On the Linear Differential Equation of the Second Order.

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1. The linear differential equation of the second order

$$d^2y/dx^2 + P(x)\,dy/dx + Q(x)\,y = R(x) \quad \ldots\ldots\ldots\ldots\ldots (1)$$

is not in general integrable by any method at present available. At the same time, several equations of this type have been integrated, either in terms of finite functions or by means of expansions in series. Some properties of the integrals of the general equation have also been obtained. It is the object of this paper to develop some general properties of these integrals, which throw some light on the nature of the solutions, even if not obtainable in explicit terms.

We need only consider a simplification of the equation (1). Thus if we can integrate the equation

$$d^2y/dx^2 + P(x)\,dy/dx + Q(x)\,y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

we can at once integrate the more general form with $R(x)$ on the right hand side. Further, we can reduce (2) to the canonical form:

$$d^2y'/dx^2 = I(x)\,y' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

by putting $\qquad y = y' \exp\left(-\tfrac{1}{2}\int P(x)\,dx\right).$

The function $I(x)$ is given by

$$I = \tfrac{1}{4}P^2 + \tfrac{1}{2}\,dP/dx - Q.$$

We shall therefore restrict ourselves to the type

$$d^2y/dx^2 = Iy, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in which I is any function of x.

2. Let us suppose that $y(x)$ is a solution of the differential equation (4). Expand $y(x+\xi)$ by means of a Taylor series. Of

course we suppose this done only within the domain of convergence of the expansion. We have

$$y(x+\xi) = y(x) + \xi\, y_1(x) + \frac{\xi^2}{2\,!}\, y_2(x) + \;\dots\; + \frac{\xi^n}{n\,!}\, y_n(x) + \dots,$$

where $\qquad y_1 = dy/dx,\; y_2 = d^2y/dx^2,\; \dots,\; y_n = d^ny/dx^n,$ etc.

Now $\qquad y_2 = Iy.$

Hence $\qquad y_3 = Iy_1 + I_1 y;$

$$y_4 = (I^2 + I_2)\, y + 2I_1 y_1;$$

and so on, where $I_1 = dI/dx$, $I_2 = d^2I/dx^2$, etc. In other words, any differential coefficient of $y(x)$ can be expressed as a linear function of $y(x)$ and $y_1(x)$ by means of the relation (4). Thus we can write

$$y(x+\xi) = y_1(x)\, Y_1(\xi) + y_1(x)\, Y_1(\xi)$$

where y is any solution of (4), and Y, Y_1 are two functions of ξ. In reality they will be infinite series depending upon the form of the function I. Let us suppose Y and Y_1 expanded in powers of ξ, and put

$$Y(\xi) = \Sigma A_n \xi^n/n\,!\,; \quad Y_1(\xi) = \Sigma B_n \xi^n/n\,!\,. \;\dots\dots\dots\dots\dots (5)$$

The quantities A and B are of course functions of x. We thus obtain

$$y(x+\xi) = \Sigma\,(A_n y + B_n y_1)\, \xi^n/n\,! \;\dots\dots\dots\dots\dots\dots\; (6)$$

If we differentiate the equation (6) twice with respect to ξ, we get

$$\partial^2 y(x+\xi)\,/\,\partial \xi^2 = \Sigma\,(A_n y + B_n y_1)\, \xi^{n-2}/(n-2)\,!\,;$$

i.e. $\qquad y_2(x+\xi) = \Sigma\,(A_n y + B_n y_1)\, \xi^{n-2}/(n-2)\,!$

But from (4) we have

$$y_2(x+\xi) = I(x+\xi)\, y(x+\xi).$$

Hence we deduce the relation :

$$\Sigma\,(A_n y + B_n y_1)\, \xi^{n-2}/(n-2)\,! = I(x+\xi)\, \Sigma\,(A_n y + B_n y_1)\, \xi^n/n\,!\,, \;\dots\; (7)$$

in which the A's and B's are functions of x. Expand $I(x+\xi)$ by Taylor's series in the form

$$I(x+\xi) = I(x) + \xi\, I_1(x) + \xi^2/2\,!\,.\, I_2(x) + \dots,$$

and equate coefficients in equation (7). We obtain

$$A_{n+2}\, y + B_{n+2}\, y_1$$
$$= I_n(A_0 y + B_0 y_1) + nI_{n-1}(A_1 y + B_1 y_1) + \dots + I(A_n y + B_n y_1).$$

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This equation can be written

$$\{A_{n+2} - [I_n A_0 + n I_{n-1} A_1 + \frac{n(n-1)}{1.2} I_{n-2} A_2 + \dots + I A_n]\} y$$
$$+ \{B_{n+2} - [I_n B_0 + \dots + I B_n]\} y_1 = 0. \quad (8)$$

If $y'(x)$ is another solution of (4) linearly independent of $y(x)$, we have similarly the equation

$$\{A_{n+2} - [I_n A_0 + \dots + I A_n]\} y' + \{B_{n+2} - [I_n B_0 + \dots + I B_n]\} y_1 = 0. \quad (9)$$

