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ON A THEOREM OF KUIPER

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1. Introduction. Let Δ_{n+1} be the standard $(n+1)$ simplex with its standard triangulation. By the Generalized Poincare Conjecture, if $n \geqq 5$ and Σ^n is a smooth homotopy n-sphere, then there exists a smooth triangulation $f : K \to \Sigma^n$, where K is a suitable subdivision of $\partial \Delta_{n+1}$. On the other hand, in [**3**], N. Kuiper proves the following theorem.

THEOREM (Kuiper). *If Σ^n is a smooth homotopy n-sphere and there exists a smooth triangulation $f : \partial \Delta_{n+1} \to \Sigma^n$, then Σ^n is diffeomorphic to the standard sphere.*

The object of this paper is to give an easier proof of Kuiper's Theorem, and to extend that theorem in a rather special setting. To arrive at that setting, we define a subset $S(n+1) \subset R^{n+1}$ = Euclidean $(n+1)$-space by induction on n: For $n = 0$ we set $S(1) = [0; \infty)$; assuming $S(n) \subset R^n$ has been defined, we set

$$S(n+1) = (S(n) \times [0, 1]) \cup R^n \times (-\infty, 0] \subset R^n \times R = R^{n+1}.$$

The set $S(n+1)$ is an $(n+1)$-submanifold of R^{n+1} and we call it the *solid model* in dimension $n+1$. We set $M(n) = \partial S(n+1)$, and we call $M(n)$ the *model* in dimension n. Let $\mathscr{M}(n)$ be the pseudogroup defined by $\mathscr{M}(n) = \{\varphi | \varphi : U \to \varphi(U)$ is a homeomorphism, U and $\varphi(U)$ are open in $M(n)$, and φ extends to an affine isomorphism of $R^{n+1}\}$. Similarly, let $\mathscr{S}(n+1)$ be the pseudogroup defined by $\mathscr{S}(n+1) = \{\varphi | \varphi : U \to \varphi(U)$ is a homeomorphism, U and $\varphi(U)$ are open in $S(n+1)$, and φ extends to an affine isomorphism of $R^{n+1}\}$. Then we say that an $M(n)$ *manifold* P is an n-manifold $|P|$ together with a maximal atlas $\mathscr{P}$ modelling $|P|$ on $M(n)$ with coordinate transformations in $\mathscr{M}(n)$; thus $P = (|P|, \mathscr{P})$. Similarly, an $S(n+1)$ *manifold* X is an $(n+1)$-manifold $|X|$ together with a maximal atlas $\mathscr{X}$ modelling $|X|$ on $S(n+1)$ with coordinate transformations in $\mathscr{S}(n+1)$; thus $X = (|X|, \mathscr{X})$. Clearly the boundary of an $S(n+1)$ manifold is an $M(n)$ manifold. In the usual categories, every closed manifold is the boundary of a manifold, but since the product of an $M(n)$ manifold with $[0, 1)$ does not appear to have a canonical $S(n+1)$-structure, it is not clear that every $M(n)$ manifold is the boundary of some $S(n+1)$ manifold. To repair this deficiency, we introduce the notion of a *sided* $M(n)$ manifold. To begin with, for $x \in M(n)$ we say that dim $(x) \geqq r$

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if there exists an affine r-plane H such that $x \in \text{int}_H\ (H \cap M(n))$, and we set $\dim(x) = \max\{r|\dim(x) \geqq r\}$. If P is an $M(n)$ manifold and $y \in P$, we set $\dim(y) = \dim(\varphi(x))$ where $y \in U$ and $(U, \varphi) \in \mathscr{P}$ is a chart of P. Clearly $\dim(y)$ is well defined. Then we set $P^r = \{y \in P|\dim(y) \leqq r\}$. Clearly $\phi = P^{-1} \subset P^0 \subset \ldots \subset P^n = P$ is a filtration of P by closed subsets; $P^r - P^{r-1}$ is an r manifold and $(P^r - P^{r-1})^r = \phi$ for $i < r$. Suppose $y \in P^{n-1}$ and $(U, \varphi), (V, \psi) \in \mathscr{P}$ with $y \in U \cap V$. Then the homeomorphism

$$\varphi(U \cap V) \xrightarrow{\psi \circ \varphi^{-1}} \psi(U \cap V)$$

extends to a *unique* affine isomorphism $A : R^{n+1} \rightarrow R^{n+1}$ and for W a sufficiently small open neighborhood of $\varphi(y)$ in R^{n+1}, we will have either

$$A(\text{int}\, S(n+1) \cap W) \subset \text{int}\, S(n+1) \cap A(W)$$

or

$$A(\text{int}\, S(n+1) \cap W) \subset A(W) - S(n+1).$$

In the first case we set $s(\psi, \varphi) = +1$ and in the second case we set $s(\psi, \varphi) = -1$. In the standard way, the function s determines a $\{+1, -1\}$-bundle $\sigma(P)$ over P^{n-1}. If this bundle is trivial, we say that P is sideable; in that case a section $\mathscr{S}$ of P is a *side* and the other section $-\mathscr{S}$ is the *opposite side*. A sideable $M(n)$ manifold P together with a side $\mathscr{S}$ is called a *sided $M(n)$ manifold*; we will abuse notation sometimes by writing $(P, \mathscr{S}) = P$ and $-P = (P, -\mathscr{S})$. Clearly, if X is an $S(n+1)$-manifold and $P = \partial X$, then P inherits a side from X. Examples of sided $M(n)$ manifolds are $\partial\Delta_{n+1}$, $\partial[-1, 1]^{n+1}$, and $\partial[-1, 1]^{n+1}/(-1)$.

If X is an $S(n+1)$ manifold, then the ring

$$C^\infty(X) = \{f : X \rightarrow R|f \circ \varphi^{-1} : \varphi(U) \rightarrow R \text{ is } C^\infty \text{ for any } (\varphi, U) \in \mathscr{X}\}$$

is well defined. If P is an $M(n)$ manifold, we say that an *open r-facet* of P is a component of $P^r - P^{r-1}$ and a *closed r-facet* is the closure of an open r-facet; a closed r-facet inherits an $S(r)$ structure, and with that structure we call it an r-facet. Let the ring $\mathscr{S}m(P) = \{f : P \rightarrow R|\, f|F \in C^\infty(F) \text{ for } F \text{ any facet of } P\}$. Similarly, if N is a smooth manifold or an $S(k)$ manifold, we may define $C^\infty(X, N)$ and $\mathscr{S}m(P, N)$. For $y \in P$, let $\mathscr{D}_y(P)$ be the set of derivations of $\mathscr{S}m(P)$ at y. It follows from Thom's Lemma below that

$$\mathscr{S}m(M(n)) = \{f : M(n) \rightarrow R|f = g|M(n),\ g : R^{n+1} \rightarrow R \text{ is } C^\infty\};$$

then for $x \in M(n)$ we have that $\mathscr{D}_x(M(n))$ is a real vector race of dimension $n+1$ if $\dim(x) \leqq n-1$ and of dimension n if $\dim(x) = n$. If (U, φ) is a chart of P with $y \in U$, then we define $d\varphi(y) : \mathscr{D}_y(P) \rightarrow \mathscr{D}_{\varphi(y)}(M(n))$ in the usual way; clearly $d\varphi(y)$ is an isomorphism, so $\mathscr{D}_y(P)$ is a real vector space of dimension $n+1$ if $y \in P^{n-1}$ and of dimension n if $y \in P - P^{n-1}$. For $x \in$

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$M(n)$ we may identify the tangent cone to $M(n)$ at x with a subset $\tau C_x(M(n))$ of $\mathscr{D}_x(M(n))$; then for $y \in P$ and (U, φ) as above we set

$$\tau C_y(P) = d\varphi(y)\tau C_{\varphi(y)}(M(n)),$$

and $\tau C_y(P)$ is well defined. Then $\tau C_y(P)$ is a subcone of $\mathscr{D}_y(P)$, piecewise linearly isomorphic to R^n. For N a smooth manifold and $f \in \mathscr{S}m(P, N)$, the linear map df $(y) : \mathscr{D}_y(P) \rightarrow \tau y(N)$ is defined in the usual manner. We will say that $P \rightarrow N$ *smooths* P *to* N if

i) $f \in \mathscr{S}m(P, N)$,

ii) f is a homeomorphism, and

iii) df $(y) : \tau C_y(P) \rightarrow \tau_y(N)$ is $1 - 1$ onto.

In that case we will say that P *subdivides* N, that P is a *subdivision* of N, and that N is a *smoothing* of P. We may extend the notion of subdivision to a pair of $M(n)$ manifolds. If P and Q are $M(n)$ manifolds, we set

$$\text{Aff }(P, Q) = \{f : P \rightarrow Q | \text{ the map } \varphi(U \cap f^{-1}(V)) \xrightarrow{\psi \circ f \circ \varphi^{-1}} \psi(V)$$
$$\text{extends to an affine map } R^{n+1} \rightarrow R^{n+1} \text{ for } (U, \varphi) \text{ a chart of } P \text{ and}$$
$$(V, \psi) \text{ a chart of } Q\} .$$

For such f, the map df $(y) : \tau C_y(P) \rightarrow \tau C_{f(y)}(Q)$ is defined. If $f \in$ Aff (P, Q) we will say that if *subdivides* Q if

i) f is a homeomorphism,

ii) for each open facet 0 of P there is an open facet $0'$ of Q with $f(0) \subset 0'$, and

iii) df $(y) : \tau C_y(P) \rightarrow \tau C_{f(y)}(Q)$ is $1 - 1$ onto.

