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Growth Estimates on Positive Solutions of the Equation $\Delta u + K u^{\frac{n+2}{n-2}} = 0$ in $\mathbb{R}^n$

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Abstract. We construct unbounded positive C^2-solutions of the equation $\Delta u + K u^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n$ (equipped with Euclidean metric g_o) such that K is bounded between two positive numbers in $\mathbb{R}^n$, the conformal metric $g = u^{4/(n-2)} g_o$ is complete, and the volume growth of g can be arbitrarily fast or reasonably slow according to the constructions. By imposing natural conditions on u, we obtain growth estimate on the $L^{2n/(n-2)}$-norm of the solution and show that it has slow decay.

1 Introduction

In this article we derive L^p-estimates on positive solutions of the conformal scalar curvature equation

$$\Delta u + K u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $n \geq 3$ is an integer, Δ the standard Laplacian on $\mathbb{R}^n$, K a smooth function. Equation (1.1) relates the scalar curvature of the conformal metric $g = u^{4/(n-2)} g_o$ to $4K(n-1)/(n-2)$, where g_o is Euclidean metric [10]. It is assumed throughout this note that

$$0 < a^2 \leq K(x) \leq b^2 \quad \text{for large } |x| \tag{1.2}$$

and for some positive constants a and b. The following estimates are known for any positive smooth solution u of equation (1.1) with condition (1.2).

$$\int_{S^{n-1}} u(r,\theta)\, d\theta \leq C_1 r^{\frac{2-n}{2}} \tag{1.3}$$

$$\int_{B_o(r)} u^{\frac{n+2}{n-2}}(x)\, dx \leq C_2 r^{\frac{n-2}{2}} \tag{1.4}$$

for large r and for some positive constants C_1 and C_2 depending on u (see, for example, [11]). Here $B_o(r)$ is the ball with center at the origin and radius r, and S^{n-1} is the unit sphere in $\mathbb{R}^n$. We seek to obtain higher order estimates of the forms

$$\int_{S^{n-1}} u^p(r,\theta)\, d\theta \leq C_3 r^{(2-n)p/2}, \quad p > 1; \tag{1.5}$$

$$\int_{B_o(r)} u^q(x)\, dx \leq \begin{cases} C_4 r^{n-(n-2)q/2} & q > \frac{n+2}{n-2},\ q \neq \frac{2n}{n-2}; \\ C_5 \ln r & q = \frac{2n}{n-2}, \end{cases} \tag{1.6}$$

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for large r, where C_3, C_4 and C_5 are positive constants. The above estimates are based on the *slow decay* of u, that is,

$$u(x) \leq C_6|x|^{(2-n)/2} \quad \text{for large } |x|, \tag{1.7}$$

where C_6 is a positive constant.

Our first observation is that, in general, (1.5), (1.6) or (1.7) do not hold. Taliaferro [13] shows that positive solution of (1.1) outside a ball in $\mathbb{R}^n$ with condition (1.2) may not have slow decay. We modify the construction in [13] to obtain positive C^2-solutions of (1.1) in $\mathbb{R}^n$ with K bounded between two positive numbers in $\mathbb{R}^n$, such that the conformal metric $g = u^{4/(n-2)}g_o$ is complete and the volume growth of $(\mathbb{R}^n, g)$ can be arbitrarily fast or reasonably slow with respect to the constructions. This suggests that the geometric structure of complete manifolds of bounded positive scalar curvature could be very complicated (*cf.* [9]). It is observed in [6] that if estimate (1.5) holds for some number $p \geq 2n/(n-2)$, then u has slow decay and hence (1.5) and (1.6) hold for all $p, q > 1$. The integral in estimate (1.6) is the volume growth of $(\mathbb{R}^n, g)$ when $q = 2n/(n-2)$. In order to obtain (1.5) and (1.6) for large p and q, additional conditions on K or u are required. By using a novel version of the moving plane method, Chen-Lin ([2] [3] and [4]) and Lin [12] examine, among other things, slow decay of u under the condition

$$0 < \frac{C_7}{|x|^{1+\alpha}} \leq |\bigtriangledown K(x)| \leq \frac{C_8}{|x|^{1+\alpha}} \quad \text{for large } |x| \tag{1.8}$$

and for some positive constants α, C_7 and C_8.