Since $$y_1/y \neq y_1'/y',$$

it follows that (8) and (9) cannot both be satisfied unless the functions in the double brackets severally vanish. We obtain the following results:

$$\left. \begin{array}{l} A_{n+2} = I_n A_0 + n I_{n-1} A_1 + \dots + I A_n ; \\ B_{n+2} = I_n B_0 + n I_{n-1} B_1 + \dots + I B_n \end{array} \right\} . \quad \dots\dots\dots\dots (10)$$

3. The four functions A_0, B_0, A_1, B_1 are clearly given by

$$A_0(x) \equiv 1 ; \quad A_1(x) \equiv 0 ;$$
$$B_0(x) \equiv 0 ; \quad B_1(x) \equiv 1 ;$$

since $$y(x) \equiv y(x) ; \quad y_1(x) \equiv y_1(x).$$

Now $y_n(x)$ is the coefficient of $\xi^n/n!$ in the expansion of $y(x+\xi)$. Hence by (6) we have

$$y_n(x) = A_n y + B_n y_1 .$$

Differentiating, we get

$$y_{n+1}(x) = (A_n + dB_n/dx) y + (IB_n + dA_n/dx) y_1.$$

But also $$y_{n+1}(x) = A_{n+1} y + B_{n+1} y_1.$$

Thus we get the relations:

$$\left. \begin{array}{l} A_{n+1} = dA_n/dx + IB_n \\ B_{n+1} = dB_n/dx + A_n \end{array} \right\} . \quad \dots\dots\dots\dots\dots\dots\dots (11)$$

It is easy to verify that the equations (11) lead to equations (10). In the equation (8) substitute the values

$$A_{n+2} = B_{n+3} - dB_{n+2}/dx ;$$
$$A_n \quad = B_{n+1} - dB_n/dx ;$$
$$A_{n-1} = B_n \quad - dB_{n-1}/dx, \text{ etc.}$$

We get the equation

$$\left[\left(B_{n+3} - \frac{d}{dx} B_{n+2}\right) - \left\{ I\left(B_{n+1} - \frac{d}{dx} B_n\right) + nI_1\left(B_n - \frac{d}{dx} B_{n-1}\right) + \dots \right\}\right] y$$
$$+ [B_{n+2} - \{IB_n + nI_1 B_{n-1} + \dots\}] y_1 = 0.$$

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The coefficient of y can be written

$$B_{n+3} - \left\{IB_{n+1} + nI_1 B_n + \frac{n(n-1)}{1.2} I_2 B_{n-1} + \dots\right\}$$

$$-\frac{d}{dx}\left[B_{n+2} - \left\{IB_n + nI_1 B_{n-1} + \dots\right\}\right]$$

$$-\left\{I_1 B_n + nI_2 B_{n-1} + \frac{n(n-1)}{1.2} I_3 B_{n-2} + \dots\right\},$$

i.e. $$\left[B_{n+3} - \left\{IB_{n+1} + (n+1) I_1 B_n + \frac{(n+1)n}{1.2} I_2 B_{n-1} + \dots + I_{n+1} B_0\right\}\right]$$

$$-\frac{d}{dx}\left[B_{n+2} - \left\{IB_n + nI_1 B_{n-1} + \frac{n(n-1)}{1.2} I_2 B_{n-2} + \dots + I_n B_0\right\}\right]$$

Hence we get the equation :

$$\left\{[B_{n+3} - \{IB_{n+1} + (n+1) I_1 B_n + \dots\}] - \frac{d}{dx}[B_{n+2} - \{IB_n + nI_1 B_{n-1} + \dots\}]\right\} y$$
$$+ [B_{n+2} - \{IB_n + nI_1 B_{n-1} + \dots\}] y_1 = 0.$$

Thus

$$\frac{B_{n+3} - \{IB_{n+1} + (n+1) I_1 B_n + \dots\}}{y} = \frac{d}{dx} \frac{B_{n+2} - \{IB_n + nI_1 B_{n-1} + \dots\}}{y}.$$

It follows that each fraction

$$= \frac{d^2}{dx^2} \frac{B_{n+1} - \{IB_{n-1} + (n-1) I_1 B_{n-2} + \dots\}}{y}$$

$$= \text{etc.}$$

$$= \frac{d^n}{dx^n} \frac{B_3 - IB_1 - I_1 B_0}{y}.$$

But $$y_3 = Iy_1 + I_1 y\,;$$

Hence $$B_3 = I.$$

Also $$B_0 = 0\,;\; B_1 = I.$$

Thus $$B_3 - IB_1 - I_1 B = 0.$$

It follows that in general

$$B_{n+2} = IB_n + nI_1 B_{n-1} + \frac{n(n-1)}{1.2} I_2 B_{n-2} + \dots + nI_{n-1} B_1 + I_n B_0.$$

The corresponding equation for the A's follows similarly

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4. We now introduce the following definitions. Let

$$\left.\begin{aligned} E(+I^{\frac{1}{2}}, \xi) &= \Sigma (A_n + I^{\frac{1}{2}} B_n) \xi^n/n!; \\ \text{and} \qquad E(-I^{\frac{1}{2}}, \xi) &= \Sigma (A_n - I^{\frac{1}{2}} B_n) \xi^n/n!. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (12)$$