If $(P, \mathscr{S})$ and $(Q, \mathscr{T})$ are sided $M(n)$ manifolds, and $f : P \rightarrow Q$ subdivides Q, then f pulls the side $\mathscr{T}$ of Q back to a side $f^*\mathscr{T}$ of P. If $f^*\mathscr{T} = \mathscr{S}$, we say of the map f that it $M(n)$-*subdivides* $(Q, \mathscr{T})$, and we say that $(P, \mathscr{S})$ is an $M(n)$-*subdivision* of $(Q, \mathscr{T})$. It is straightforward that if $g : P \rightarrow Q$ subdivides or $M(n)$ subdivides Q and $f : Q \rightarrow N$ smooths Q, then $f \circ g$ smooths P. The natural equivalence relations on $M(n)$ manifolds are $M(n)$-*equivalence* and *equivalence*: $(Q, \mathscr{T})$ is $M(n)$-*equivalent* to $(Q', \mathscr{T}')$ if there exists $(P, \mathscr{S})$ that is an $M(n)$ subdivision of both $(Q, \mathscr{T})$ and $(Q', \mathscr{T}')$; the definition of equivalence is similar except that sides do not enter. Neither of these relations is very tractable, so we will introduce a coarser (by Proposition 3 below) equivalence relation on a certain class of sided $M(n)$ manifolds. To introduce the coarser equivalence relation, we let

$$\bar{\mathscr{M}}(n) = \{\varphi | \varphi : U \rightarrow \varphi(U) \text{ is a diffeomorphism and } U, \varphi(U) \text{ open} \subset M(n)\}$$
$$\text{and } \bar{\mathscr{S}}(n + 1) = \{\varphi | \varphi : U \rightarrow \varphi(U) \text{ is a diffeomorphism and } U, \varphi(U)$$
$$\text{open} \subset S(n + 1)\}.$$

Then *smooth* $M(n)$ *manifolds* are those modelled on $M(n)$ with coordinate transformations in $\bar{\mathscr{M}}(n)$ and *smooth* $S(n+1)$ *manifolds* are those modelled on $S(n+1)$ with coordinate transformations in $\bar{\mathscr{S}}(n+1)$. As in the affine case above, we may introduce the dimension filtration, siding, facets, tangent cones, smoothing and subdivision. Moreover, an $M(n)$ or $S(n+1)$ manifold relaxes to a unique smooth $M(n)$ or smooth $S(n+1)$ manifold, and closed smooth manifolds are automatically smooth $M(n)$ manifolds. If P and Q are compact sided smooth $M(n)$ manifolds, we will say that P is *strongly cobordant* to Q if there is a smooth $S(n+1)$ manifold X such that $X = P \coprod -Q$, and X is PL isomorphic to $P \times [0, 1]$. Let $\mathscr{C} = \{P | P$ is strongly cobordant to a smooth manifold$\}$. Suppose $P \in \mathscr{C}$ and that X is a strong cobordism from P to a smooth manifold N. There is a smooth vector field A on X, transverse to P. By the Cairns Hirsch Theorem, there is a smooth submanifold $N' \subset \operatorname{int} X$ which is transverse to A. We may push P into the region of X between N' and N by means of a solution of A. Thus we have a copy P' of P between N and N'. Let Y be the closure of the region between N and P', and let Z be the closure of the region between P and P'. Then Y defines a strong cobordism from $-P$ to N and Z from P to P. Thus, writing $\sim$ for strong cobordism we have $P \in \mathscr{C}$ implies $-P \in \mathscr{C}$ and $P \in \mathscr{C}$ implies $P \sim P$. Suppose X is a strong cobordism from P to Q. As above, we may insert a smooth manifold N in int X (transverse to a smooth field transverse to P). We may put a copy P' of P between N and Q, and a copy Q' of Q between P and N so that the closure of the region between P' and Q' is a strong cobordism from P' to Q'. But with the inherited sides, it is a strong cobordism from $-P$ to $-Q$; that is, a strong cobordism from Q to P. Thus $P \sim Q$ implies $Q \sim P$. Finally, if $P \sim Q$ via X and $Q \sim T$ via Y, we may put smooth manifolds N and N' in int X and int Y respectively so that the closures X_0, X_1, Y_0, Y_1 of the regions between P and N, between N and Q, between Q and N' and between N' and T are strong cobordisms. From Proposition 3 below we conclude that N and N' are diffeomorphic. Then glueing X_0 and Y_1 smoothly by a diffeomorphism $N \to N'$, we obtain a strong cobordism Z from P to T. Thus $\sim$ is transitive. Finally, if $P \sim N$ via X with N smooth, $X \cup_N X$ is a strong cobordism from P to $-P$. Thus $\sim$ is an equivalence relation on $\mathscr{C}$ and $P \sim -P$ for $P \in \mathscr{C}$.

Now, the theorem we wish to prove is most naturally stated in five propositions.

PROPOSITION 1. *If two compact smooth manifolds are strongly cobordant to the same $M(n)$ manifold, and $n \geqq 6$, then they are diffeomorphic.*

PROPOSITION 2. *Let P be a sideable $M(n)$ manifold, and M a smooth manifold. Then there is a smoothing from P to M if and only if P and M are strongly cobordant.*

PROPOSITION 3. *If two smoothable sided $M(n)$ manifolds are $M(n)$-equivalent, then they are strongly cobordant.*

PROPOSITION 4. *If $n \geqq 5$ and M is an orientable compact closed smooth n-*

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manifold smoothly immersible in R^{n+1}, then there exists an $M(n)$ manifold strongly cobordant to M.

PROPOSITION 5. *If the smooth compact closed homotopy n-sphere Σ bounds a smooth compact parallelizable manifold, then there exist a polyhedron $P \subset R^{n+2}$ which is an $M(n)$ manifold strongly cobordant to Σ.*

From these propositions we conclude that for each smooth homotopy n-sphere Σ, the classes

$$K(\Sigma) = \{Q | Q \text{ is an } M(n) \text{ manifold, } Q \text{ strongly cobordant to } \Sigma\}$$

are each non-empty, and mutually disjoint. Also, if Σ is a non-standard bP_{n+1} sphere, then the polyhedron P supplied by Proposition 5 supplies two examples: 1) the cone CP is a polyhedron, PL isomorphic to I^{n+1}, but not smoothable, and 2) the suspension $SP = CP \cup_P CP$ is a polyhedron, PL isomorphic to ∂I^{n+1}, but not smoothable.

2. Proofs. Proposition 1 is the result that an $M(n)$ manifold has at most one diffeomorphism class of smoothings. It may be obtained as a corollary of a "Boundary Collar Theorem" for smooth $S(n+1)$ manifolds, and that in turn is an immediate consequence of a lemma of Thom [**4**]. In addition, we will require a simple proposition about $S(n+1)$.

PROPOSITION 6. *Suppose $p \in S(n+1)$ with $\dim_{S(n+1)} p = r \leqq n$. Then there is a basis $e_1, \ldots, e_{n+1}$ of $\tau_p(R^{n+1})$ such that the n-facets of $S(n+1)$ containing p are $F_1, \ldots, F_{n+1-r}$ with $\tau_p(F_i) = \text{span}\ (e_1, \ldots, \hat{e}_i, \ldots . e_{n+1})$.*

Proof. The proposition is true for $n = 0$. We prove it inductively in dimension $n + 1$. We may write $S(n+1) = S = T \times [0, 1] \cup R^n \times (-\infty, 0]$ with $T = S(n)$. If $\dim_S(p) = n$, the proposition is immediate. If $\dim_S(p) = r < n$, then $p = (q, t)$ with $q \in T$ and $0 \leqq t \leqq 1$. If $0 < t < 1$, then $\dim_S(p) = 1 + \dim_T(q)$. Let $e_1', \ldots, e_n'$ be the basis of $\tau_q(R^n)$ given by the proposition in dimension n. Let $e_1, \ldots, e_n$ be the parallel vectors at $p = (q, t)$ and let e_{n+1} be the vertical vector at p. Then near p, the n-facets are $F_1' \times [0, 1], \ldots, F_{n-(r-1)}' \times [0, 1]$ where $F_1', \ldots, F_{n-(r-1)}'$ are the $(n-1)$-facets of T containing q, and clearly we have $\tau_p(F_i' \times [0, 1]) = \text{span}\ (e_1, \ldots, \hat{e}_i, \ldots, e_{n+1})$. If $t = 0$ or 1, then $\dim_S(q, t) = \dim_T(q)$; let $e_1', \ldots, e_n'$ be the basis given by the proposition in dimension n, for $\tau_q(R^n)$. Let $e_1, \ldots, \hat{e}_{n+r-1}, \ldots, e_{n+1}$ be the parallel basis at p, and let e_{n-r+1} be the vertical vector there. Then, near p, the n-facets of S are $F_1' \times [0, 1], \ldots, F_{n-r}' \times [0, 1]$, F_{n-r+1} where $F_{n-r+1} = T$ 1 if $t = 1$ and $F_{n-r+1} = \text{clos}\ (R^n - T) \times 0$ if $t = 0$. But $\tau_p(F_i' \times [0, 1]) = \text{span}\ (e_1, \ldots, \hat{e}_i, \ldots, e_{n+1})$ and $\tau_p(F_{n-r+1}) = \text{span}\ (e_1, \ldots, \hat{e}_{n-r+1}, \ldots, e_{n+1})$, so the proposition is proved.

