, there exists an A_i such that d_n^i strictly increases at least once for elements of A_i in the interval $[\min I_k^i, \max I_k^i]$.

We first look at L_r, the left most of the L_i. According to the last paragraph, since there are $M_r/3r$ blocks I_k^r in L_r, there are at least $M_r/3r$ increases of the d_n^i among $A_1, A_2, \dots A_r$. Thus there exists i_0 such that, for points of A_{i_0} within the interval $[\min L_r, \max L_r]$, $d_n^{i_0}$ increases at least $M_r/3r^2$ times. Let $d_{n_r}^{i_0}$ be the largest difference of A_{i_0} in the interval $[\min L_r, \max L_r]$. Clearly

$$d_{n_r}^{i_0} > M_r/3r^2.$$

On the other hand

$$M_r/3r^2 = (3r)M_r/9r^3 = (3r)M_r/G$$
$$= (3r)(H_r + B) = (3r)(2H_r + 3r - 1)/2.$$

This last number is the diameter of the interval determined by $I_{M_r/3r}^r$. That is,

$$d_{M_r}^{i_0} > \max I_{M_r/3r}^r - \min I_{M_r/3r}^r.$$

Evidently this diameter exceeds any diameter of the interval determined by I_j^t when $t < r$. It follows that

$$|A_{i_0} \cap I_j^t| \leqq 1$$

for any j, t where $t < r$. Without loss of generality we may assume $i_0 = 1$.

Now we look at L_{r-1}. Again, for the $M_{r-1}/3r$ blocks, I_k^{r-1}, there is an A_i such that

$$|A_i \cap I_k^{r-1}| \geqq 3.$$

Clearly $i \neq 1$ and it follows, as before, that there is an i_1 ($\neq 1$) such that, for points of A_{i_1} within the interval [min L_{r-1}, max L_{r-1}], $d_n^{i_1}$ increases at least $M_{r-1}/3r^2$ times. We may assume $i_1 = 2$. The largest difference $d_{n_{r-1}}^2$ thus exceeds $M_{r-1}/3r^2$. So that, as before,

$$|A_2 \cap I_j^t| \leqq 1 \quad \text{for } t < r - 1.$$

We repeat this process r times and then look at $L_0 = I_1^0$. It follows from the above that

$$|A_i \cap I_1^0| \leqq 1 \quad \text{for all } i = 1, 2, \dots, r.$$

But this implies that

$$3r + 1 = |I_1^0| = |I_1^0 \cap (A_1 \cup A_2 \cup \dots \cup A_r)| \leqq r$$

a contradiction.

PROPOSITION 12. $\mathscr{L}_{M_1}^* = \mathscr{L}$.

Proof. Let $A = \{a_i\} \in \mathscr{L}$ and set $N_0 = 1$. For any $k \geqq 1$, there exists $N_k > N_{k-1}$ such that $d_n > k^2$ whenever $n > N_k$. For each n with $N_k < n \leqq N_{k+1}$, we let

$$d_n = q_n k + r_n, \quad \text{where } 0 \leqq r_n < k.$$

Thus

$$q_n k = d_n - r_n > k^2 - k = (k - 1)k.$$

Hence $q_n > k - 1$ and $d_n = (q_n - r_n)k + (k + 1)r_n$ where $q_n - r_n$ is positive. Let

$$\alpha_n = (\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nq_n})$$

be the finite sequence $(k, k, \dots, k, k + 1, k + 1, \dots, k + 1)$ where there are $q_n - r_n$ many k and r_n many $k + 1$. Let

$$(e_m) = (\alpha_1, \alpha_2, \alpha_3, \dots)$$
$$= (\alpha_{11}, \alpha_{12}, \dots, \alpha_{1q_1}, \alpha_{21}, \dots, \alpha_{2q_2}, \dots, \alpha_{n1}, \dots, \alpha_{nq_n}, \dots).$$

It follows from the definition of α_n that, for $n \leqq m$,

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$$\alpha_{ni} \leqq \alpha_{mj} + 1 \quad \text{for any } i \text{ and } j.$$

Hence, letting $b_1 = a_1$ and $b_{m+1} = b_m + e_m$, the set $B = \{b_m : m \in I\} \in \mathscr{L}_{M_1}$.