By comparing these general definitions with some special cases, we can obtain a general notion of their significance. Suppose I is a constant, in particular unity. Then I_1, I_2, ... all vanish. It follows from (10) that all the odd A's and even B's are zero, whilst the even A's and odd B's are all unity. Thus, for this case

$$E(+I^{\frac{1}{2}}, \xi) \text{ becomes } \exp(+\xi),$$

and

$$E(-I^{\frac{1}{2}}, \xi) \text{ becomes } \exp(-\xi).$$

Similarly, if I is a constant but not unity, we get

$$E(+I^{\frac{1}{2}}, \xi) \equiv \exp(+I^{\frac{1}{2}}\xi),$$

and

$$E(-I^{\frac{1}{2}}, \xi) \equiv \exp(-I^{\frac{1}{2}}\xi).$$

E.g. if I is -1, we have

$$E(+I^{\frac{1}{2}}, \xi) \equiv \cos\xi + \iota \sin\xi,$$

and

$$E(-I^{\frac{1}{2}}, \xi) \equiv \cos\xi - \iota \sin\xi.$$

where ι is $(-1)^{\frac{1}{2}}$. Thus we can consider the equation (12) to define extended exponential functions, involving two variables, x and ξ, the former being present in the coefficients of the powers of the latter.

If further we define the following functions:

$$\left.\begin{aligned} c(I^{\frac{1}{2}}, \xi) &= \Sigma A_n \xi^n/n!, \\ \text{and} \qquad s(I^{\frac{1}{2}}, \xi) &= \Sigma B_n \xi^n/n!. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

we obtain the following relations:

$$\left.\begin{aligned} E(+I^{\frac{1}{2}}, \xi) &= c(I^{\frac{1}{2}}, \xi) + I^{\frac{1}{2}} s(I^{\frac{1}{2}}, \xi), \\ \text{and} \qquad E(-I^{\frac{1}{2}}, \xi) &= c(I^{\frac{1}{2}}, \xi) - I^{\frac{1}{2}} s(I^{\frac{1}{2}}, \xi). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (14)$$

In other words, the functions c and s bear the same relations to the function E as the cosine and sine functions bear to the exponential function of an imaginary quantity, corresponding to $I = -1$, or the cosh and sinh bear to the exponential of a real quantity, corresponding to $I = +1$. We are therefore justified in calling c and s extended cosine and sine functions of ξ, corresponding to the general function $I(x)$ in the equation (4). In c and s, as in E, x is present in the coefficients of the powers of ξ.

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We shall show that the integrals of equation (4) can be expressed linearly in terms of the functions c and s. It will also be seen that c and s and their derivates have many properties analogous to those of cos and sin, and cosh and sinh, and their derivatives.

5. The functions E, c, and s involve two variables x and ξ, so that we can differentiate these functions in two ways—with respect to x or with respect to ξ. In what follows a suffix 1, 2, ... after c, s, and $E(\pm)$ will denote differentiations with respect to ξ, whilst a suffix x will denote differentiation with respect to x.

Let us now form the differential coefficients. We have

$$\begin{aligned} E_1(+I^{\frac{1}{2}}, \xi) &= \Sigma(A_n + I^{\frac{1}{2}}B_n)\,\xi^{n-1}/(n-1)\,! \\ &= \Sigma(A_{n+1} + I^{\frac{1}{2}}B_{n+1})\,\xi^n/n\,! \\ &= \Sigma\,(dA_n/dx + IB_n + I^{\frac{1}{2}}\,dB_n/dx + I^{\frac{1}{2}}A_n)\,\xi^n/n\,! \\ \text{by (11),}\quad &= I^{\frac{1}{2}}\Sigma(A_n + I^{\frac{1}{2}}B_n)\,\xi^n/n\,! + \Sigma(dA_n/dx + I^{\frac{1}{2}}dB_n/dx)\xi^n/n\,! \end{aligned}$$

Also $E_x(+I^{\frac{1}{2}}, \xi) = \Sigma\,(dA_n/dx + I^{\frac{1}{2}}dB_n/dx + I_1B_n/2I^{\frac{1}{2}})\,\xi^n/n\,!$

Thus we obtain the relation

$$E_1(+I^{\frac{1}{2}}, \xi) - E_x(+I^{\frac{1}{2}}, \xi) = I^{\frac{1}{2}}E(+I^{\frac{1}{2}}, \xi) - \Sigma\frac{I_1}{2I^{\frac{1}{2}}}B_n\,\xi^n/n\,!$$

If we change $+I^{\frac{1}{2}}$ into $-I^{\frac{1}{2}}$ we get these results:

$$\left.\begin{aligned} E_1(+) - E_x(+) &= I^{\frac{1}{2}}E(+) - I_1 s/2I^{\frac{1}{2}}, \\ E_1(-) - E_x(-) &= -I^{\frac{1}{2}}E(-) + I_1 s/2I^{\frac{1}{2}}; \end{aligned}\right\} \quad \ldots\ldots\ldots (15)$$

where the $+$ sign refers to $+I^{\frac{1}{2}}$ and the $-$ sign to $-I^{\frac{1}{2}}$.