PROPOSITION 7 (Thom's Lemma). *Let $e_1, \ldots, e_{n+1}$ be a base of R^{n+1}, let $C \subset U\{\text{span}\ (e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}) | 1 \leqq i \leqq r\}$, and let $f : C \to R$ be such that* *each restriction* $f|C \cap \text{span}(e_1, \ldots, \hat{e}_i, \ldots, e_{n+1})$ *is* C^∞ *for* $1 \leqq i \leqq r$. *Then there is a* C^∞ *function* $F : R^{n+1} \to R$ *which restricts to* f.

Proof. The proof proceeds by induction on r. For $r = 1$, there is almost nothing to prove. Suppose the lemma has been proved for $r - 1$. Then

$$g = f|C \cap U\{\text{span}(e_1, \ldots, \hat{e}_i, \ldots, e_{n+1})|1 \leqq i \leqq r - 1\}$$

extends to a C^∞ function $G : R^{n+1} \to R$. To extend f, it suffices $f - G|C$. Thus we may assume that $f|C \cap \text{span}(e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}) = 0$ for $1 \leqq i < r$. But then we may assume in addition $\text{span}(e_1, \ldots, \hat{e}_i, \ldots, e_{n+1}) \subset C$ for $1 \leqq i < r$. And in this case $F(x_1, \ldots, x_{n+1}) = f(x_1, \ldots, x_{r-1}, 0, x_{r+1}, \ldots, x_{n+1})$ is the desired extension, and the lemma is proved.

THEOREM 1. *Suppose M_1 and M_2 are smooth $S(n + 1)$ manifolds; N_1 and N_2 are components of ∂M_1 and ∂M_2 respectively; and $f : N_1 \to N_2$ is an isomorphism of smooth sided $M(n)$ manifolds. Then f extends to an isomorphism of smooth $S(n + 1)$ manifolds from an open neighborhood of N_1 in M_1 to an open neighborhood of N_2 in M_2.*

Proof. Suppose $x \in N_1$ with $\dim_{N_1}(x) < n$. Then there exist charts (U, φ) of N_1 at x and (V, ψ) of N_2 at $f(x)$ such that $f(U) \subset V$ and $\varphi(U) \subset M(n)$ and $\psi(V) \subset M(n)$. Then $\dim_{N_1}(x) = \dim_{M(n)}\varphi(x) = \dim_{M(n)}\psi(f(x))$ and f induces a smooth map $g : \varphi(U) \to \psi(V)$; that is, g is C^∞ on each facet. By Proposition 6, there is a basis $(e_1, \ldots, e_{n+1})$ of R^{n+1} at $\varphi(x)$ such that the hyperplanes spanned by the n-facets of $\varphi(U)$ at $\varphi(x)$ are $\text{span}(e_1, \ldots, \hat{e}_i, \ldots, e_{n+1})$ for $1 \leqq i \leqq r$. Regarding $(e_1, \ldots, e_{n+1})$ as a basis of $\tau_{\varphi(x)}R^{n+1}$, we see that for $i = 1, \ldots, n + 1$ the vectors $dg(\varphi(x))e_i = e_i'$ are defined, that $(e_1', \ldots, e_{n+1}')$ is a basis of R^{n+1} at $g(\varphi(x)) = \psi(f(x))$, and that the hyperplanes spanned by the n-facets of $\psi(V)$ at $g(\varphi(x))$ are $\text{span}(e_1', \ldots, \hat{e}_i', \ldots, e_{n+1}')$ for $1 \leqq i \leqq r$. Now $\varphi(U) \subset U\{\text{span}(e_1, \ldots, \hat{e}_i, \ldots, e_{n+1})|1 \leqq i \leqq r\}$ and $g|\varphi(U) \cap \text{span}(e_1, \ldots, \hat{e}_i, \ldots, e_{n+1})$ is C^∞ for $1 \leqq i \leqq r$. By Thom's Lemma, there is a C^∞ extension $G : R^{n+1} \to R^{n+1}$. Returning to the charts (U, φ) and (V, ψ), we may assume that there exist charts $(0, \Phi)$ of M_1 at x and (P, Ψ) of M_2 at $f(x)$ such that $\Phi(0) \subset S(n + 1)$, and $0 \cap N_1 = U$ with $\Phi|U = \varphi$, and similarly for (P, Ψ) and (V, ψ). By means of the Euclidean metric and its exponential map we see that it follows from the hypothesis that f preserves siding that $G(\Phi(0)) \subset \Psi(P)$ so that $f|U$ extends to a C^∞ map $0 \to P$. It follows that there exist open neighborhoods $\mathcal{N}_1'$ and $\mathcal{N}_2'$ of N_1^{n-1} and N_2^{n-1} in M_1 and M_2 respectively, and a C^∞ map $F' : \mathcal{N}_1' \to \mathcal{N}_2'$ extending $f|\mathcal{N}_1 \cap \mathrm{N}_1$. Since $x \in N_1^{n-1}$ was arbitrary and $dg(\varphi(x))$ carried the base e to the base e', it follows that $dF'(x)$ is non-singular for $x \in N_1^{n-1}$. Thus we may assume that F' is a diffeomorphism $\mathcal{N}_1' \to \mathcal{N}_2'$. Finally, by means of open collars of the open n-facets we see that F' may be extended to a diffeomorphism $F : \mathcal{N}_1 \to \mathcal{N}_2$, where $\mathcal{N}_i$ is an open neighborhood of N_i in M_i. The theorem is now proved.

COROLLARY (Proposition 1). *If two compact smooth manifolds are strongly cobordant to the same $M(n)$ manifold and $n \geqq 6$, then they are diffeomorphic.*

Proof. Let the two smooth manifolds be N_1 and N_2. We are assuming that N_1 is strongly cobordant to the $M(n)$ manifold N and that N_2 is strongly cobordant to $\pm N$. Replacing N_2 if necessary with $-N_2$, we may assume that N_1 and N_2 are strongly cobordant to N. Let M_i be the strong cobordism from N_i to N. By Theorem 1, the identity map $N \to N$ extends to a diffeomorphism

$$\mathcal{N}_1 \xrightarrow{\varphi} \mathcal{N}_2$$

where $\mathcal{N}_i$ is an open neighborhood of N in M_i. By Siebenmann's Collaring Theorem, we may find A_1 compact $\subset \mathcal{N}_1$ such that $\partial A_1 = N \cup N_1'$ with N_1' a smooth boundary of A_1 and $N_1' = \text{fr}\mathcal{N}_1 - A_1 \subset \overline{\mathcal{N}_1 - A_1}$ a homotopy equivalence. We may assume that $\overline{M_1 - A_1}$ is a smooth s-cobordism from N_1' to N_1. Then N_1' and N_1 are diffeomorphic by the h-cobordism theorem. Passing to PL structures, we see that A_1 is an s-cobordism from N to N_1', so $\varphi(A_1) = A_2$ is an s-cobordism from N to $\varphi(N_1') = N_2'$. But since M_2 is an s-cobordism from N to N_2, it follows that $\overline{M_2 - A_2}$ is a smooth s-cobordism from N_2' to N_2. Thus N_1 and N_2 are diffeomorphic, and the corollary is proved.

Next we obtain Proposition 2 and half of Proposition 3 as corollaries of a theorem on subdivision of smooth $M(n)$ manifolds. Notice that subdivision becomes smooth subdivision upon relaxing $M(n)$ structures to smooth $M(n)$ structures, and that if the map $f : P \to N$ smooths P to N, then it (smoothly) subdivides N.

THEOREM 2. *Suppose M is a compact sided smooth $M(n)$ manifold, N is a smooth manifold, and $f : M \to N$ is a map that smoothly subdivides N. Then M and N are strongly cobordant.*

Proof. Suppose (U, φ) is a chart of M and $\gamma : N \to (0, \infty)$ is a function on N. Let $\Gamma(\gamma) : N \to N \times (0, \infty)$ be the graph of γ, and for $X \subset N$, let $L(\gamma)(X) = \{(x, t) | x \in X,\ t \geqq \gamma(x)\}$. Then we have a bijection $g : \varphi(U) \to \partial L(\gamma)(\varphi(U))$ defined by $g = \Gamma(\gamma) \circ f \circ \varphi^{-1}$. We will say that γ is *admissible over* (U, φ) if g extends to a diffeomorphism $G : V' \to V$ where V' is an open neighborhood of $\varphi(U)$ in $S(n + 1)$ and V is an open neighborhood of $\partial L(\gamma)(\varphi(U))$ in $L(\gamma)(\varphi(U))$.