To gain better understanding on u, consider the case when K is equal to a positive constant, say $K = n(n-2)/4$, outside a compact subset of $\mathbb{R}^n$. We express u as an associated function on the cylinder $\mathbb{R} \times S^{n-1}$ by letting

$$v(s, \theta) = |x|^{\frac{n-2}{2}} u(x), \quad \text{where } |x| = e^s \text{ and } \theta = x/|x| \in S^{n-1}, x \neq 0. \tag{1.9}$$

Then v satisfies the equation

$$\frac{\partial^2 v}{\partial s^2} + \Delta_\theta v - \frac{(n-2)^2}{4} v + K v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R} \times S^{n-1}, \tag{1.10}$$

where Δ_θ is the standard Laplacian on S^{n-1}. Here K is interpreted as a function on $\mathbb{R} \times S^{n-1}$ such that $(s, \theta) \mapsto K(e^s, \theta)$ for $s \in \mathbb{R}$ and $\theta \in S^{n-1}$. By a result of Caffarelli, Gidas and Spruck [1], with improvements by Korevaar, Mazzeo, Pacard and Schoen [8], either g can be realized as a smooth metric on S^n (in this case u is said to have *fast decay*), or

$$v(s, \theta) = v_\varepsilon(s + T)[1 + O(e^{-\kappa s})] \quad \text{for large } s, \ \theta \in S^{n-1} \tag{1.11}$$

and for some constants $\kappa > 0$ and $T \in \mathbb{R}$. Here v_ε, $\varepsilon \in (0, [(n-2)/n]^{(n-2)/4}]$, is one of a one-parameter family of positive solutions of the O.D.E.

$$v'' - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}, \tag{1.12}$$

and $\varepsilon = \min_{t\in\mathbb{R}} v(t)$ is referred as the necksize of the solution [8]. As O.D.E. (1.12) is autonomous, $|v_\varepsilon'|$ is bounded in $\mathbb{R}$. Furthermore, the Pohozaev number

$$(1.13)\quad P(u) = \lim_{r\to+\infty} P(u,r) \quad \text{where } P(u,r) = \frac{n-2}{2n}\int_{B_o(r)} x\cdot\nabla K(x)u^{\frac{2n}{n-2}}(x)\,dx$$

is a negative number [8]. When K may not be a constant outside a compact subset of $\mathbb{R}^n$, we have the following results.

Theorem A *Let u be a positive smooth solution of equation (1.1) with condition (1.2), and v given by (1.9). Assume that there exist positive constants C_9 and C_{10} such that*

$$(1.14)\qquad \int_{S^{n-1}}\Big(\frac{\partial v}{\partial s}\Big)^2(s,\theta)\,d\theta \le C_9 + C_{10}\int_{S^{n-1}} v^2(s,\theta)\,d\theta$$

for large s. If $P(u,r) \ge -\delta^2$ for large r and for a positive constant δ, then

$$(1.15)\qquad \int_{B_o(r)} u^{\frac{2n}{n-2}}\,dx \le C'\ln r \quad \text{and} \quad \int_{B_o(r)} |\nabla u|^2\,dx \le C''\ln r$$

for large r and for some positive constants C' and C''.

Theorem B *Assume that there exist positive constants C_{11} and C_{12} such that*

$$(1.16)\qquad \Big|\frac{\partial v}{\partial s}\Big|(s,\theta) \le C_{11} + C_{12}v(s,\theta) \quad \text{for large } s \text{ and } \theta\in S^{n-1}.$$

If $P(u,r) \ge -\delta^2$ for large r and for a positive constant δ, then

$$(1.17)\qquad \int_{S^{n-1}} u^{\frac{2n}{n-2}}(r,\theta)\,d\theta \le Cr^{-n}$$

for large s and for some positive constant C. Moreover, u has slow decay.

We prove theorems A and B in Section 4. Lower bounds on $P(u,r)$ are obtained in Section 3, and examples are constructed in Section 2. We use $c, C, C_1, C_2, \ldots$ to denote positive constants, which may be different from section to section.

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2 Examples

We begin with a construction of positive C^2-solution u of equation (1.1) with K bounded between two positive constants in $\mathbb{R}^n$, such that u is unbounded from above

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in $\mathbb{R}^n$ (and hence does not have slow decay), and the conformal metric g is complete. Throughout this note $n \geq 3$ is an integer. Let

$$\bar{u}(r,\lambda) = \alpha_n\Big(\frac{\lambda}{\lambda^2 + r^2}\Big)^{(n-2)/2} \quad \text{for } r \geq 0 \quad \text{and} \quad \lambda > 0, \tag{2.1}$$

where $\alpha_n = [n(n-2)]^{(n-2)/4}$, and

$$u_o(x) = \bar{u}(|x|, 1) = \frac{\alpha_n}{(1 + |x|^2)^{(n-2)/2}} \quad \text{for } x \in \mathbb{R}^n. \tag{2.2}$$