The equations (15) are obvious extensions of the differential properties of the exponential functions as ordinarily defined, since for these I is constant so that I_1 vanishes.

Again, equations (15) in conjunction with (14) give:

$$2c = E(+) + E(-),$$

so that $\qquad c_1 - c_x = I^{\frac{1}{2}}\{E(+) - E(-)\}/2;$

but $\qquad 2s = \{E(+) - E(-)\}/I^{\frac{1}{2}};$

thus $\qquad c_1 - c_x = Is. \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$

In the same way we get, after a little calculation,

$$s_1 - s_x = c. \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

For $I = -1$, or $+1$, the functions c and s do not contain x, and equations (16) become respectively:

$$\frac{d}{d\xi}\cos\xi = -\sin\xi;\ \frac{d}{d\xi}\sin\xi = \cos\xi;$$
$$\frac{d}{d\xi}\cosh\xi = \sinh\xi;\ \frac{d}{d\xi}\sinh\xi = \cosh\xi.$$

Hence the equations (16) give us properties of the generalised cosine and sine functions, exactly analogous to those of the circular and hyperbolic functions, which can be considered as special cases.

The equations (16), as might be expected, can be obtained directly from the definitions of c and s in (13), using (11).

6. It is well known that the circular and hyperbolic functions are linearly independent integrals of their differential equations. We shall now show that the extended functions c and s here defined are also independent integrals of the differential equation (4), ξ being the independent variable. Further, the functions $E(+)$ and $E(-)$ are linear functions of c and s.

From (15) we have:

$$c_2 = \Sigma A_n \xi^{n-2}/(n-2)!$$
$$= \Sigma A_{n+2}\xi^n/n!.$$

Substitute from (10). We get

$$c_2 = \Sigma\left[IA_n + nI_1A_{n-1} + \frac{n(n-1)}{1.2}I_2A_{n-2}\ \ldots\right]\xi^n/n!$$

If we pick out the coefficient of A_n, we find it to be

$$I\frac{\xi^n}{n!} + (n+1)\frac{I_1\xi^{n+1}}{(n+1)!} + \frac{(n+1)(n+2)}{1.2}\frac{I_2\xi^{n+2}}{(n+2)!} + \ldots$$

i.e.
$$\frac{\xi^n}{n!}\left[I + I_1\xi + \frac{I_2\xi^2}{2!} + \frac{I_3\xi^3}{3!} + \ldots\right]$$

i.e.
$$\xi^n/n!\,.\,I(x+\xi).$$

Hence
$$c_2 = I(x+\xi)\,c. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

In exactly the same way we find that

$$\left.\begin{aligned} s_2 &= I(x+\xi)\,s,\\ \text{and}\quad E_2(\pm) &= I(x+\xi)\,E(\pm).\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (17)$$

The equations (17) have an obvious interpretation If we plot c, s, and $E(\pm)$ as functions of ξ, placing the axis of ξ along the

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axis of x, but with the origin of ξ at the point $x = x$, then c, s, and $E(\pm)$ are integrals of the differential equation

$$y_2 = I(\xi)\, y.$$

In other words, the functions introduced in this paper are integrals of a differential equation in ξ, the same as equation (4), but differing from (4) in having the origin of ξ not at $x = 0$, but at $x = x$. In the case of I constant, the equation (4) is unaltered by shifting the origin, so that the circular and hyperbolic functions are the same, no matter where we start. But in the general form the integrals have different coefficients according to the position of the origin of ξ.

Further, it is easily seen that c and s are linearly independent as regards ξ. For if possible let

$$Xc \equiv X's + X'',$$

where X, X', X'' are functions of x. Then from (13)

$$XA_n = X'B_n$$

for all values of n. Thus we should have

$$A_1/B_1 = A_2/B_2 = A_3/B_3 = \ldots = A_n/B_n = \text{etc}$$

But $\qquad A_1 = 0,\ B_1 = 1,\ A_2 = I,\ B_2 = 0.$

We should therefore get

$$0/1 = I/0,$$

which is impossible, unless $I = 0$. Even if $I = 0$, we have

$$c = 1,\ s = \xi,$$

which are also linearly independent.

Since it has thus been established that c and s are linearly independent integrals of a modified form of (4), it follows that $E(\pm)$ are linear functions of these independent integrals.