LEMMA 1. *Suppose $p \in M(n)$. Then there is an open set of n-planes H through p such that the orthogonal projection $\pi_H : R^{n+1} \to H$ carries a neighborhood* 0 *of p in $M(n)$ homeomorphically onto a neighborhood* 0′ *of p in H so that $\pi_H | 0$ smoothly subdivides* 0′.

Proof. The lemma is clear for $n = 1$. The existence of such planes may be established inductively, and the openness is clear.

Given such a plane H, there is a (unique) unit normal u_H at p which points into $S(n + 1)$. Then there is a continuous function $\gamma_H : 0 \to R$ such that $\{x + \gamma_H(x) u_H | x \in 0'\} = 0$ and such that near p the two sets $S(n + 1)$ and

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$\{x + tu_H | x \in 0', t \geqq \gamma_H(x)\}$ are equal. Since the manifold M is $M(n)$ oriented, we have an atlas $\mathscr{A}$ of charts of M such that $(U, \varphi) \in \mathscr{A}$ implies $\varphi(U) \subset M(n)$ and such that $(U, \varphi), (V, \psi) \in \mathscr{A}$ with $U \cap V \cap M^{n-1} \neq \phi$ implies that the map

$$\varphi(U \cap V) \xrightarrow{\varphi^{-1}} U \cap V \xrightarrow{\psi} \psi(U \cap V)$$

extends to a diffeomorphism from an open set of $S(n + 1)$ to an open set of $S(n + 1)$. Let $(U, \varphi) \in \mathscr{A}$ and $x \in M^{n-1} \cap U$. Consider the composition

$$\varphi(U) \xrightarrow{\varphi^{-1}} U \xrightarrow{f} f(U) \text{ open} \subset N.$$

Since $f \circ \varphi^{-1}$ is smooth on $\varphi(U)$, by Thom's Lemma it extends to a C^∞ map $F : V' \to f(U)$. The differential $dF(\varphi(x)) : \tau_{\varphi(x)}R^{n+1} \to \tau_{f(x)}N$ is onto. By Lemma 1, we may choose an n-plane H through $p = \varphi(x)$ so that $\pi_H : 0 \to 0'$ is a homeomorphism, $0, 0' \subset \varphi(U)$, and $d(F|H)(\varphi(x)) : \tau_{\varphi(x)}H \to \tau_{f(x)}N$ is an isomorphism. Thus we may assume that $F : 0 \to 0'$ is a diffeomorphism. Let W be an open neighborhood of $\varphi(x)$ in R^{n+1} on which F is defined. We may assume W is small enough that $\dim \ker dF(y) = 1$ for $y \in W$, and (by reducing 0 and $0'$ about $\varphi(x)$) that $0' = W \cap H$. Then $\ker dF$ is spanned by a smooth unit vector field with solution σ_s; we may assume that $\sigma_s(y)$ is defined for $|s| < \epsilon$ for some $\epsilon > 0$ and $y \in 0 \cup 0'$, and that for $y \in 0'$ there is $t(y)$ such that $|t(y)| < \epsilon$ and $\sigma_{t(y)}y \in 0$. Let π be the map $\pi : 0 \to 0'$ defined by $\pi(y) = \sigma_{t(y)}y$; we may assume π is a smooth homeomorphism. Notice that the function $\gamma_1 : 0' \to R$ defined by $\gamma_1 : y \to -t(\pi^{-1}(y))$ has the property that $0 = \{\pi_{\gamma_1(y)}(y) | y \in 0'\}$ and that, after reversing the direction of the vector field if necessary, $\{\sigma_t(y) | y \in 0', \gamma_1(y) \leqq t, \sigma_t(y)\}$ defined is an open neighborhood V' of 0 in $S(n + 1)$. Now define a function $\gamma_2 : F(0') \to R$ by $\gamma_2(F(y)) = \gamma_1(y)$. By reducing $0'$ again, to a relatively compact subset, we may assume that for some $c > 0$ we have $\gamma = \gamma_2 + c : F(0') \to (0, \infty)$. It is straightforward to see that $F \circ \pi = f \circ (\varphi^{-1}|0)$. Then it is clear that γ is admissible over $(\varphi^{-1}(0), \varphi|\varphi^{-1}(0))$ with G defined by $G({}_t(y)) = (F(y), t + c)$ for $\gamma_1(y) \leqq t$ with $\sigma_t(y)$ defined and $y \in 0'$. Since we may assume $(\varphi^{-1}(0), \varphi|\varphi^{-1}(0)) \in \mathscr{A}$, we have obtained Lemma 2 (notice that it is immediate for $x \in M - M^{n-1}$):

LEMMA 2. *Let $\mathscr{A}$ be the orientation atlas of M chosen above. Then for any $x \in M$ there exist a chart at x, $(U, \varphi) \in \mathscr{A}$, and $\gamma : N \to (0, \infty)$ admissible over (U, φ).*

This lemma states that locally admissible functions exist. We wish to glue locally admissible functions to obtain globally admissible functions. For that purpose we use Lemma 3:

LEMMA 3. *Suppose $\gamma, \gamma' : N \to (0, \infty)$ are both admissible over (U, φ); then for any $x \in U, \gamma + \gamma'$ is admissible over (V, φ) where $x \in V$ open $\subset U$. Suppose $\mu : N \to (0, \infty)$ is C^∞. Then $\mu\gamma$ is admissible over (U, φ).*

Proof. As in the discussion before Lemma 2, by taking V small enough about x, we may assume that there exist a n-plane H through $\varphi(x) \in M(n)$, on open set $W \subset R^{n+1}$ containing $\varphi(V) = 0$, open subset $0'$ of H containing $\varphi(x)$, and a C^∞ extension $F : W \to f(V)$ of $f \circ \varphi^{-1}$. As in that discussion, dim ker $(dF(y)) = 1$ for $y \in W$ so that ker dF is spanned by a C^∞ unit vector field whose direction we may choose so that it points into $S(n+1)$ on $M(n) \cap W$; we may assume that vector field is transverse to 0 and $0'$, and we may assume that the solution φ_t of that vector field is defined for $|t| < \epsilon$ on $0 \cup 0'$, that for each $y \in 0$ (respectively $y \in 0'$) there is $t(y)$ with $|t(y)| < \epsilon$ (respectively $t'(y)$ with $|t'(y)| < \epsilon$) such that $\varphi_{t(y)}(y) \in 0'$ (respectively $\varphi_{t'(y)}(y) \in 0$). We may assume $F : 0' \to F(0')$ is a diffeomorphism. Finally, we may assume that a map $\pi : W \to 0'$ is defined by $\pi(y) =$ the unique point on $0'$ that is on the integral curve through y. Then π is C^∞ and $\pi|0 : 0 \to 0'$ is a smooth homeomorphism such that $F \circ \pi = f \circ \varphi^{-1}$. Granted these constructions, let

$$\bar{G} : (W, W \cap S(n+1)) \\ \to (\bar{G}(W), L(\bar{\gamma}) \cap \bar{G}(W) \subset (N \times (0, \infty), N \times (0, \infty))$$

be the diffeomorphism defined by $\bar{G}(y) = (F(\pi(y)),\ c + t'(\pi(y)))$ where $c > 0$ is sufficiently large that $\bar{\gamma} = c + t' \circ \pi : W \to (0, \infty)$. Let $G : (W, W \cap S(n+1)) \to (G(W), L(\gamma) \cap G(W))$ be a diffeomorphism making γ admissible over $(F(0'), \varphi)$ so that $G(y) = (F(y), (F(y)))$ for $y \in 0$. Consider the diffeomorphism $G \circ (\bar{G})^{-1}$; it satisfies

$$G \circ (\bar{G})^{-1}(z, \bar{\gamma}(z)) = (z, \gamma(z)) \text{ for } z \in F(0').$$