Let $\{\varepsilon_k\}_{k=1}^{\infty} \subset (0,1)$ be a sequence of decreasing numbers such that

$$\sum_{k=1}^{\infty} \varepsilon_k = 1, \tag{2.3}$$

$\{r_k\}_{k=1}^{\infty}$ a sequence of positive numbers such that $r_1 \geq 1$, $r_{k+1} - r_k \geq 1$ for $k = 1, 2, \ldots$, and $\{M_k\}$ a sequence of positive numbers such that $M_k \to +\infty$ as $k \to +\infty$. For $x^{1,k} := (r_k, 0, \ldots, 0) \in \mathbb{R}^n$, $k = 1, 2, \ldots$, there exist positive numbers λ_k, $k = 1, 2, \ldots$, such that

$$u_k(x) := \bar{u}(|x - x^{1,k}|, \lambda_k) \quad \text{for } x \in \mathbb{R}^n \tag{2.4}$$

satisfies

$$\Delta u_k + u_k^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \tag{2.5}$$

$$u_k(x) \leq \varepsilon_k u_o(x) \text{ and } |\bigtriangledown u_k(x)| < \varepsilon_k \quad \text{for } |x - x^{1,k}| \geq \frac{1}{4}, \quad \text{and} \tag{2.6}$$

$$u_k(x^{1,k}) = \alpha_n \lambda_k^{(2-n)/2} \geq M_k \tag{2.7}$$

for $k = 1, 2, \ldots$. Using (2.3) and (2.6), it follows as in [13] that $\sum_{k=0}^{\infty} u_k$ converges uniformly on compact subsets of $\mathbb{R}^n$ to a positive C^2-function. For a positive number b, let

$$\tilde{u}_b(x) = (|x|^2 + b^2)^{(2-n)/4} \quad \text{for } x \in \mathbb{R}^n. \tag{2.8}$$

We have

$$\Delta \tilde{u}_b + K_b \tilde{u}_b^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \tag{2.9}$$

where

$$K_b(x) = \frac{n(n-2)}{2}\Big(1 - \frac{n+2}{2n}\frac{|x|^2}{|x|^2 + b^2}\Big) \quad \text{for } x \in \mathbb{R}^n. \tag{2.10}$$

In particular

$$\frac{n(n-2)^2}{4n} \le K_b(x) \le \frac{n(n-2)}{2} \quad \text{for } x \in \mathbb{R}^n. \tag{2.11}$$

Let

$$u(x) = \tilde{u}_b(x) + \sum_{k=0}^{\infty} u_k(x) \quad \text{for } x \in \mathbb{R}^n. \tag{2.12}$$

It follows from (2.5), (2.9) and (2.11) that

$$-\Delta u(x) = \Big[K_b(x)\tilde{u}_b^{\frac{n+2}{n-2}}(x) + \sum_{k=0}^{\infty} u_k^{\frac{n+2}{n-2}}(x)\Big] \le \frac{n(n-2)}{2} u^{\frac{n+2}{n-2}}(x) \tag{2.13}$$

for $x \in \mathbb{R}^n$. Assume that $x \in B_{x_{k'}}(1/4)$ for some positive integer k'. Using (2.3), (2.6) and the inequality $(a+b)^p \le 2^{p-1}(a^p + b^p)$ for $a, b \ge 0$ and $p \ge 1$, we have

$$\begin{aligned} u^{\frac{n+2}{n-2}}(x) &= \Big[\tilde{u}_b(x) + u_{k'}(x) + \sum_{k \ne k'} u_k(x)\Big]^{\frac{n+2}{n-2}} \le [\tilde{u}_b(x) + u_{k'}(x) + 2u_o(x)]^{\frac{n+2}{n-2}} \\ &\le c_1[\tilde{u}_b^{\frac{n+2}{n-2}}(x) + u_{k'}^{\frac{n+2}{n-2}}(x) + u_o^{\frac{n+2}{n-2}}(x)] \le -c_2 \Delta u(x), \end{aligned} \tag{2.14}$$

where c_1 and c_2 are positive constants depending on n only. Similar estimate holds for $x \notin B_{x_{k'}}(1/4)$ for $k' = 1, 2, \ldots$, if we choose c_2 to be large enough, which depends on n only. Thus u satisfies the equation $\Delta u + K u^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n$, where

$$K(x) = [-\Delta u(x)][u^{\frac{n+2}{n-2}}(x)]^{-1} \quad \text{for } x \in \mathbb{R}^n$$

is a continuous function which is bounded in $\mathbb{R}^n$ between two positive constants by (2.13) and (2.14). (2.7) shows that u is not bounded from above in $\mathbb{R}^n$. The conformal metric $u^{4/(n-2)} g_o$ is complete because

$$u^{4/(n-2)}(x) \ge \tilde{u}_b^{4/(n-2)}(x) \ge (1/2)|x|^{-2} \tag{2.15}$$

for large $|x|$. Let

$$V_n := \omega_n \int_0^{\infty} \left(\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + r^2}\right)^n r^{n-1}\, dr = \omega_n \int_0^{\infty} \left(\frac{\sqrt{n(n-2)}}{1+t^2}\right)^n t^{n-1}\, dt \tag{2.16}$$

for $\lambda > 0$, where $t = \lambda^{-1} r$ and ω_n is the volume of the unit sphere in $\mathbb{R}^n$. By choosing r_k suitably far from each other, together with (2.16) and the fact that the first integral in (2.16) concentrates more on a neighborhood of 0 for smaller λ, we have

$$\int_{B_o(r)} u^{\frac{2n}{n-2}}(x)\, dx \le C_2 \ln r \tag{2.17}$$

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for large r and for a positive constant C_2.