7. Further analogies between the c and s functions and the elementary circular and hyperbolic functions are easily derived. In (13) we have defined the c and s functions for the positive value of the square root of I. Writing the second equation of (14) in a form analogous to that of the first, we may define

$$E(-I^{\frac{1}{2}}, \xi) = c(-I^{\frac{1}{2}}, \xi) - I^{\frac{1}{2}} s(-I^{\frac{1}{2}}, \xi).$$

It follows that and

$$\left.\begin{aligned} c(-) &= c(+), \\ s(-) &= s(+). \end{aligned}\right\} \qquad (18)$$

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These results do *not*, of course, correspond to

$$\left.\begin{aligned} \cos(-\xi) &= \cos\xi, \ \cosh(-\xi) = \cosh\xi; \\ \text{and} \qquad \sin(-\xi) &= -\sin\xi, \ \sinh(-\xi) = -\sinh\xi. \end{aligned}\right\} \qquad (19)$$

As a matter of fact, we shall see in the sequel that the analogues of (19) are other and very important results. Really, the equations (18) correspond to the fact that in solving the equations

$$y_2 = -y\,;\ y_2 = +y,$$

we can use the positive or negative values of $(-1)^{\frac{1}{2}}$ and $(+1)^{\frac{1}{2}}$ indifferently. Thus

$$\tfrac{1}{2}\left(e^{\iota\xi} + e^{-\iota\xi}\right) = \tfrac{1}{2}\left(e^{-\iota\xi} + e^{\iota\xi}\right);$$

$$\tfrac{1}{2}\left(e^{\xi} + e^{-\xi}\right) = \tfrac{1}{2}\left(e^{-\xi} + e^{\xi}\right);$$

$$\frac{1}{2\iota}\left(e^{\iota\xi} - e^{-\iota\xi}\right) = \frac{1}{2(-\iota)}\left(e^{-\iota\xi} - e^{\iota\xi}\right);$$

$$\frac{1}{2(+1)}\left(e^{\xi} - e^{-\xi}\right) = \frac{1}{2(-1)}\left(e^{-\xi} - e^{\xi}\right);$$

remembering that the sinh like the sine has, in the denominator, the square root of the coefficient of y in its differential equation. If this fact is ignored, as is usually done, the analogy between the sinh and the sine is somewhat masked.

In what follows we shall therefore ignore the sign of $I^{\frac{1}{2}}$ in c and s.

8. We have so far developed our results on the basis of Taylor expansions. Such expansions necessarily involve the question of convergency, and are to this extent unsatisfactory in a general investigation like the present one. It will therefore be an advantage to remove this restriction, which we proceed to do as follows.

Starting off with the equation (4), let us define functions c and s, both involving the variables x and ξ, such that, if y is any solution of (4), we may have

$$y(x+\xi) = c(x,\xi)\,y(x) + s(x,\xi)\,y_1(x). \qquad (20)$$

Then $\qquad y_1(x+\xi) = c_1(x,\xi)\,y(x) + s_1(x,\xi)\,y_1(x).$

But

$$\begin{aligned} y_1(x+\xi) &= y_x(x+\xi) \\ &= c_x(x,\xi)\,y(x) + s_x(x,\xi)\,y_1(x) + c(x,\xi)\,y_1(x) \\ &\qquad + I\,s(x,\xi)\,y(x) \end{aligned}$$

since $\qquad y_2(x) = Iy(x).$

Hence $\qquad c_1 y + s_1 y_1 = c_x y + Is y + s_x y_1 + cy_1 ;$

i.e. $\qquad (c_1 - c_x - Is) y + (s_1 - s_x - c) y_1 = 0. \quad \text{.................. (21)}$

If y' is another integral of (4) independent of y, we have also

$$(c_1 - c_x - Is) y' + (s_1 - s_x - c) y_1' = 0. \quad \text{.................. (22)}$$

Equations (21) and (22) are consistent only if each expression in brackets vanishes, for otherwise y and y' would be connected by a linear relation. We therefore deduce for c and s the properties:

$$\left.\begin{aligned} c_1 - c_x &= Is ; \\ \text{and} \qquad s_1 - s_x &= c ; \end{aligned}\right\} \quad \text{........................ (16)}$$

the results established in Art. 5.

Again, differentiate (20) twice with respect to ξ. We get

$$y_2(x+\xi) = c_2(x, \xi) y(x) + s_2(x, \xi) y_1(x).$$

But $\qquad y_2(x+\xi) = I(x+\xi) y(x+\xi)$

$$= I(x+\xi)(cy + sy_1).$$

We deduce $\qquad [c_2(x, \xi) - I(x+\xi) c(x, \xi)] y(x)$

$$+ [s_2(x, \xi) - I(x+\xi) s(x, \xi)] y_1(x) = 0.$$

Using another solution $y'(x)$, we get by the same argument as before

$$\left.\begin{aligned} c_2 &= I(x+\xi) c, \\ \text{and} \qquad s_2 &= I(x+\xi) s, \end{aligned}\right\} \quad \text{..................... (17)}$$

as in Art. 6. It follows that c and s as defined in (20) are integrals of (4) with ξ as the independent variable, the origin being at $x = x$.

We can now define $E(\pm)$ by means of (14).

It remains to show that c and s are linearly independent as regards ξ. If not, let

$$Xc \equiv X's + X'',$$

where X, X', X'' are functions of x. Differentiate twice with respect to ξ. Then

$$Xc_2 \equiv X's_2 ;$$

i e. $\qquad XI(x+\xi) c \equiv X'I(x+\xi) s,$

by (17). Unless $\qquad I(x+\xi) \equiv 0, \qquad$ *i.e.* I is identically zero, we must have $\qquad Xc \equiv X's.$

It follows that $X'' \equiv 0$, so that if c and s are not linearly independent as regards ξ, we should have

$$s/c = \text{a function of } x \text{ only.}$$

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Now in (20) put $\xi = 0$, and we get

$$s(x, 0) = 0 \; ; \; c(x, 0) = 1 \; , \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (23)$$

since (20) is true for any solution of (4). Thus, if s/c is independent of ξ, it must be identically zero, which is easily seen to be impossible. We get then that c and s must be linearly independent integrals.