It follows that there is a horizontal vector field Λ on $G(W)$ such that $\Lambda = 0$ on $G(W) \cap \Gamma(\bar{\gamma})(F'(0))$ and $\exp \Lambda(z, t) = \text{pr}\ (\bar{G} \circ (G)^{-1}(z, t)),\ t)$ where $\text{pr} : N \times (0, \infty) \to N$ is the projection (of course, it may be necessary to reduce the size of W about $\varphi(x)$). Notice that on $M(n) \cap W$ we have $(\exp \Lambda)^{-1} \circ G = G = (f \circ \varphi^{-1}) \times (\gamma \circ f \circ \varphi^{-1})$, and that on all W we have $\text{pr} \circ (\exp \Lambda)^{-1} \circ G = F \circ \pi$. Thus, replacing G with $(\exp \Lambda)^{-1} \circ G$ we see that we may assume that $\text{pr} \circ G = F \circ \pi$. Doing the same for γ' and (U, φ), we see that we may assume $\text{pr} \circ G = F \circ \pi$. But then $G' \circ G^{-1} : (G(W), L(\gamma)(F(0'))) \to (G'(W), L(\gamma')(F(0')))$ is a diffeomorphism and $G' \circ G^{-1}(z, t) = (z, h(z, t))$ for some C^∞ function h. Since $L(\gamma)(F'(0))$ is carried to $L(\gamma')(F(0'))$, we have $\partial_t h(z, \gamma(z)) > 0$ for all $z \in F(0')$. Consider the map $H(z, t) = (z, t + h(z, t))$ defined on $G(W)$. Clearly H is smooth, and at any point $(z, \gamma(z))$ we have $dH(z, \gamma(z))\partial_t = a\partial_t$ with $a > 0$. Since $\text{pr} \circ H = \text{pr}$, it follows that $dH(z, \gamma(z))$ is non-singular for $z \in F(0')$. Thus, there is an open set $W' \subset W$ such that $0 \subset W'$ and such that $H : G(W') \to H \circ G(W')$ is a diffeomorphism; thus $H \circ G : W' \to H \circ G(W')$ is a diffeomorphism. But for $y \in M(n) \cap W' = M(n) \cap W = 0$, we have $H \circ G(y) = H(F(\pi(y)), \gamma F(\pi(y))) = (f \circ \varphi^{-1}(y), \gamma(f \circ \varphi^{-1}(y))) = (f \circ \varphi^{-1}(y), \gamma(f \circ \varphi^{-1}(y)) + h(f \circ \varphi^{-1}(y), \gamma(f \circ \varphi^{-1}(y)))) = (f \circ \varphi^{-1}(y), \gamma(f \circ \varphi^{-1}(y)) + \gamma'(f \circ \varphi^{-1} \int y)))$. Similarly one

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checks that $H \circ G(W' \cap S(n+1)) \subset L(\gamma+\gamma')(F(0'))$ so that $H \circ G$ makes $\gamma+\gamma'$ admissible over (V, φ) where $V = f^{-1}(F(0'))$, and the first half of Lemma 3 is proved. The proof of the second half of Lemma 3 is straightforward.

Now Lemmas 2 and 3 fit together with a suitable C^∞ partition of unity of N to complete the proof of Theorem 2.

Corollary 1. (Proposition 2). *Let P be a compact $M(n)$ oriented manifold and N a smooth manifold. Then there is a smoothing from P to N if and only if P and N are strongly cobordant.*

Proof. Let $f : P \to N$ be a smoothing. Relax the $M(n)$ structure on P to a smooth $M(n)$ structure. Then f smoothly subdivides N, and Theorem 2 applies to imply that P and N are strongly cobordant. The other direction is an application of the Cavins-Hirsch Theorem: Let X be the strong cobordism from P (relaxed to a smooth $M(n)$ manifold) to N. There exists a smooth vector field transverse to P, and pointing into X along P. By the Cairns-Hirsch Theorem there is a smooth compact manifold $N' \subset \operatorname{int} X$ transverse to the field, and the solution curves of the field define a map $P \xrightarrow{g} N$ that smoothly subdivides N'. Relaxing further to PL structure, we see that $X = X_1 \cup X_2$ where $X_1 \cap X_2 = N'$ and both X_1 and X_2 are cobordisms, from P to N' and N' to N respectively. But we see that X_1 is PL isomorphic to $P \times [0, 1]$ by means of the integral curves, and X also is PL isomorphic to $P \times [0, 1]$. It follows that both X and X_1 are regular neighborhoods of P, so that X_2 is PL isomorphic to $N' \times [0, 1]$. Then there is a unique smoothing on $N' \times [0, 1]$ extending that on N' so X_2 is diffeomorphic to $N' \times [0, 1]$. Finally, if $\varphi : N' \to N$ is a diffeomorphism, $\varphi \circ f$ is a smoothing from P to N and the corollary is proved.

Corollary 2 (Proposition 3). *Suppose that P_1 and P_2 are compact sided $M(n)$-manifolds such that each admits a smoothing. If they are $M(n)$-equivalent, then they are strongly cobordant.*

Proof. We may assume that there exists a map $f : P_1 \to P_2$ which $M(n)$-subdivides P_2. Let $g : P_2 \to N$ be a map which smooths P_2 to N. Then $g \circ f$ smooths P_1 to N. By Theorem 2, both P_1 and P_2 are strongly cobordant to N. Since strong cobordism is an equivalence relation, P_1 and P_2 are strongly cobordant, and the corollary is proved.

To prove Proposition 4, we need to introduce some constructions and terminology. We will say that a set of the form $[a_0, b_0] \times \ldots \times [a_n, b_n] \subset R \times \ldots \times R = R^{n+1}$ is a *hyper-rectangle*. Suppose $\mathcal{O}$ is an (open) $(n+1)$-manifold and $F : \mathcal{O} \to R^{n+1}$ an immersion. We will say that a subset $C \subset \mathcal{O}$ that F maps homeomorphically onto a hyper-rectangle is an *F-hyper-rectangle*. Then a finite union P of F-hyper-rectangles has an obvious generalized polyhedral structure making $F|P$ an affine map.

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Suppose that $\mathcal{O}$ and F are smooth, and that M is a compact smooth submanifold of $\mathcal{O}$. An *F-simple neighborhood* of M will be a finite union N of F-hyper-rectangles such that (1) N is a manifold, (2) $M \subset \text{int } N$, and (3) the inclusion $M \subset N$ is a simple homotopy equivalence. For a relative version of this definition let $\zeta : R^{n+1} \to R$ be projection on the last factor. Suppose that both $\zeta \circ F|M$ and $\zeta \circ F|\partial M$ are Morse functions with neither $\alpha, \beta \in R$ a critical value. Let $g = \zeta \circ F|M$. Then an *F-simple neighborhood* of $g^{-1}[\alpha, \beta]$ is a finite union of hyper-rectangles in $(\zeta \circ F)^{-1}[\alpha, \beta]$ such that [1] N is a manifold, (2) $g^{-1}[\alpha, \beta] \subset \text{int } N$, where the interior is with respect to the topology of $(\zeta \circ F)^{-1}[\alpha, \beta]$, and (3) the inclusion $(g^{-1}[\alpha, \beta], g^{-1}\{\alpha, \beta\}) \subset (N, N \cap (\zeta \circ F)^{-1}\{\alpha, \beta\})$ is a simple homotopy equivalence.

It seems intuitively clear that at least codimension 1 closed compact smooth submanifolds of $\mathcal{O}$ have F-simple neighborhoods – in fact arbitrarily small simple neighborhoods. But we will settle for less.

From now on M is always a compact smooth submanifold of $\mathcal{O}$. Let $\mathcal{U}$ be an open subset of $\mathcal{O}$ containing M. Let the pair of rotation groups $(SO(n + 1), SO(n))$ act on R^{n+1}, R^n in the usual way, where $R^n = R^n \times 0 \subset R^{n+1}$. Notice that for $B \in SO(n + 1)$ the composition $BF = B \circ F$ is also a smooth immersion of $\mathcal{O}$. Then define the open subset $U(M, F, \mathcal{U})$ of $SO(n + 1)$ to be

$$\{B \in SO(n + 1) | \text{There is a } BF\text{-simple neighborhood } N \text{ of } M \text{ with } N \subset \mathcal{U}\}.$$

Instead of proving that arbitrarily small F-simple neighborhoods of M exist, we will prove the following theorem. Then Proposition 4 will follow as a corollary.

THEOREM 3. *If M is a smooth closed compact n-submanifold of the smooth open $(n + 1)$-manifold $\mathcal{O}$, and $F : \mathcal{O} \to R^{n+1}$ is a smooth immersion, then for $\mathcal{U}$ an open neighborhood of M in $\mathcal{O}$ the set $U(M, F, \mathcal{U})$ is open and dense in $SO(n + 1)$.*

Proof. Clearly $U(M, F, \mathcal{U})$ is open, and clearly the theorem is true in the zero dimensional case $(n = 0)$. From now on we make the inductive hypothesis that the theorem has been proved in the $(n - 1)$ dimensional case.

It is straightforward to see that $\{C \in SO(n + 1) | \zeta \circ C \circ F|M \text{ is Morse}\}$ is an open dense subset of $SO(n + 1)$. We fix C in that set and write $g_C = \zeta \circ C \circ F|M$. For $\alpha, \beta \in R$ such that neither is a critical value of g_C, write

$$V([\alpha, \beta], F, C, \mathcal{U}) = \{B \in SO(n - 1) | \text{There is a } \begin{bmatrix} B0 \\ 01 \end{bmatrix} \circ C \circ F\text{-simple neighborhood of } g_C^{-1}[\alpha, \beta] \text{ in } \mathcal{U}\}.$$

LEMMA 1. *If $\mathscr{P} \xrightarrow{G} R^n$ is a smooth immersion of a smooth n-manifold $\mathscr{P}$, and P is a smooth compact n-submanifold of $\mathscr{P}$, and $\mathcal{O}$ is an open neighborhood of P in $\mathscr{P}$, then $U(P, G, \mathcal{O})$ is an open dense subset of $SO(n)$.*

Proof. By the induction hypothesis, $U(\partial P, G, \mathcal{O})$ is open dense in $SO(n)$. Suppose $B \in U(\partial P, G, \mathcal{O})$. Then there is a BG-simple neighborhood N of ∂P in $\mathcal{O}$. But then $P \subset N \cup P$, and $N \cup P$ is a BG-simple neighborhood of P in $\mathcal{O}$, and the lemma is proved.