Next, given a function $\phi\colon [0,\infty) \to [0,\infty)$, we construct a positive C^2-solution u of equation (1.1) with K bounded between two positive constants in $\mathbb{R}^n$, such that the conformal metric $g = u^{4/(n-2)} g_o$ is complete and

$$\int_{B_o(r)} u^{\frac{2n}{n-2}}(x)\,dx \ge \phi(r) \quad \text{for } r > 2. \tag{2.18}$$

Without loss of generality, we may assume that ϕ is increasing and $\phi(0) \ge 10V_n$. For $k = 1, 2, \ldots$, let N_k be a positive integer such that

$$N_k \ge 2V_n^{-1}\phi(k+2) \quad \text{for } k = 1, 2, \ldots \tag{2.19}$$

Let $\{\epsilon_k\}_{k=1}^{\infty} \subset (0,1)$ be a sequence of decreasing numbers such that

$$\sum_{k=1}^{\infty} N_k \epsilon_k \le 1. \tag{2.20}$$

Let $\theta_k = 2\pi/N_k$. Let

$$x_{k,j} = \big(k\sin(j\theta_k), k\cos(j\theta_k), 0, \ldots, 0\big) \in \mathbb{R}^n \quad \text{for } j = 1, 2, \ldots, N_k, \tag{2.21}$$

and

$$u_{k,j}(x) = \bar{u}(|x - x_{k,j}|, \lambda_k) \quad \text{for } x \in \mathbb{R}^n \text{ and } j = 1, 2, \ldots, N_k. \tag{2.22}$$

We choose λ_k to be small so that

$$u_{k,j}(x) \le \epsilon_k u_o(x) \text{ and } |\bigtriangledown u_{k,j}(x)| < \epsilon_k \quad \text{for } |x - x_{k,j}| \ge \pi/(10N_k), \tag{2.23}$$

and

$$\int_{B_{x_{k,j}}(\pi/(10N_k))} u_{k,j}^{\frac{2n}{n-2}}(x)\,dx \ge \frac{V_n}{2} \quad \text{for } j = 1, 2, \ldots, N_k, \tag{2.24}$$

where $B_{x_{k,j}}\big(\pi/(10N_k)\big)$ is the ball with center at $x_{k,j}$ and radius equal to $\pi/(10N_k)$. (2.24) is possible because, when λ is smaller, the first integral in (2.16) concentrates more on a neighborhood of the origin. It follows from (2.20) and (2.23) that the series $\sum_{k=1}^{\infty}\sum_{j=1}^{N_k} u_{k,j}$ converges uniformly on compact subsets of $\mathbb{R}^n$ to a positive C^2-function. Let

$$u = \tilde{u}_b + u_o + \sum_{k=1}^{\infty}\sum_{j=1}^{N_k} u_{k,j} \quad \text{in } \mathbb{R}^n.$$

As above, we have $\Delta u + K u^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n$, where K is a continuous function on $\mathbb{R}^n$ that is bounded between two positive constants. For any $r > 2$, let k be the integer such that $k + 1 \le r < k + 2$. By (2.19) we have

$$\begin{aligned}\phi(r) \le \phi(k+2) \le \frac{V_n N_k}{2} &\le \sum_{j=1}^{N_k} \int_{B_{x_{k,j}}(\pi/(10N_k))} u_{k,j}^{\frac{2n}{n-2}}(x)\,dx \\ &\le \int_{B_o(k+1)} u^{\frac{2n}{n-2}}(x)\,dx \le \int_{B_o(r)} u^{\frac{2n}{n-2}}(x)\,dx.\end{aligned}$$

3 Estimates on $P(u,r)$

Let $P(u,r)$ be given by (1.13) in the introduction. The Pohozaev identity (see, for example, [7]) states that

$$(3.1) \qquad P(u,r) = \int_{S_r} \Big[r\Big(\frac{\partial u}{\partial r}\Big)^2 - \frac{r}{2} |\bigtriangledown u|^2 + \frac{n-2}{2n} r K u^{\frac{2n}{n-2}} + \frac{n-2}{2} u \frac{\partial u}{\partial r} \Big] \, dS$$

for $r > 0$, where $S_r = \partial B_o(r)$ is the sphere of radius r.

Theorem 3.2 *Let u be a positive smooth solution of equation (1.1) with condition (1.2). Assume that u is bounded from above in $\mathbb{R}^n$ and*

$$(3.2) \qquad \frac{\partial K}{\partial r}(x) \ge -\frac{C_1}{|x|^{(n+2)/2} (\ln |x|)^{1+\epsilon}}$$

for large $|x|$ and for some positive constants C_1 and ϵ. Then $P(u,r) \ge -\delta^2$ for large r and for a positive constant δ.