For any n,

$$d_n = \alpha_{n1} + \ldots + \alpha_{nq_n}.$$

Thus, for $a_n \in A$,

$$a_n = a_1 + \sum_{i=1}^{n-1} d_i = a_1 + \sum_{i=1}^{n-1} \left(\sum_{j=1}^{q_i} \alpha_{ij}\right) = b_m,$$

where

$$m = 1 + \sum_{i=1}^{n} q_i.$$

Hence $A \subset B$ and $\mathscr{L} \subset \mathscr{L}^*_{M_1}$. The reverse inclusion is immediate.

Next we show that $[\mathscr{L}_{M_i}] \subsetneq [\mathscr{L}_{M_j}]$ for $i < j$. We need a lemma:

LEMMA 13. *Suppose that d, m, s, t, u, v, i and j are nonnegative integers such that $d > m^2 + m$, $i < j < m$, $s \leqq m$, $1 \leqq v \leqq m$ and $1 \leqq t \leqq m$, then*

1) *$v(d + j) \leqq t(d + j) + i$ implies $v \leqq t$ and $v(d + j) \leqq t(d + j)$,*

2) *$vd \leqq td + i$ implies $v \leqq t$ and $vd \leqq td$,*

3) *$v(d + j) \leqq s(d + j) + td + i$ implies $v < s + t$ and $v(d + j) < s(d + j) + td$,*

4) *$v(d + j) + sd \leqq td + i$ implies $v + s < t$ and $v(d + j) + sd < td$,*

5) *$vd \leqq sd + t(d + j) + i$ implies $v \leqq s + t$ and $vd < sd + t(d + j)$,*

6) *$vd + s(d + j) \leqq t(d + j) + i$ implies $v + s \leqq t$ and $vd + s(d + j) < t(d + j)$.*

Proof. The proofs of 2), 4), 6) are similar to those of 1), 3), 5) respectively. We prove only 1), 3) and 5):

1) $v(d + j) \leqq t(d + j) + i < t(d + j) + d + j = (t + 1)(d + j)$. Hence $v < t + 1$ so that $v \leqq t$.

3) $v(d + j) \leqq s(d + j) + td + i < s(d + j) + t(d + j) = (s + t)(d + j)$ which proves the first part. Now

$$\begin{aligned} v(d + j) &\leqq (s + t - 1)(d + j) \\ &= s(d + j) + td + (t-1)j - d < s(d + j) + td \end{aligned}$$

since

$$(t - 1)j - d < m^2 - (m^2 + m) < 0.$$

5) Since $vd \leqq sd + t(d + j) + i$ is equivalent to $-i - tj \leqq (s + t - v)d$, we have

$$-d < -m - m^2 < -i - tj \leqq (s + t - v)d.$$

Thus we get $-1 < (s + t - v)$ or, $v \leqq s + t$. If $vd \geqq sd + t(d + j)$ then we have

$$(v - s)d \geqq t(d + j).$$

This implies $v - s > t$ which is a contradiction.

PROPOSITION 14. $[\mathscr{L}_{M_i}] \subsetneq [\mathscr{L}_{M_j}]$ *for* $0 \leqq i < j$.

Proof. We make the following definitions:
$L_m = (m^3 + j, m^3 + j, \dots, m^3 + j)$, m repetitions of $m^3 + j$,
$R_m = (m^3, m^3, \dots, m^3)$, m repetitions of m^3,
$B_m = (L_m, R_m, L_m, R_m, \dots, L_m, R_m)$, m repetitions of L_m, R_m,
$(d_n) = (B_1, B_2, \dots, B_m, \dots)$,
$A = \{a_n\}$ where $a_n = 1 + d_1 + \dots + d_{n-1}$,
$A[a_m, a_n] = \{a_r : m \leqq r \leqq n\}$,
$A(a_m, a_n) = \{a_r : m < r < n\}$,
$\alpha(m, t) = 1 + 2(1^2 + 2^2 + \dots + (m - 1)^2) + 2(t - 1)m$