LEMMA 2. *The intersection of* $V([\alpha, \beta], F, C, \mathcal{U})$, $V([\beta, \gamma], F, C, \mathcal{U})$ *and* $V([\alpha, \gamma], F, C, \mathcal{U})$ *is dense in* $V([\alpha, \beta], F, C, \mathcal{U}) \cap V([\beta, \gamma], F, C, \mathcal{U})$.

Proof. Suppose that $B \in V([\alpha, \beta], F, C, \mathcal{U}) \cap V([\beta, \gamma], F, C, \mathcal{U})$ and let O be any open neighborhood of B in that intersection. Then there exist $\begin{bmatrix} B0 \\ 01 \end{bmatrix} \circ C \circ F$-simple neighborhoods N_1 of $g_C^{-1}[\alpha, \beta]$ and N_2 of $g_C^{-1}[\beta, \gamma]$ in $\mathcal{U}$. Let $G = \begin{bmatrix} B0 \\ 01 \end{bmatrix} \circ C \circ F$ and let $\sigma = \zeta \circ \begin{bmatrix} B0 \\ 01 \end{bmatrix} \circ C \circ F = \zeta \circ C \circ F$. Notice that $g_C^{-1}(\beta)$ is an open smooth n-manifold $\mathcal{P}$, and that $\begin{bmatrix} B0 \\ 10 \end{bmatrix} \circ C \circ F|\mathcal{P} = G|\mathcal{P} : \mathcal{P} \to \zeta^{-1}(\beta)$ is a smooth immersion; and the $SO(n)$ space $\zeta^{-1}(\beta)$ identifies canonically with the $SO(n)$ space R^n. Recall the basis $(e_0, \ldots, e_n)$, and for $x \in \sigma^{-1}(\beta)$ define $(x, t) \in \mathcal{O}$ by $G(x, t) = G(x) + te_n$. This point is well defined for t sufficiently near β; if X is compact and ϵ, δ are sufficiently near β, then $X \times [\epsilon, \delta]$ is well defined by $X \times [\epsilon, \delta] = \{(x, t) | x \in X,\ r \in [\epsilon, \delta]\}$. A similar construction is this: for $D \in SO(n + 1)$ and $x \in \mathcal{O}$, then Dx is well defined by $G(Dx) = DG(x)$ provided D is sufficiently near the identity. And for X compact $\subset \mathcal{O}$, there is a neighborhood of the identity such that $D \cdot X$ is well defined in that neighborhood. In the same way, for $A \in SO(n)$ near the identity and X compact $\subset \mathcal{P}$, $A \cdot X = \{Ax | x \in X\}$ is well defined by $A(G|\mathcal{P})(x) = G|\mathcal{P}(Ax)$. Now, there exist ϵ and δ with $\alpha < \epsilon < \beta$ and $\beta < \delta < \gamma$, sufficiently near β that

(1) $N_1 \cap \sigma^{-1}[\alpha, \delta]$ and $N_2 \cap \sigma^{-1}[\epsilon, \gamma]$ are $\begin{bmatrix} B0 \\ 01 \end{bmatrix} \circ C \circ F$-simple neighborhoods of $g_C^{-1}[\alpha, \delta]$ and $g_C^{-1}[\epsilon, \delta]$ respectively.

(2) There is a compact n-submanifold P of $\mathcal{P}$ such that
 (i) $g^{-1}[\delta, \epsilon] \subset \text{int } Q \times [\delta, \epsilon] \subset (\text{int}_{\mathcal{P}} N_1' \cap \text{int}_{\mathcal{P}} N_2') \times [\delta, \epsilon]$ where $N_i' = N_i \cap \sigma^{-1}(\beta)$, and
 (ii) the inclusion $g_C^{-1}(\beta) \subset Q$ is a simple homotopy equivalence.

Then by shrinking O about B suitably, we may extend (1) and (2) to the following:

(1′) For $E \in O$, $\begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} N_1 \cap \sigma^{-1}[\alpha, \delta]$ and $\begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} N_2 \cap \sigma^{-1}[\epsilon, \gamma]$ are $\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix} \circ C \circ F$-simple neighborhoods of $g_C^{-1}[\alpha, \delta]$ and $g_C^{-1}[\epsilon, \gamma]$ respectively.

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(2′) For $E \in O$,

$$\text{(i) } g_C^{-1}[\delta, \epsilon] \subset \begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} (\text{int } Q \times [\delta, \epsilon]) \subset \begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} [(\text{int}_{\mathscr{P}} N_1') \cap (\text{int}_{\mathscr{P}} N_2') \times [\delta, \epsilon]]$$

(ii) the inclusion $g_C^{-1}(\beta) \subset \begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} Q$ is a simple homotopy equivalence.

Let $\mathscr{V}$ be open in $\mathscr{P}$, such that $\overline{\mathscr{V}}$ is compact, and $Q \times [\delta, \epsilon] \subset \mathscr{V} \times [\delta, \epsilon] \subset \overline{\mathscr{V}} \times [\delta, \epsilon] \subset (\text{int}_{\mathscr{P}} N_1') \cap (\text{int}_{\mathscr{P}} N_2') \times [\delta, \epsilon]$. By shrinking O about B again, we may assume $E \in O$ implies

$$\overline{\mathscr{V}} \times [\delta, \epsilon] \subset \begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} [(\text{int}_{\mathscr{P}} N_1') \cap (\text{int}_{\mathscr{P}} N_2') \times [\delta, \epsilon]].$$

By Lemma 1 carried over to $\zeta^{-1}(\beta)$ in place of R^n, and $B^{-1} \circ (G|\mathscr{P}) : \mathscr{P} \to \zeta^{-1}(\beta)$, we have that $U(Q, B^{-1} \circ (G|\mathscr{P}), \mathscr{V})$ is open dense in $SO(n)$. Thus, there is some $E \in 0 \cap U(Q, B^{-1} \circ (G|\mathscr{P}), \mathscr{V})$. It follows that there is an $EB^{-1} \circ (G|\mathscr{P})$-simple neighborhood N'' of Q in $\mathscr{V}$.

From (1) it follows that $\left(\begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} N_1\right) \cap \sigma^{-1}[\alpha, \delta]$ and $\left(\begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} N_2\right) \cap \sigma^{-1}[\epsilon, \delta]$ are $\begin{bmatrix} EB^{-1} & 0 \\ 0 & 1 \end{bmatrix} \circ G$-simple neighborhoods of $g_C^{-1}[\alpha, \delta]$ and $g_C^{-1}[\alpha, \gamma]$ respectively. But $N'' \times [\delta, \epsilon]$ is a union of $\begin{bmatrix} EB^{-1} & 0 \\ 0 & 1 \end{bmatrix} \circ G$-hyper-rectangles, so

$$N = \left[\left(\begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} N_1\right) \cap \sigma^{-1}[\alpha, \delta]\right] \cup N'' \times [\delta, \epsilon] \cup \left[\left(\begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} N_2\right) \cap \sigma^{-1}[\epsilon, \gamma]\right]$$

is a union of $\begin{bmatrix} EB^{-1} & 0 \\ 0 & 1 \end{bmatrix} \circ G$-hyper-rectangles. And since

$$N'' \times [\delta, \epsilon] \cap \begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} N_1 \cap \sigma^{-1}[\alpha, \delta] = N'' \times \delta \subset (\text{int}_{\mathscr{P}} N_1') \times \delta,$$

and

$$N''[\delta, \epsilon] \cap \begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} N_2 \cap \sigma^{-1}[\epsilon, \gamma] = N'' \times \epsilon \subset (\text{int}_{\mathscr{P}} N_2') \times \epsilon,$$

it follows that the $\begin{bmatrix} EB^{-1} & 0 \\ 0 & 1 \end{bmatrix} \circ G$-polyhedral structure on N makes it an $M(n)$-manifold. Finally, the inclusion

$$(g_C^{-1}[\alpha, \gamma], g_C^{-1}\{\alpha, \gamma\}) \subset (N, N \cap \sigma^{-1}\{\alpha, \gamma\})$$

is a homotopy equivalence, so N is a (relative) $\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix} \circ C \circ F$-simple neighborhood of $g_C^{-1}[\alpha, \gamma]$. Thus $E \in V([\alpha, \gamma], F, C, \mathscr{U})$ and Lemma 2 is proved.