9. The equations (23) are of very great importance. They correspond to $\cos 0 = 1$, $\cosh 0 = 1$; and $\sin 0 = 0$, $\sinh 0 = 0$. They are quite independent of x, and are easily verifiable by means of (13). The two independent integrals, c and s, of equation (4) with variable ξ and shifted origin, thus have the values 1 and 0 respectively at the origin, independently of x and even of the form of $I(x)$. But we can go even further. Differentiate (20) once with respect to ξ, and put $\xi = 0$ in the result. We get

$$c_1(x, 0) = 0 \; ; \; s_1(x, 0) = 1 \; ; \; \ldots\ldots\ldots\ldots\ldots\ldots \; (24)$$

for all values of x and for all forms of $I(x)$. In other words, the functions c and s not only commence with the same values 1 and 0 respectively for all origins and all forms of I, but they also have the same slopes 0 and 1 respectively, corresponding to

$$(D \cos \xi)_0 = (D \cosh \xi)_0 = 0 \; ,$$

and

$$(D \sin \xi)_0 = (D \sinh \xi)_0 = 1 \; .$$

In view of these properties of c and s, we may call them the *principal integrals* of the differential equation (4) in ξ with shifted origin.

The equation (20) therefore states that the integral of the equation (4) with argument $x + \xi$, is obtained from the principal integrals c and s in ξ with origin at $x = x$ by multiplying them respectively by the value of y at the new origin and the slope of y at this origin. Considering ξ as the variable and x as a constant, any solution is thus expressible linearly in terms of the principal integrals c and s.

10. Consider the equations (17). They give at once

$$cs_2 = sc_2.$$

Integrate once with respect to ξ. We get

$$cs_1 - sc_1 = \text{a function of } x \text{ only}.$$

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Put $\xi = 0$. Then by (23) and (24) we obtain unity for the function of x on the right-hand side, for all values of x. Hence

$$cs_1 - sc_1 = 1, \quad \text{.......................... (25)}$$

for all values of x and all forms of $I(x)$. Equation (25) is a simpler form of the well-known relation between any two solutions of the linear differential equation (2). Using (16) it becomes

$$c^2 - Is^2 = 1 + sc_x - cs_x. \quad \text{...................... (26)}$$

The circular and hyperbolic functions of ξ are independent of x, so that $c_x = s_x = 0$, and we get the well-known results

$$\cos^2 \xi + \sin^2 \xi = 1;$$

and

$$\cosh^2 \xi - \sinh^2 \xi = 1.$$

It will be seen later that I = constant is the only case in which the right-hand side of (26) is identically unity.

From (25) we deduce

$$\frac{\partial}{\partial \xi} \frac{s(x, \xi)}{c(x, \xi)} = \frac{1}{c^2(x, \xi)}$$

where we write $c^2(x, \xi)$ for the square of the c function, etc. Hence

$$\frac{s(x, \xi)}{c(x, \xi)} = \int^{\xi} \frac{d\xi}{c^2(x, \xi)} + X_1(x)$$

where $X_1(x)$ is some function of x. Similarly

$$\frac{c(x, \xi)}{s(x, \xi)} = -\int^{\xi} \frac{d\xi}{s^2(x, \xi)} + X_2(x),$$

where $X_2(x)$ is a function of x. If c and s are supposed expanded in powers of ξ as in (13), we find that X_1 and X_2 are identically zero. We therefore have the simple results

$$\left.\begin{aligned} s &= c \int d\xi / c^2; \\ \text{and} \quad c &= -s \int d\xi / s^2. \end{aligned}\right\} \quad \text{...................... (27)}$$

Equations (27) are easily verifiable for the circular and hyperbolic functions.

11. Article 10 suggests the definitions of fresh functions analogous to the tan and cot, tanh and coth, respectively. Put

$$\left.\begin{aligned} t(x, \xi) &= s(x, \xi)/c(x, \xi); \\ \text{and} \quad ct(x, \xi) &= 1/t(x, \xi) = c(x, \xi) / s(x, \xi). \end{aligned}\right\} \quad \text{........... (28)}$$

Then from (27) we get

$$t_1 = 1/c^2,$$

and

$$ct_1 = -1/s^2. \quad \text{.......................... (29)}$$

These results are of course identical with those for the circular and hyperbolic function.