for $1 \leqq m$, $1 \leqq t \leqq m + 1$,

$\beta(m, t) = \alpha(m, t) + m$.
Note that $\alpha(m + 1, 1) = \alpha(m, m + 1)$. For $1 \leqq t \leqq m + 1$ define

$$A_{Lmt} = A[a_{\alpha(m,t)}, a_{\beta(m,t)}], \quad A^{\circ}_{Lmt} = A(a_{\alpha(m,t)}, a_{\beta(m,t)})$$

$$A_{Rmt} = A[a_{\beta(m,t)}, a_{\alpha(m,t+1)}], \quad A^{\circ}_{Rmt} = A(a_{\beta(m,t)}, a_{\alpha(m,t+1)}).$$

If we let

$$A_m = A_{Lm1} \cup A_{Rm1} \cup A_{Lm2} \cup A_{Rm2} \cup \dots \cup A_{Lmm} \cup A_{Rmm},$$

then A_m is the m-th part of A corresponding B_m. It is clear that

$$A \in \mathscr{L}_{M_j} \subset [\mathscr{L}_{M_j}].$$

Suppose that $X = \{x_q\} \in \mathscr{L}_{M_i}$ and $X \subset A$. We will show that, if $j < m$ and $d = m^3 > m^2 + m$, then

$$|X \cap A_{Rmm}| \leqq 2.$$

Let $\{y_q\}$ be the difference sequence of $\{x_q\}$ and f be the function on I such that $x_q = a_{f(q)}$. Then $f(s + 1) - f(s)$ equals the number of terms in the sum

$$y_s = d_{f(s)} + d_{f(s)+1} + \dots + d_{f(s+1)-1}.$$

At first we will consider the following six cases.

(i) If

$$a_{\alpha(m,t)} \leqq x_q < x_{q+1} < x_{q+2} \leqq a_{\beta(m,t)}$$

(i.e., three consecutive elements of x are in A_{Lmt}), then, since $x \in \mathscr{L}_{M_i}$ so that $y_q \leqq y_{q+1} + i$, we have

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$$x_{q+1} - x_q \leqq x_{q+2} - x_{q+1} + i,$$
$$a_{f(q+1)} - a_{f(q)} \leqq a_{f(q+2)} + i,$$
$$(f(q + 1) - f(q))(d + j) \leqq (f(q + 2) - f(q + 1))(d + j) + i,$$

where $d = m^3$. By Lemma 13, case 1), we conclude that

$$f(q + 1) - f(q) \leqq f(q + 2) - f(q + 1) \quad \text{and} \quad y_q \leqq y_{q+1}.$$

(ii) Similarly, if $x_q < x_{q+1} < x_{q+2}$ are in the interval A_{Rmt}, then we apply Lemma 13 case 2) and we get

$$f(q + 1) - f(q) \leqq f(q + 2) - f(q + 1) \quad \text{and} \quad y_q \leqq y_{q+1}.$$

(iii) If

$$a_{\alpha(m,t)} \leqq x_q < x_{q+1} \leqq a_{\beta(m,t)} < x_{q+2} \leqq a_{\alpha(m,t+1)}$$

that is, x_q, x_{q+1} are in A_{Lmt} and x_{q+2} is in A_{Rmt}, then, since

$$x_{q+1} - x_q \leqq x_{q+2} - x_{q+1} + i,$$

it follows that

$$a_{f(q+1)} - a_{f(q)} \leqq a_{f(q+2)} - a_{f(q+1)} + i$$
$$= a_{\beta(m,t)} - a_{f(q+1)} + a_{f(q+2)} - a_{\beta(m,t)} + i,$$

which is equivalent to

$$(f(q + 1) - f(q))(d + j)$$
$$\leqq (\beta(m, t) - f(q + 1))(d + j) + (f(q + 2) - \beta(m, t))d + i.$$

Now we apply Lemma 13 case 3) and get

$$f(q + 1) - f(q) < f(q + 2) - f(q+1)$$

and so $y_q < y_{q+1}$.