LEMMA 3. *If* $[\alpha, \beta] \subset g_C(M)$ *contains no critical values of* g_C, *then* $V([\alpha, \beta], F, C, \mathscr{U})$ *is open and dense in* $SO(n)$.

Proof. Clearly $V([\alpha, \beta], F, C, \mathscr{U})$ is open. We set

$$\Gamma = \{x \in [\alpha, \beta] | V([\alpha, x], F, C, \mathscr{U}) \text{ is open dense in } SO(n)\}.$$

By the induction hypothesis, $\alpha \in \Gamma$. Now we show that Γ is open in $[\alpha, \beta]$. Suppose $x \in \Gamma$; we may assume that $x < \beta$. Let $G = C \circ F : \mathscr{O} \to R^{n+1}$ and $\sigma = \zeta \circ G$, and $g = \zeta \circ G|M$. Let $\mathscr{V}$ be an open subset of $\mathscr{P} = \sigma^{-1}(x)$ with $g^{-1}(x) \subset \mathscr{V} \subset \overline{\mathscr{V}}$ compact $\subset \mathscr{U}$. We may define $\overline{\mathscr{V}} \times [x, b] \subset \mathscr{O}$ as in the proof of Lemma 2, for b sufficiently near x. Then for some b with $x < b \leqq \beta$ and $\overline{\mathscr{V}} \times [x, b] \subset \mathscr{U}$ there exists a compact smooth n submanifold Q of $\mathscr{P}$ such that

(i) $g^{-1}[x, b] \subset \text{int } Q \times [x, b]$ $Q \times [x, b] \subset \mathscr{V} \times [x, b]$, and

(ii) the inclusion $g^{-1}(x) \subset Q$ is a simple homotopy equivalence.

By Lemma 1, the set $U(Q, G|\mathscr{P}, \mathscr{V})$ is open dense in $SO(n)$. Suppose $B \in U(Q, G|\mathscr{P}, \mathscr{V})$. Then there is a $\begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} \circ (G|\mathscr{P})$-simple neighborhood N' of Q in $\mathscr{V}$. But then $N' \times [x, b]$ is a $\begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} \circ G$-simple (relative) neighborhood of $g^{-1}[x, b]$ in $\mathscr{V} \times [x, b]$. Thus $U(Q, G|\mathscr{P}, \mathscr{V}) \subset V([x, b], F, C, \mathscr{U})$ and the right hand set is open dense in $SO(n)$. But already $V([\alpha, x], V, C, \mathscr{U})$ is open dense, so $V([\alpha, x], F, C, \mathscr{U}) \subset V([x, b], F, C, \mathscr{U})$ is open dense. Finally, an application of Lemma 2 shows that $V([\alpha, b], F, C, \mathscr{U})$ is open in $SO(n)$. Thus $b \in \Gamma$, and Γ must be open in $[\alpha, \beta]$.

To see that Γ is closed, suppose $a_1 < a_2 < a_3 < \ldots$ is an increasing sequence in Γ with limit y. We must show that $y \in \Gamma$; we have $\alpha < y < \beta$. As above, there will be some a with $\alpha < a < y$ such that $V([a, y], F, C, \mathscr{U})$ is open dense in $SO(n)$. Since $a \in \Gamma$, we have that $V([\alpha, a], F, C, \mathscr{U})$ is already open dense in $SO(n)$, and an application of Lemma 2 shows that $V([\alpha, y], F, C, \mathscr{U})$ is open dense. Consequently $y \in \Gamma$, and Γ is closed in $[\alpha, \beta]$.

Since Γ was already non-empty and open, it follows that $\Gamma = [\alpha, \beta]$, and the lemma is proved.

LEMMA 4. *Suppose* x *is a critical point of* $g = \zeta \circ C \circ F|M$. *Then there exists* $\epsilon > 0$ *such that* $V([x - \epsilon, x + \epsilon], F, C, \mathscr{U})$ *is open and dense in* $SO(n)$.

Proof. Let $G = C \circ F$ and $\sigma = \zeta \circ G$ and $g = \zeta \circ G|M$. The canonical form of a Morse function at a critical point allows us to find a compact smooth n-submanifold P of $\mathscr{P} = \sigma^{-1}(x)$ and $\gamma > 0$ such that x is the only critical value in $[x - \gamma, x + \gamma]$, and

$$g^{-1}[x - \gamma, x + \gamma] \subset \operatorname{int} P \times [x - \gamma, x + \gamma] \subset P \times [x - \gamma, x + \gamma] \subset \mathscr{U},$$

and such that $g^{-1}(x) \subset P$ is a simple homotopy equivalence. Let $\mathscr{V}$ be an open subset of $\mathscr{P}$ such that $P \times [x - \gamma, x + \gamma] \subset \mathscr{V} \times [x - \gamma, x + \gamma] \subset \mathscr{U}$. Then by Lemma 2, we have that $U_0 = U(P, G|\mathscr{P}, \mathscr{V})$ is open dense in $SO(n)$. Now choose $\epsilon > \gamma$ such that $[x - \epsilon, x - \gamma] \cup [x + \gamma, x + \epsilon]$ contains no critical values of g. Then $U_- = V([x - \gamma, x - \epsilon], C, F, \mathscr{U})$ and $U_+ = V([x + \gamma, x + \epsilon], C, F, \mathscr{U})$ are open dense in $SO(n)$ by Lemma 3. Now we argue as in the proof of Lemma 2: Suppose $B \in U_- \cap U_0 \cap U_+$. Then there exist $\begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} \circ G$-simple neighborhoods N_-, $N \times [x - \gamma, x + \gamma]$, and N_+ of $g^{-1}[x - \epsilon, x - \gamma]$, $P \times [x - \gamma, x + \gamma]$, and $g^{-1}[x + \gamma, x + \epsilon]$ respectively. Let $N_-' = N_- \cap \sigma^{-1}(x - \gamma)$ and $N_+' = N_+ \cap \sigma^{-1}(x + \gamma)$. Now we need to complicate notation somewhat more: There exist a, b such that $0 < a < \gamma < b < \epsilon$ and compact smooth n-submanifolds Q_- and Q_+ of $\sigma^{-1}(x - \gamma)$ and $\sigma^{-1}(x + \gamma)$ respectively, such that

(i) $g^{-1}[x - b, x - a] \subset (\operatorname{int} Q_-) \times [x - b, x - a] \subset Q_- \times [x - b, x - a] \subset (\operatorname{int} N_-' \cap \operatorname{int} N \times (x - \gamma) \times [x - b, x - a]$,

the same for $+$ in place of $-$, and

(ii) the inclusions $g^{-1}(x - \gamma) \subset Q_-$ and $g^{-1}(x + \gamma) \subset Q_+$ are simple homotopy equivalences.

Let $\mathscr{V}_-$ and $\mathscr{V}_+$ be open in $\sigma^{-1}(x - \gamma) = \mathscr{P}_-$ and $\sigma^{-1}(x + \gamma) = \mathscr{P}_+$ respectively, such that $\overline{\mathscr{V}}_\pm$ are compact and $Q_\pm \subset \mathscr{V}_\pm \subset \overline{\mathscr{V}}_\pm \subset \operatorname{int} N_\pm' \cap \operatorname{int} N \times (x \pm \gamma)$. Let 0 be an open neighborhood of B in $U_- \cap U_0 \cap U_+$. By shrinking 0 about B suitably, we may assume that for $E \in 0$ we have

(1) $\begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} N_- \cap \sigma^{-1}[x - \epsilon, x - b]$ and $\begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} N_+ \cap \sigma^{-1}[x + b, x + \epsilon]$ are $\begin{bmatrix} E & 0 \\ 0 & 1 \end{bmatrix} \circ G$-simple (relative) neighborhoods of $g^{-1}[x - \epsilon, x - b]$ and $g^{-1}[x + b, x + \epsilon]$ respectively in $\mathscr{U}$.

(2) $N_\pm \cap \sigma^{-1}[x \pm b, x \pm \gamma] = N_\pm' \times [x \pm b, x \pm \gamma]$. In particular, $N_\pm' \times [x \pm \gamma, x \pm b]$ is a $\begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} \circ G$-simple neighborhoods of $g^{-1}[x \pm \gamma, x \pm b]$ in $\mathscr{U}$.

(3) $\mathscr{V}_\pm \subset \begin{bmatrix} BE^{-1} & 0 \\ 0 & 1 \end{bmatrix} \operatorname{int} N_\pm' \cap \operatorname{int} N \times (x \pm \gamma)$.