Let us further define

$$\sigma(x, \xi) = 1/c(x, \xi),$$

and

$$c\sigma(x, \xi) = 1/s(x, \xi); \quad \text{..................... (30)}$$

so that the functions t, ct, σ, $c\sigma$ bear the same relations to the c and s functions as tan, cot, sec, cosec bear to sin and cos, or as tanh, coth, sech, cosech bear to sinh and cosh. Equations (29) can be written

$$t_1 = \sigma^2,$$

and

$$ct_1 = -c\sigma^2. \quad \text{.......... .... (31)}$$

Again, by (16) we have

$$c(s_1 - s_x) - s(c_1 - c_x) = c^2 - Is^2,$$

i.e.

$$c^2\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial x}\right)\frac{s}{c} = c^2 - Is^2,$$

and

$$s^2\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial x}\right)\frac{c}{s} = -(c^2 - Is^2).$$

Hence

$$t_1 - t_x = 1 - It^2,$$

and

$$ct_1 - ct_x = I - ct^2. \quad \text{........................ (32)}$$

These results correspond to the differential properties of the circular and hyperbolic tangent and cotangent.

Further,

$$\sigma_1 - \sigma_x = -(c_1 - c_x)/c^2,$$

$$= -Is/c^2$$

by (16),

$$= -It\sigma. \quad \text{.......................... (33)}$$

Also

$$c\sigma_1 - c\sigma_x = -(s_1 - s_x)/c^2,$$

$$= -c/s^2$$

$$= -ct \, . \, c\sigma. \quad \text{......................... (33)}$$

The circular and hyperbolic analogies are obvious.

12. The comparison of equations (31) and (32) is of some interest. For $I = \pm 1$, c and s do not involve x at all, so that t and ct are independent of x, and t_x and ct_x are zero. Thus for $I = \pm 1$ we get

$$1 - It^2 = \sigma^2$$

and

$$I - ct^2 = -c\sigma^2.$$

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Both these equations are equivalent to

$$c^2 - Is^2 = 1. \qquad \text{.............................} \quad (34)$$

Thus for $I = \pm 1$ we get the well-known properties of the circular and hyperbolic functions, as already remarked in Art. 10. We shall now prove the converse theorem, viz. that (34) is true only for I constant, *i.e.* that t and ct must involve x for any other form of I.

From (16) we have at once that

$$\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial x}\right)(c^2 - Is^2) = 2c(c_1 - c_x) - 2Is(s_1 - s_x) + I_1 s^2$$
$$= I_1 s^2.$$

Hence (34) is true only if I_1 vanishes identically, *i.e.* $I = \text{const.}$ The exact significance of constant values of I other than ± 1 will present no difficulty to the reader.

It follows that in general t and ct involve x, so that t_x and ct_x do not vanish. We have, in fact,

$$\left.\begin{aligned} t_x &= \frac{1 - (c^2 - Is^2)}{c^2}, \\ \text{and} \qquad ct_x &= \frac{(c^2 - Is^2) - 1}{s^2}. \end{aligned}\right\} \qquad \text{........................} \quad (35)$$

Equations (16) also give us

$$\left.\begin{aligned} (sc)_1 - (sc)_x &= c^2 + Is^2, \\ \text{and} \qquad (c^2 + Is^2)_1 - (c^2 + Is^2)_x &= 4Isc - I_1 s^2. \end{aligned}\right\} \qquad \text{...........} \quad (36)$$

These are extensions of well-known results for circular and hyperbolic functions.

13. Having established considerable analogy between the c and s functions and the circular and hyperbolic functions, we now return to the fundamental equation (20).

We have $y(x + \xi) = c(x, \xi)\, y(x) + s(x, \xi)\, y_1(x)$. .... (20)

Hence $\quad y(x + \xi + \xi') = c(x, \xi + \xi')\, y(x) + s(x, \xi + \xi')\, y_1(x)$,

and also

$$= c(x + \xi, \xi')\, y(x + \xi) + s(x + \xi, \xi')\, y_1(x + \xi)$$
$$= c(x + \xi, \xi')\, \{c(x, \xi)\, y(x) + s(x, \xi)\, y_1(x)\}$$
$$+ s(x + \xi, \xi')\, \{c_1(x, \xi)\, y(x) + s_1(x, \xi)\, y_1(x)\}.$$

Comparing these two values we obtain

$$y(x)\left[c(x, \xi + \xi') - \{c(x + \xi, \xi')\, c(x, \xi) + s(x + \xi, \xi')\, c_1(x, \xi)\}\right]$$
$$+ y_1(x)\left[s(x, \xi + \xi') - \{c(x + \xi, \xi')\, s(x, \xi) + s(x + \xi, \xi')\, s_1(x, \xi)\}\right] = 0.$$

This is true for any other solution of the equation (4), and therefore we must have the two results:

$$\left.\begin{aligned} c(x, \xi+\xi') &= c(x, \xi)\, c(x+\xi, \xi') + c_1(x, \xi)\, s(x+\xi, \xi');\\ \text{and}\qquad s(x, \xi+\xi') &= s(x, \xi)\, c(x+\xi, \xi') + s_1(x, \xi)\, s(x+\xi, \xi'). \end{aligned}\right\} \quad (37)$$

The equations (37) are addition formulae analogous to the addition formulae for the circular and hyperbolic functions.