(iv) Similarly, if

$$a_{\alpha(m,t)} \leqq x_q < a_{\beta(m,t)} \leqq x_{q+1} < x_q \leqq a_{\alpha(m,t+1)},$$

that is, x_q is in A_{Lmt} and x_{q+1}, x_{q+2} are in A_{Rmt}, then we can apply Lemma 13 case 4) and get

$$f(q + 1) - f(q) < f(q + 2) - f(q + 1) \quad \text{and} \quad y_q < y_{q+1}.$$

(v) If

$$a_{\beta(m,t)} \leqq x_q < x_{q+1} \leqq a_{\alpha(m,t+1)} < x_{q+2} \leqq a_{\beta(m,t+1)}$$

where $t \leqq m$, that is, x_q and x_{q+1} are in A_{Rmt} and x_{q+2} is in $A_{Lm(t+1)}$, then we have

$$a_{f(q+1)} - a_{f(q)} \leqq a_{f(q+2)} - a_{f(q+1)} + i$$
$$= a_{\alpha(m,t+1)} - a_{f(q+1)} + a_{f(q+2)} - a_{\alpha(m,t+1)} + i$$

or, equivalently,

$$(f(q+1) - f(q))d \leqq (\alpha(m, t+1) - f(q+1))d$$
$$+ (f(q+2) - \alpha(m, t+1))(d+j) + i.$$

By Lemma 13 case 5) we get

$$f(q+1) - f(q) \leqq f(q+2) - f(q+1) \quad \text{and} \quad y_q < y_{q+1}.$$

(vi) Finally, if

$$a_{\beta(m,t)} \leqq x_q < a_{\alpha(m,t+1)} \leqq x_{q+1} < x_{q+2} \leqq a_{\beta(m,t+1)}$$

then we can apply the previous lemma case 6) and obtain

$$f(q+1) - f(q) \leqq f(q+2) - f(q+1) \quad \text{and} \quad y_q < y_{q+1}.$$

Now assume that $|X \cap A_{Rmm}| \geqq 3$ and so there exist three consecutive elements x_w, x_{w+1}, x_{w+2} of x in A_{Rmm}. By case (ii)

$$f(w+1) - f(w) \leqq f(w+2) - f(w+1)$$

and so

$$2(f(w+1) - f(w)) \leqq f(w+1) - f(w) + f(w+2) - f(w+1)$$
$$= f(w+2) - f(w) \leqq m.$$

Thus

$$f(w+1) - f(w) \leqq (1/2)m \quad \text{and}$$
$$y_w = (f(w+1) - f(w))d \leqq (1/2)md = (1/2)m^4,$$

the half diameter of A_{Rmm}.

We claim, for any $u < w$ and $x_u \geqq a_{\alpha(m,1)}$, that $y_u \leqq y_w$.

Proof of claim: Since $X \in \mathscr{L}_{M_i}$, we have $y_u \leqq y_w + i$. We may write

$$y_u = t(d+j) + vd \quad \text{and} \quad y_w = qd$$

and get

$$t(d+j) + vd \leqq qd + i.$$

If $t > 0$ (resp. $t = 0$), then we apply the previous lemma case 4) (resp. case 2) and get $y_u \leqq y_w$.

By this claim we conclude that for any $u \leqq w$ and $x_u \geqq a_{\alpha(m,1)}$ we have $y_u \leqq y_w \leqq (1/2)m^4 = (1/2)$ diameter of $A_{Rmt} \leqq (1/2)$ diameter of A_{Lmt} for $t = 1, 2, \ldots, m$.

Hence, for any $t \leqq m$, A_{Rmt} and A_{Lmt} each contain at least two elements of X.