Proof Fixing a large number R and using (1.4) we have

$$\begin{aligned} &\int_{B_o((m+1)R)\setminus B_o(mR)} r \frac{\partial K}{\partial r}(x) u^{\frac{2n}{n-2}}(x)\, dx \\ &\quad \ge -\frac{C_1}{(mR)^{\frac{n}{2}} [\ln(mR)]^{1+\epsilon}} \int_{B_o((m+1)R)\setminus B_o(mR)} u^{\frac{2n}{n-2}}(x)\, dx \\ &\quad \ge -\frac{C_2}{(mR)^{\frac{n}{2}} [\ln(mR)]^{1+\epsilon}} \int_{B_o((m+1)R)\setminus B_o(mR)} u^{\frac{n+2}{n-2}}(x)\, dx \\ &\quad \ge -\frac{C_3 [(m+1)R]^{\frac{n-2}{2}}}{(mR)^{\frac{n}{2}} [\ln(mR)]^{1+\epsilon}} \ge -\frac{C_4}{m (\ln m)^{1+\epsilon}} \end{aligned}$$

for any positive integer m larger than 1, where $r = |x|$. Here C_2, C_3 and C_4 are positive constants. As the series

$$\sum_{m=2}^{\infty} \frac{1}{m (\ln m)^{1+\epsilon}}$$

converges, we conclude that there exists a positive constant δ such that $P(u,r) \ge -\delta^2$ for large r. ■

Theorem 3.4 *Let u be a positive smooth solution of equation (1.1) with condition (1.2). Assume that there exists a positive constant c such that*

$$(3.3) \qquad \frac{\partial K}{\partial r}(r,\theta) \ge -\frac{c}{r^2} \quad \text{for large } r \text{ and } \theta \in S^{n-1}.$$

If there exist positive constants C and $\lambda \in (0,1)$ such that

$$(3.4) \qquad \int_{S^{n-1}} \Big(\frac{\partial v}{\partial s}\Big)^{\frac{2n}{n-2}} (s,\theta)\, d\theta \le C e^{\lambda s}$$

for large s, then $P(u,r) \ge -\delta^2$ for large r and for a positive constant δ.

Proof For a positive number $\varepsilon > 0$ such that $\varepsilon + \lambda < 1$, using Young's inequality we have

$$\begin{aligned}
&\frac{d}{dr}\Big(\int_{S_r} r^{\varepsilon} u^{\frac{2n}{n-2}}(r,\theta)\,dS\Big)\\
&\quad = \frac{d}{dr}\Big(\int_{S^{n-1}} r^{n-1+\varepsilon} u^{\frac{2n}{n-2}}(r,\theta)\,d\theta\Big)\\
&\quad = \frac{n-1+\varepsilon}{r}\int_{S^{n-1}} r^{n-1+\varepsilon} u^{\frac{2n}{n-2}}(r,\theta)\,d\theta + \frac{2n}{n-2}\int_{S^{n-1}} u^{\frac{n+2}{n-2}}(r,\theta)\frac{\partial u}{\partial r}(r,\theta) r^{n-1+\varepsilon}\,d\theta\\
&\quad = \frac{-1+\varepsilon}{r}\int_{S^{n-1}} r^{n-1+\varepsilon} u^{\frac{2n}{n-2}}(r,\theta)\,d\theta\\
&\qquad + \frac{2n}{n-2}\int_{S^{n-1}} u^{\frac{n+2}{n-2}}(r,\theta)\Big[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\Big](r,\theta) r^{n-1+\varepsilon}\,d\theta\\
&\quad \le \frac{C_5}{r^{2-\varepsilon}}\int_{S^{n-1}}\Big\{r^{\frac{n}{2}}\Big[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\Big](r,\theta)\Big\}^{\frac{2n}{n-2}}\,d\theta\\
&\quad = \frac{C_5}{r^{2-\varepsilon}}\int_{S^{n-1}}\Big(\frac{\partial v}{\partial s}\Big)^{\frac{2n}{n-2}}(s,\theta)\,d\theta \le \frac{C_6}{r^{2-\lambda-\varepsilon}}
\end{aligned}$$

for large r, where $r = e^s$ and C_5 and C_6 are positive constants. It follows that there exists a positive constant C_7 such that

$$\int_{S_r} r^{\varepsilon} u^{\frac{2n}{n-2}}\,dS \le C_7 \quad \text{or} \quad \int_{S_r} u^{\frac{2n}{n-2}}\,dS \le C_7 r^{-\varepsilon} \tag{3.5}$$

for large r. For a fixed large number R_o, we have

$$\frac{2n}{n-2}P(u,R) = \int_{B_o(R)} r\frac{\partial K}{\partial r} u^{\frac{2n}{n-2}}\,dx \ge -C_8 - C_9\int_{R_o}^{R} r^{-1}\int_{S_r} u^{\frac{2n}{n-2}}\,dS\,dr \ge -C_{10}$$

for large R with $R_o < R$. Here C_8, C_9 and C_{10} are positive constants. ■

4 Proofs of Theorem A and B

Proof of Theorem A Let

$$w(s) = \frac{1}{2}\int_{S^{n-1}} v^2(s,\theta)\,d\theta \quad \text{for } s \in \mathbb{R}, \tag{4.1}$$

where v is defined in (1.9). Using equation (1.10) we have

$$\begin{aligned}
w''(s) = &\int_{S^{n-1}}\Big(\frac{\partial v}{\partial s}\Big)^2(s,\theta)\,d\theta + \int_{S^{n-1}} |\bigtriangledown_\theta v(s,\theta)|^2\,d\theta\\
&+ \Big(\frac{n-2}{2}\Big)^2\int_{S^{n-1}} v^2(s,\theta)\,d\theta - \int_{S^{n-1}} K(e^s,\theta) v^{\frac{2n}{n-2}}(s,\theta)\,d\theta
\end{aligned} \tag{4.2}$$