Now

$$U_- \cap U_0 \cap U_+ \cap U\left(Q_-, \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} \circ G|\mathscr{P}_-, \mathscr{V}_-\right) \cap U\left(Q_+, \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} \circ G|\mathscr{P}_+, \mathscr{V}_+\right)$$

is open dense in $SO(n)$, so the intersection of this set with 0 is non-empty; let E be in that intersection. We apply Lemma 1 to $G_\pm = G|\mathscr{P}_\pm : \mathscr{P}_\pm \to \zeta^{-1}(x \pm \gamma)$ and we see that we may assume in addition that there exist $E \circ G_\pm$-simple neighborhoods $N_\pm''$ of $Q_\pm$ in $\mathscr{V}_\pm$. Finally then, the inclusion

$$\begin{aligned}(g^{-1}[x-\epsilon, x+\epsilon], g^{-1}\{x-\epsilon, x+\epsilon\}) \subset \Bigg(\Bigg[\begin{bmatrix} BE^{-1} & 0 \\ 0 & 1\end{bmatrix} N_- \\ \cap\, \sigma^{-1}[x-\epsilon, x-b]\Bigg] \cup N_-'' \times [x-b, x-a] \cup \Bigg[\begin{bmatrix} BE^{-1} & 0 \\ 0 & 1\end{bmatrix} N \\ \times [x-a, x+a]\Bigg] \cup N_+'' \times [x+a, x+b] \cup \begin{bmatrix} BE^{-1} & 0 \\ 0 & 1\end{bmatrix} N_+ \\ \cap\, \sigma[x+b, x+\epsilon], \begin{bmatrix} BE^{-1} & 0 \\ 0 & 1\end{bmatrix} (N_- \cap \sigma^{-1}(x-\epsilon)) \cup (N_+ \\ \cap\, \sigma^{-1}(x+\epsilon))\Bigg)\end{aligned}$$

is a simple homotopy equivalence. But then $E \in V([x-\epsilon, x+\epsilon], F, C, \mathscr{U})$. Thus $V([x-\epsilon, x+\epsilon], F, C, \mathscr{U})$ is dense; since it is already open, the lemma is proved.

Proof of theorem. By Lemmas 3 and 4, we may write $g(M)$ as a finite union of consecutive intervals $[\alpha, \beta]$ such that for each $[\alpha, \beta]$ the set $V([\alpha, \beta], F, C, \mathscr{U})$ is open and dense in $SO(n)$. It follows that their intersection is open and dense, so we may choose B in their intersection so that $\begin{bmatrix} B & 0 \\ 0 & 1\end{bmatrix} \cdot C$ is arbitrarily close to C and there exists a $\begin{bmatrix} B & 0 \\ 0 & 1\end{bmatrix} \circ C \circ F$-simple neighborhood of M in $\mathscr{U}$. Thus $U(M, F, \mathscr{U})$ is dense. Since it is already open, the theorem is proved.

COROLLARY (Proposition 4). *If $n \geqq 5$ and M is an orientable closed compact smooth n-manifold that immerses smoothly in R^{n+1}, then there exists an $M(n)$-oriented manifold strongly cobordant to M.*

Proof. By taking the normal bundle of a smooth immersion $f : M \to R^{n+1}$, we obtain a smooth open $(n+1)$ manifold $\mathscr{O} \supset M$ and a smooth immersion $F : \mathscr{O} \to R^{n+1}$. By the theorem, there is $C \in SO(n+1)$ such that there exists $C \circ G$-simple neighborhood N of M in $\mathscr{O}$. Then N is an $S(n+1)$ manifold and ∂N is an $M(n)$ manifold. Moreover $\partial N = \partial_0 N \cup \partial_1 N$ and $N = N_0 \cup N_1$ with N_0 an s-cobordism from M to $\partial_0 N$. Since $n \geqq 5$, N_0 is a strong cobordism and the corollary is proved.

Finally, we sketch the proof of Proposition 5 since the tilting details are fairly similar in technique to those of Theorem 3.

PROPOSITION. *Let Σ be a smooth homotopy n-sphere that bounds a parallelizable manifold. Then there is a polyhedron $P \subset R^{n+2}$ that is an $M(n)$ manifold strongly cobordant to Σ.*

Proof. If $n \leqq 6$ there is nothing to prove so we may assume $n \geqq 7$. We have $n = 2r - 1$ and $\Sigma = \partial X$ where X consists of an $(n + 1)$ disk with r-handles attached so that X is parallelizable. We may immerse X in R^{n+1} so that the disk lies in $R^n \times (-\infty, 0]$ and contains $D^n \times (-1, 0]$, so that each handle H is embedded and near $D^n \times 0$ coincides with $\Gamma_H \times [0, \infty)$ for some copy $\Gamma_H \subset \text{int } D^n$ of $S^{r-1} \times D^r$. We may assume that two handles intersect crosswise in a disjoint union of copies of $D^r \times D^r$ so that the double point manifold of the immersion $F : X \to R^{n+1}$ consists of a disjoint union of copies of $D^r \times D^r$, which are pairwise interchanged by the double point involution. We may assume further, by cutting the embedded handles with affine n-spaces parallel to $R^n \times 0$ that there exist $\Gamma_1, \Gamma_2, \ldots, \Gamma_k \subset X$ such that each $F(\Gamma_i)$ is the translate of some Γ_H, and such that each component of $X - \Gamma_1 - \Gamma_2 - \ldots - \Gamma_k$ contains exactly one component of the double point manifold. For each pair of components of the double point manifold paired by the double point involution, assign $+1$ to one member and -1 to the other. Thus we may assign $+1$ or -1 to the corresponding component of $X - \Gamma_1 - \ldots - \Gamma_k$; to obtain a smooth embedding $X \subset R^{n+2}$ we may find a C^∞ function $h : X \to R$, positive on each $+1$ component of $X - \Gamma_1, - \ldots - \Gamma_k$ and negative on each -1 component. Then $x \to (F(x), h(x))$ is an embedding. Instead we let $\mathscr{O} = \text{int } X$ and we identify Σ with the boundary of an open collar of X. We may assume that Σ meets each Γ_i transversally in a copy of $S^{r-1} \times S^{r-1}$. After suitable tilting, we find $F|\mathscr{O} \cap \Gamma_i$-simple neighborhoods $N_1, \ldots, N_k$ of $\Sigma \cap \Gamma_1, \ldots, \Sigma \cap \Gamma_k$. These give rise to relative F-simple neighborhoods $N_1 \times [a_1, b_1], \ldots, N_k \times [a_k, b_k]$ of $(\Gamma_1 \times [a_1, b_1]) \cap \Sigma, \ldots, (\Gamma_k \times [a_k, b_k]) \cap \Sigma$ respectively, where $[a_i, b_i]$ is a suitable closed neighborhood of x_i, and $\Gamma_i \subset R^n \times x_i$. After another tilt, we may suppose that we have as well a relative F-simple neighborhood M of $\Sigma \cap [\mathscr{O} - \Gamma_1 \times (a_1', b_1') - \ldots - \Gamma_k \times (a_k', b_k')]$ where (a_i', b_i') is a suitable open interval containing $[a_i, b_i]$. Finally, we have relative F-simple neighborhoods $R_1 \times [a_1', a_1], \ldots, R_k \times [a_k', a_k]$ of $\Sigma \cap (\Gamma_1 \times [a_1', a_1]), \ldots, \Sigma \cap (\Gamma_k \times [a_k', a_k])$ respectively, and $L_1 \times [b_1, b_1'], \ldots, L_k \times [b_k, b_k']$ of $\Sigma \cap (\Gamma_1 \times [b_1, b_1']), \ldots, \Sigma \cap (\Gamma_k \times [b_k, b_k'])$ respectively. We may assume that each $R_i \times a_i'$ and $L_j \times b_j'$ is contained in the interior of a corresponding n-facet of M, and that $R_i \times a_i \cup \text{int } N_i \times a_i$ and $L_j \times b_j \subset \text{int } N_j \times b_j$. Then

$$(\bigcup\{R_i \times [a_i', a_i] \cup N_i \times [a_i, b_i] \cup L_i \times [b_i, b_i'] | i = 1, \ldots, k\}) \cup M = Y$$

is an F-simple neighborhood of Σ, and its boundary is strongly cobordant to Σ. Notice that each component of M is in some component of $X - \Gamma_1 - \ldots - \Gamma_k$ and so inherits $+1$ or -1. Let M_+ be the union of all those components inheriting $+1$ and M_- the union of all those inheriting -1. Each R_i and L_j is in one of these components and so inherits a $+1$ or a -1, which we write as $\mathscr{O}(R_i)$ or $\mathscr{O}(L_j)$. Define a map $G : Y \to R^{n+1} \times R$ by $G(x) = (F(x), +1)$ if $x \in M_+$ and $G(x) = (F(x), -1)$ if $x \in M_-$, and $G(x) = (F(x), 0)$ if $x \in$

$\cup \{N_i \times [a_i, b_i] | i = 1, \ldots, k\}$. For $(x, t) \in R_i \times [a_i', a_i]$, set

$$G(x, t) = (F(x, t), 0) + \left(0, \frac{\sigma(R_i)}{a_i' - a_i}(t - a_i)\right)$$

and for $(x, t) \in L_j \times [b_j, b_j']$, set

$$G(x, t) = (F(x, t), 0) + \left(0, \frac{\sigma(L_j)}{b_j' - b_j}(t - b_j)\right).$$

Then G determines an affine isomorphism from ∂Y to $P = G(\partial Y)$, and P is a subpolyhedron of R^{n+1}. The proof of Proposition 5 is complete.

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