Therefore we conclude that: By cases (iii) and (iv) above, if $x_q \in A^\circ_{Lmt}$ and $x_{q+2} \in A^\circ_{Rmt}$ then

$$f(q + 1) - f(q) < f(q + 2) - f(q + 1).$$

By cases (v) and (vi), if $x_q \in A^\circ_{Rmt}$ and $x_{q+2} \in A^\circ_{Lm(t+1)}$ then

$$f(q + 1) - f(q) \leqq f(q + 2) - f(q + 1).$$

By cases (i) and (ii) if $x_q, x_{q+1}, x_{q+2} \in A_{Lmt}$ or $x_q, x_{q+1}, x_{q+2} \in A_{Rmt}$, then

$$f(q + 1) - f(q) \leqq f(q + 2) - f(q + 1).$$

Now if we let x_{s_q} be an element of X such that $x_{s_q} \in A^\circ_{Lmq}$ and $x_{s_q+2} \in A^\circ_{Rmq}$ for $q = 1, 2, \ldots, m$. Then we have for $q = 1, 2, \ldots, m - 1$,

$$f(s_q + 1) - f(s_q) < f(s_{q+1} + 1) - f(s_{q+1}).$$

Therefore we get

$$1 \leqq f(s_1 + 1) - f(s_1) < f(s_2 + 1) - f(s_2) < \ldots$$
$$< f(s_m + 1) - f(s_m) \leqq f(w + 1) - f(w) \leqq (1/2)m.$$

Since there are $m - 1$ strict inequalities, we get a contradiction. Therefore we conclude that $|X \cap A_{Rmm}| \leqq 2$.

Finally we show that $A \notin [\mathscr{L}_{M_i}]$. Suppose that $A = X_1 \cup X_2 \cup \ldots \cup X_n$ where $X_s \in \mathscr{L}_{M_i}$ for $s = 1, 2, \ldots, n$. Since

$$A_{Rmm} = \cup_{i=1}^n (A_{Rmm} \cap X_i),$$

for any m with $m^3 > m^2 + m$ and $m > j$, we have

$$m = |A_{Rmm}| \leqq \sum_{i=1}^n |A_{Rmm} \cap X_i| \leqq 2n.$$

Thus m is bounded above, a contradiction.

COROLLARY 15. *For all* $i \geqq 0$, $[\mathscr{L}_{M_i}] \subsetneq [\mathscr{L}]$. *In particular* $[\mathscr{L}_1] \subsetneq [\mathscr{L}]$.

PROPOSITION 16. $[\mathscr{L}_2] \subsetneq [\mathscr{L}_1]$.

Proof. Obviously $[\mathscr{L}_2] \subset [\mathscr{L}_1]$. Strictness is proved by observing that $\{n^2\} \in \mathscr{L}_1$ but $\{n^2\} \notin [\mathscr{L}_2]$.

At this point we have completed the proofs of all diagrams given at the beginning of this section. Some further interesting inclusions concerning $\mathscr{L}_1$ follow.

PROPOSITION 17. $\mathscr{L}_1^* \subsetneq [\mathscr{L}_1^*]$.

Proof. Let $A = \{n^2\}$ and $B = \{n^2 + 1\}$. Then

$$A \cup B \in [\mathscr{L}_1] \subset [\mathscr{L}_1^*].$$

But $A \cup B$ is not lacunary. Thus $A \cup B \notin \mathscr{L}_1^*$.

Finally we will prove $[\mathscr{L}_1] \subsetneq [\mathscr{L}_1^*]$. First, we define some terms and prove a lemma.

Definition 3. Let $\{a_n\} = A$ be a sequence and $(a_s, a_{s+1}, \dots, a_{s+r})$ be a part of $\{a_n\}$.

If $(d_s, d_{s+1}, \dots, d_{s+r-1})$ is a strictly decreasing sequence, where $d_i = a_{i+1} - a_i$, then we say that $(a_s, a_{s+1}, \dots, a_{s+r})$ is a *consecutive descending wave* of *length* $r + 1$ in A. Further, the d_t are called the (*decreasing*) *steps* of the wave. (Note that the definition of descending wave in **[1]** is more general.)

LEMMA 18. *There exists a function $f(n)$ (depending only on n) such that, for any sets $A_1, A_2, \dots, A_n \in \mathscr{L}_1$, and for any consecutive descending wave X in $A_1 \cup A_2 \cup \dots \cup A_n$, $|X| \leqq f(n)$.*

Proof. We take $f(1) = 2$ which clearly works.

Suppose there exists $f(n - 1)$ such that for any $A_1, A_2, \dots, A_{n-1}$ in $\mathscr{L}_1$, and any consecutive descending wave X in $A_1 \cup A_2 \cup \dots \cup A_{n-1}$, we have $|X| \leqq f(n - 1)$.