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for $s \in \mathbb{R}$. The Pohozaev identity can be expressed as

(4.3)
$$2P(u,e^s) = \int_{S^{n-1}} \Big(\frac{\partial v}{\partial s}\Big)^2 (s,\theta)\, d\theta - \int_{S^{n-1}} |\bigtriangledown_\theta\, v(s,\theta)|^2\, d\theta \\ - \Big(\frac{n-2}{2}\Big)^2 \int_{S^{n-1}} v^2(s,\theta)\, d\theta + \frac{n-2}{n}\int_{S^{n-1}} K(e^s,\theta) v^{\frac{2n}{n-2}}(s,\theta)\, d\theta$$

for $s \in \mathbb{R}$ [6]. It follows from (1.2), (1.4), (4.2) and (4.3) that

$$w''(s) \le C_1 + C_2 \int_{S^{n-1}} v^2(s,\theta)\, d\theta - \frac{2a^2}{n}\int_{S^{n-1}} v^{\frac{2n}{n-2}}(s,\theta)\, d\theta \tag{4.4}$$

for large s, where C_1 and C_2 are positive constants. Applying Young's inequality we obtain

$$-\frac{2a^2}{n}\int_{S^{n-1}} v^{\frac{2n}{n-2}}(s,\theta)\, d\theta \le C_3 - C_4 \int_{S^{n-1}} v^2(s,\theta)\, d\theta \tag{4.5}$$

for large s, where C_3 and C_4 are positive constants. Furthermore, by choosing C_3 to be large, we can take C_4 to be large as well. Hence there exists a positive constant C_5 such that

$$w''(s) \le C_5 - w(s) \quad \text{for large } s. \tag{4.6}$$

From (4.6) it is easy to see that $w(s)$ is uniformly bounded from above for large s. To prove this assertion, assume that there is a large s' such that $w(s') \ge C_5 + 1$. (4.6) implies that $w''(s') \le -1$. Let s_o be a number larger than s' such that $w(s_o) < C_5 + 1$ and $w'(s_o) \le 0$. If $w(s) < C_5 + 1$ for all $s > s_o$, then we are done. Assume that s_1 is the smallest number larger than s_o such that $w(s_1) = C_5 + 1$. We claim that

$$D := w'(s_1) < 2(C_5 + 1). \tag{4.7}$$

Let $\bar{s} \in (s_o, s_1)$ be the largest number such that $w'(\bar{s}) = D/2$. As $w'' \le C_5$ on (s_o, s_1), we have $s_1 - \bar{s} \ge D/(2C_5)$. On the other hand, $w' \ge D/2$ on $(\bar{s}, s_1)$. Therefore we have

$$C_5 + 1 \ge w(s_1) - w(\bar{s}) \ge \frac{D}{2C_5}\cdot\frac{D}{2} \Rightarrow D^2 \le 4C_5(C_5+1).$$

Hence we have (4.7). From s_1, $w(s)$ can become no larger than $(C_5 + 1) + [2(C_5 + 1)]^2$ before $w'(s)$ becomes negative again. Hence we conclude that $w(s)$ is uniformly bounded from above for large s.

From Pohozaev identity (3.1) we obtain

$$\int_{S_r} r|\bigtriangledown u|^2\, dS = 2\int_{S_r}\Big[r\Big(\frac{\partial u}{\partial r}\Big)^2 + \frac{n-2}{2n} rKu^{\frac{2n}{n-2}} + \frac{n-2}{2} u\frac{\partial u}{\partial r}\Big]\, dS - 2P(u,r) \tag{4.8}$$