By division we obtain

$$t(x, \xi+\xi') = \frac{t(x, \xi) + \dfrac{s_1(x, \xi)}{c(x, \xi)}\, t(x+\xi, \xi')}{1 + \dfrac{c_1(x, \xi)}{s(x, \xi)}\, t(x, \xi)\, t(x+\xi, \xi')}; \quad \ldots\ldots (38)$$

the analogue of the tan and tanh addition formulae.

The formulae (37) and (38) have the disadvantage that in each case we have on the right-hand side principal integrals for two different origins, namely x and $x+\xi$. In order to remove this disadvantage and to obtain results more closely analogous to those for the circular and hyperbolic functions, we shall deduce difference formulae (instead of addition formulae) by solving for $c(x+\xi, \xi')$ and $s(x+\xi, \xi')$. Remembering the universal relation (25) we get

$$\left.\begin{aligned} c(x+\xi, \xi') &= c(x, \xi+\xi')\, s_1(x, \xi) - s(x, \xi+\xi')\, c_1(x, \xi);\\ \text{and}\qquad s(x+\xi, \xi') &= s(x, \xi+\xi')\, c(x, \xi) - c(x, \xi+\xi')\, s(x, \xi). \end{aligned}\right\} \quad (39)$$

The second formula in (39) is remarkably like the difference formula for the sin and sinh. The only thing to notice is that the s function of the difference refers to a different origin. The formula for c is also at once comparable with that for cos and cosh, if we remember the differential properties of the circular and hyperbolic functions.

The relative forms of the two results (39) now make it clear why the sinh difference formula is the same as that for sin, whilst the cosh formula differs from that for cos, in regard to the sign of the second term. This is made more evident if we substitute from (16), so that we get

$$\begin{aligned} c(x+\xi, \xi') = c(x, \xi+\xi')\, c(x, \xi) - I(x)\, s(x, \xi+\xi')\, s(x, \xi)&\\ + [c(x, \xi+\xi')\, s_x(x, \xi) - s(x, \xi+\xi')\, c_x(x, \xi)].& \quad \ldots(40) \end{aligned}$$

For $I = \pm 1$, we get zero for the part in square brackets since c and s are then independent of x, and the sign of the second term becomes positive for cos, $I = -1$, and negative for cosh, $I = +1$. It must be noticed that the square bracket vanishes only for I constant. For, if it vanishes, we get, putting $\xi' = 0$, and remembering (23), the relation

$$c^2 - I s^2 = 1 ,$$

and this has been proved to be true only for I constant.

14. In (39) put $\xi' = -\xi$; remembering (23–4) we get

$$\left.\begin{aligned} s(x+\xi,\ -\xi) &= -s(x,\xi) ; \\ \text{and} \qquad c(x+\xi,\ -\xi) &= s_1(x,\xi) \\ &= c(x,\xi) + s_x(x,\xi) , \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (41)$$

by (16). These results are the real analogues of the properties (19) of the circular and hyperbolic functions. Thus we see that s changes its sign but retains its arithmetical value, if the sign of the argument is changed at the same time as the origin is shifted forwards by the amount of the argument. For the c function the property is more complicated, but the fact that the differential coefficient of the sin is cos, and of sinh is cosh, has an easy analogy in the case of the general s function.

If we differentiate (39) with respect to the variable ξ', we get

$$\left.\begin{aligned} c_1(x+\xi,\xi') &= c_1(x,\xi+\xi')\, s_1(x,\xi) - s_1(x,\xi+\xi')\, c_1(x,\xi) ; \\ \text{and} \quad s_1(x+\xi,\xi') &= s_1(x,\xi+\xi')\, c(x,\xi) - c_1(x,\xi+\xi')\, s(x,\xi) . \end{aligned}\right\} \ldots (42)$$

It follows that

$$s_1(x+\xi,\xi') = c(x+\xi+\xi',\ -\xi') ,$$

which is really the second equation (41) in a slightly different form.

Again, putting $\xi' = -\xi$ in the first equation (42), we get

$$c_1(x+\xi,\ -\xi) = -c_1(x,\xi) . \ldots\ldots\ldots\ldots\ldots\ldots (43)$$

This means that the gradient of c has the same negative property as the s function. From (41) we also deduce that

$$s_1(x+\xi,\ -\xi) = s_1(x,\xi) - s_x(x,\xi), \ \ldots\ldots\ldots\ldots\ldots (44)$$

so that the gradient of s has the same negative property as the c function.

A more symmetrical way of writing the results here obtained is as follows:

$$\left.\begin{aligned} s(x+\xi,\xi_1-\xi) &= -s(x+\xi_1,\xi-\xi_1) ; \\ c(x+\xi,\xi_1-\xi) &= s_1(x+\xi_1,\xi-\xi_1) \\ &= c(x+\xi_1,\xi-\xi_1) + s_x(x+\xi_1,\xi-\xi_1) ; \\ s_1(x+\xi,\xi_1-\xi) &= c(x+\xi_1,\xi-\xi_1) \\ &= s_1(x+\xi_1,\xi-\xi_1) - s_x(x+\xi_1,\xi-\xi_1) ; \\ \text{and} \qquad c_1(x+\xi,\xi_1-\xi) &= -c_1(x+\xi_1,\xi-\xi_1). \end{aligned}\right\} \quad (45)$$