Let $A = A_1 \cup A_2 \cup \dots \cup A_{n-1}$ and $B = \{b_u\} = A_n$ where $A_1, A_2, \dots, A_n \in \mathscr{L}_1$. Further let

$$W_u = \{a \in A : b_u < a < b_{u+1}\},$$

$$V_u = \{c \in A \cup B : b_u \leqq c \leqq b_{u+1}\}.$$

Suppose that X is a consecutive descending wave in $A \cup B$, $V_u \subset X$ and $V_{u+1} \subset X$, then we prove that $|W_u| < |W_{u+1}|$.

Let $e_1 > e_2 > \dots > e_{q+1} > c_1 > c_2 > \dots > c_{p+1}$ be the decreasing steps of the consecutive descending wave $V_u \cup V_{u+1}$, where $|W_u| = q$ and $|W_{u+1}| = p$. Since $B \in \mathscr{L}_1$,

$$\begin{aligned}(q + 1)e_{q+1} &\leqq e_1 + e_2 + \dots + e_{q+1}\\ &= b_{u+1} - b_u \leqq b_{u+2} - b_{u+1}\\ &= c_1 + c_2 + \dots + c_{p+1} \leqq (p + 1)c_1 < (p + 1)e_{q+1}.\end{aligned}$$

Therefore $q + 1 < p + 1$ and so $q < p$.

Next we show, if $X \subset A \cup B$ is a consecutive descending wave then $|X \cap B| \leqq f(n - 1) + 2$.

Suppose, otherwise, that $|X \cap B| > f(n - 1) + 2$. Let

$$\{b_r, b_{r+1}, \dots, b_s\} = X \cap B,$$

where $s \geqq r + f(n - 1) + 2$. Then $V_k \subset X$ for all $r \leqq k \leqq s - 1$. By the above, $0 \leqq |W_r| < |W_{r+1}| < \dots < |W_{s-1}|$, thus we have

$$|W_{s-1}| \geqq s - r - 1 \geqq f(n - 1) + 1 > f(n - 1)$$

which is a contradiction since W_{s-1} is a consecutive descending wave in A.

Finally, let X be a descending wave of $A \cup B$. Again, writing

$$X \cap B = \{b_r, b_{r+1}, \dots, b_s\},$$

we get

$$X \subset H \cup V_r \cup \dots \cup V_{s-1} \cup J$$

where H and J are the (possibly empty) consecutive descending waves in $A \cap X$ which come before b_r, and after b_s respectively. Thus

$$|X| \leqq |H| + |J| + \textstyle\sum_{i=r}^{s-1} |V_i| \leqq (f(n-1)+3)(f(n-1)+2)$$

and so we can set

$$f(n) = (f(n-1)+2)(f(n-1)+3).$$

PROPOSITION 19. $[\mathscr{L}_1] \subsetneq [\mathscr{L}_1^*]$.

Proof. Let

$$B_n = (n^2, (n-1)n, (n-2)n, \dots, 2n, n),$$

$$(d_n) = (B_1, B_2, \dots, B_q, \dots),$$

$$a_n = 1 + d_1 + \dots + d_{n-1} \text{ for } n = 1, 2, 3, \dots,$$

$$W_m = (m, m, \dots, m), \text{ with } m(m+1)/2 \text{ repetitions of } m,$$

$$(y_m) = (W_1, W_2, \dots, W_p, \dots)$$
$$= (1, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, \dots),$$

$$x_m = 1 + y_1 + y_2 + \dots + y_{m-1} \text{ for } m = 1, 2, \dots.$$

Then $\{x_n\} \in \mathscr{L}_1$ and $\{a_n\} \subset \{x_n\}$. Thus $\{a_n\} \in \mathscr{L}_1^* \subset [\mathscr{L}_1^*]$. Since $\{a_n\}$ contains arbitrarily long consecutive descending waves, by the previous lemma, $\{a_n\} \notin [\mathscr{L}_1]$. Thus $[\mathscr{L}_1] \subsetneq [\mathscr{L}_1^*]$.

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