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for $r > 0$. We have

(4.9)
$$\int_{S_r} r\Big(\frac{\partial u}{\partial r}\Big)^2 dS = \int_{S_r} r\Big[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\Big]^2 dS - (n-2)\int_{S_r} u\frac{\partial u}{\partial r}\, dS - \Big(\frac{n-2}{2}\Big)^2 \int_{S_r}\frac{u^2}{r}\, dS$$

for $r > 0$. Using (1.14) and the fact that w is bounded from above we obtain

(4.10)
$$\begin{aligned} -\int_{S_r} u\frac{\partial u}{\partial r}\, dS &= \int_{S_r} u\Big[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\Big] dS + \frac{n-2}{2}\int_{S_r}\frac{u^2}{r}\, dS \\ &\le \int_{S_r} r\Big[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\Big]^2 dS + \frac{n}{2}\int_{S_r}\frac{u^2}{r}\, dS \\ &= \int_{S^{n-1}}\Big(\frac{\partial v}{\partial s}\Big)^2(s,\theta)\, d\theta + \frac{n}{2}\int_{S^{n-1}} v^2(s,\theta)\, d\theta \le C_6 \end{aligned}$$

for large r and for a positive constant C_6, where $r = e^s$. It follows from (4.8), (4.9) and (4.10) that

$$\int_{S_r} |\bigtriangledown u|^2\, dS \le \frac{C_7}{r} + \frac{n-2}{n}\int_{S_r} K u^{\frac{2n}{n-2}}\, dS$$

for large r, where C_7 is a positive constant. Therefore we obtain

(4.11)
$$\int_{B_o(r)} |\bigtriangledown u|^2\, dx \le C_8 \ln r + \frac{n-2}{n}\int_{B_o(r)} K u^{\frac{2n}{n-2}}\, dx$$

for large r and for a positive constant $C_8 \ge C_7$. On the other hand we have

$$\begin{aligned} \int_{B_o(r)} K u^{\frac{2n}{n-2}}\, dx &= \int_{B_o(r)} u(-\Delta u)\, dx = \int_{B_o(r)} |\bigtriangledown u|^2\, dx - \int_{S_r} u\frac{\partial u}{\partial r}\, dS \\ &\le C_8 \ln r + \frac{n-2}{n}\int_{B_o(r)} K u^{\frac{2n}{n-2}}\, dx + C_6 \end{aligned}$$

for large r, where we use (4.10). Hence there exists a positive constant C_9 such that

$$\int_{B_o(r)} K u^{\frac{2n}{n-2}}\, dx \le C_9 \ln r$$

for large r. If $u \in L^{2n/(n-2)}(\mathbb{R}^n)$, then clearly we have the first inequality in (1.15). Assume that $u \notin L^{2n/(n-2)}(\mathbb{R}^n)$. Using (1.2) we have

$$\int_{B_o(r)} u^{\frac{2n}{n-2}}\, dx \le \frac{2}{a^2}\int_{B_o(r)} K u^{\frac{2n}{n-2}}\, dx \le \frac{2C_9}{a^2}\ln r$$

for large r. Hence we have the first inequality in (1.15). The second inequality follows from (4.11). ∎

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Proof of Theorem B From the proof of theorem A we have

$$\int_{S_r} \frac{u^2(x)}{r}\, dS = \int_{S^{n-1}} v^2(s,\theta)\, d\theta = 2w(s) \le C_{10} \tag{4.12}$$

for large r, where $r = |x| = e^s$ and C_{10} is a positive constant. By using (1.16) and (4.12) we also have

$$\int_{S_r} r\Big[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\Big]^2 dS = \int_{S^{n-1}} r^n\Big[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\Big]^2 d\theta = \int_{S^{n-1}} \Big(\frac{\partial v}{\partial s}\Big)^2 d\theta \le C_{11} \tag{4.13}$$

for large r, where C_{11} is a positive constant. It follows from Pohozaev identity (3.1) that

$$\begin{aligned}(4.14)\quad \int_{S_r} r|\bigtriangledown u|^2\, dS &\le C_{12} + \frac{n-2}{n}\int_{S_r} rKu^{\frac{2n}{n-2}}\, dS + 2\int_{S_r} r\Big[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\Big]^2 dS \\ &\quad - (n-2)\int_{S_r} u\Big[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\Big]\, dS \\ &\le C_{13} + \frac{n-2}{n}\int_{S_r} rKu^{\frac{2n}{n-2}}\, dS + C_{14}\int_{S_r} r\Big[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\Big]^2 dS \\ &\quad + C_{15}\int_{S_r}\frac{u^2}{r}\, dS \le C_{16} + \frac{n-2}{n}\int_{S_r} rKu^{\frac{2n}{n-2}}\, dS\end{aligned}$$

for large r, where we use (4.12) and (4.13). Here $C_{12}, C_{13}, C_{14}, C_{15}$ and C_{16} are positive constants. (4.14) implies that there exists a positive constant C_{17} such that

$$\int_{B_o(R)} r|\bigtriangledown u|^2\, dx \le C_{17}R + \frac{n-2}{n}\int_{B_o(R)} rKu^{\frac{2n}{n-2}}\, dx \tag{4.15}$$

for large R. We have

$$\begin{aligned}(4.16)\quad \int_{B_o(R)} rKu^{\frac{2n}{n-2}}\, dx &= \int_{B_o(R)} (ru)(-\Delta u)\, dx \\ &= \int_{B_o(R)} r|\bigtriangledown u|^2\, dx + \int_{B_o(R)} u\frac{\partial u}{\partial r}\, dx - R\int_{S_R} u\frac{\partial u}{\partial r}\, dS\end{aligned}$$

for $R > 0$. Using (4.13) we obtain

$$\begin{aligned}(4.17)\quad \int_{B_o(R)} u\frac{\partial u}{\partial r}\, dx &= \int_{B_o(R)} u\Big[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\Big]\, dx - \frac{n-2}{2}\int_{B_o(R)}\frac{u^2}{r}\, dx \\ &\le C_{18}\int_{B_o(R)} r\Big[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\Big]^2 dx \le C_{19}R\end{aligned}$$

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for large R, where C_{18} and C_{19} are positive constants. As in (4.10) we have

$$\left| R \int_{S_R} u \frac{\partial u}{\partial r}\, dS \right| \le C_{20} R \tag{4.18}$$

for large R, where C_{20} is a positive constant. From (4.15), (4.16), (4.17) and (4.18) we obtain

$$\frac{2}{n} \int_{B_o(R)} rKu^{\frac{2n}{n-2}}\, dx \le C_{21} R \tag{4.19}$$

for large R, where C_{21} is a positive constant. Using (1.16) we have

$$\begin{aligned}
&\frac{d}{dr}\Big(\int_{S_r} r^2 u^{\frac{2n}{n-2}}\, dS \Big) \\
&\quad = \frac{d}{dr}\Big(\int_{S^{n-1}} r^{n+1} u^{\frac{2n}{n-2}}\, d\theta \Big) \\
&\quad = (n+1) \int_{S^{n-1}} r^n u^{\frac{2n}{n-2}}\, d\theta + \frac{2n}{n-2} \int_{S^{n-1}} r^{n+1} u^{\frac{n+2}{n-2}} \frac{\partial u}{\partial r}\, d\theta \\
&\quad = \int_{S^{n-1}} r^n u^{\frac{2n}{n-2}}\, d\theta + \frac{2n}{n-2} \int_{S^{n-1}} r^{n+1} u^{\frac{n+2}{n-2}} \Big[\frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r} \Big]\, d\theta \\
&\quad = \int_{S^{n-1}} r^n u^{\frac{2n}{n-2}}\, d\theta + \frac{2n}{n-2} \int_{S^{n-1}} (r^{\frac{n+2}{2}} u^{\frac{n+2}{n-2}}) \Big\{ r^{\frac{n}{2}} \Big[\frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r} \Big] \Big\}\, d\theta \\
&\quad \le C_{22} \int_{S^{n-1}} r^n u^{\frac{2n}{n-2}}\, d\theta + C_{23} \int_{S^{n-1}} \Big\{ r^{\frac{n}{2}} \Big[\frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r} \Big] \Big\}^{\frac{2n}{n-2}}\, d\theta \\
&\quad = C_{22} \int_{S^{n-1}} r^n u^{\frac{2n}{n-2}}\, d\theta + C_{23} \int_{S^{n-1}} \Big| \frac{\partial v}{\partial s} \Big|^{\frac{2n}{n-2}}\, d\theta \\
&\quad \le C_{24} \int_{S^{n-1}} r^n u^{\frac{2n}{n-2}}\, d\theta + C_{25}
\end{aligned}$$

for large r, and for some positive constants C_{22}, C_{23}, C_{24} and C_{25}, where $r = e^s$. Hence

$$\begin{aligned}
\int_{S_R} R^2 u^{\frac{2n}{n-2}}\, dS &= \int_0^R \Big(\int_{S_t} t^2 u^{\frac{2n}{n-2}}\, dS \Big)'\, dt \le C_{26} + \int_{r_o}^R \Big(\int_{S_t} t^2 u^{\frac{2n}{n-2}}\, dS \Big)'\, dt \\
&\le C_{26} + C_{24} \int_{r_o}^R \int_{S_t} t u^{\frac{2n}{n-2}}\, dS\, dt + C_{25}(R - r_o) \\
&\le C_{27} R + C_{28} \int_{B_o(R)} r u^{\frac{2n}{n-2}}\, dx
\end{aligned} \tag{4.20}$$

for R and r_o large, with $R > r_o$. Here C_{26}, C_{27} and C_{28} are positive constants. Consider the case when $u \notin L^{2n/(n-2)}(\mathbb{R}^n)$. It follows from (1.2) and (4.19) that

$$\int_{B_o(R)} r u^{\frac{2n}{n-2}}\, dx \le C_{29} \int_{B_o(R)} rKu^{\frac{2n}{n-2}}\, dx \le C_{30} R \tag{4.21}$$

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for large R and for some positive constants C_{29} and C_{30}. Clearly we have

$$\int_{B_o(R)} r u^{\frac{2n}{n-2}}\, dx \le C_{31} R \tag{4.22}$$

for large R and for some positive constants C_{31} if $u \in L^{2n/(n-2)}(\mathbb{R}^n)$. From (4.20), (4.21) and (4.22) we have (1.17). By the results in [11] (see also [6]), we obtain slow decay (1.7) as well. ■

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