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# A CHARACTERISTIC SUBGROUP AND KERNELS OF BRAUER CHARACTERS

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If $G$ is finite group and $P$ is a Sylow $p$-subgroup of $G$, we prove that there is a unique largest normal subgroup $L$ of $G$ such that $L \cap P = L \cap \mathbf{N}_G(P)$. If $G$ is $p$-solvable, then $L$ is the intersection of the kernels of the irreducible Brauer characters of $G$ of degree not divisible by $p$.

## 1. INTRODUCTION

Our aim in this note is to prove the following two results.

**THEOREM A.** *Let $G$ be an arbitrary finite group and let $P$ be a Sylow $p$-subgroup of $G$ for some prime $p$. Then there exists a unique largest normal subgroup $L$ of $G$ such that*

$$L \cap P = L \cap \mathbf{N}_G(P).$$

Note that the intersection property in Theorem A is equivalent to saying that $\mathbf{N}_L(P)$ is a $p$-group. Also, since this property is clearly independent of the choice of $P$ in $\mathrm{Syl}_p(G)$, it is clear that $L$ is characteristic in $G$. Our interest in this characteristic subgroup was motivated by the following.

**THEOREM B.** *Suppose that $G$ is $p$-solvable and let $L$ be the largest normal subgroup of $G$ such that $L \cap P = L \cap \mathbf{N}_G(P)$, where $P \in \mathrm{Syl}_p(G)$. Then $L$ is the intersection of the kernels of the irreducible Brauer characters of $G$ with degree not divisible by $p$.*

The assumption that $G$ is $p$-solvable in Theorem B is essential. Consider, for example, the simple group $G = M_{23}$ and take $p = 2$. Then $G$ has a self-normalising Sylow 2-subgroup, and thus the characteristic subgroup $L$ of Theorem A is the whole group $G$. But $G$ has an irreducible Brauer character of degree 11, and hence the conclusion of Theorem B fails in this case.

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## 2. Proofs

Theorem A is an immediate consequence of, [4, Lemma 5.3], and so we take this opportunity to offer a new and simpler proof of a somewhat more general result. The original lemma is the case of the following where both $H$ and $K$ are normal in $G$.

**Lemma 2.1.** *Let $G$ be a finite group and let $P \in \mathrm{Syl}_p(G)$, where $p$ is a prime. Let $H$ and $K$ be subgroups of $G$ such that $HK$, $HP$ and $KP$ are subgroups. Then*

$$\mathbf{N}_{HK}(P) = \mathbf{N}_H(P)\mathbf{N}_K(P).$$

Proof: We argue by induction on $|G:H|\,|G:K|$. Note that $|H : P\cap H| = |HP : P|$ is coprime to $p$, and so $P\cap H$ is a Sylow $p$-subgroup of $H$ and similarly, $P\cap K$ is a Sylow $p$-subgroup of $K$. It follows that

$$\big|(P\cap H)(P\cap K)\big| = \frac{|P\cap H|\,|P\cap K|}{|P\cap H\cap K|} \geqslant \frac{|H|_p|K|_p}{|H\cap K|_p} = |HK|_p \geqslant |P\cap HK|,$$

and thus $(P\cap H)(P\cap K) = P\cap HK$.

Suppose first that $P$ is not contained in $H$. We can then apply the inductive hypothesis with $PH$ in place of $H$, and we deduce that

$$\mathbf{N}_{PHK}(P) = \mathbf{N}_{PH}(P)\mathbf{N}_K(P).$$

By Dedekind's lemma, $\mathbf{N}_{PH}(P) = \mathbf{N}_H(P)P$, and thus

$$\mathbf{N}_{PHK}(P) = \mathbf{N}_H(P)P\mathbf{N}_K(P).$$

Now let $g \in \mathbf{N}_{HK}(P)$. We can then write $g = xuy$, where $x \in \mathbf{N}_H(P)$, $u \in P$ and $y \in \mathbf{N}_K(P)$. Since $g$, $x$ and $y$ are all in $HK$, we see that also $u \in HK$, and therefore $u \in P\cap HK$. By the first paragraph, we can write $u = rs$, where $r \in P\cap H$ and $s \in P\cap K$. Then

$$g = (xr)(sy) \in \mathbf{N}_H(P)\mathbf{N}_K(P),$$

and we are done in this case. Similarly the lemma is proved if $P$ is not contained in $K$.

We can now assume $P$ is contained in $H\cap K$, and we denote this intersection by $D$. Suppose that $g \in \mathbf{N}_{HK}(P)$ and write $g = hk^{-1}$, with $h \in H$ and $k \in K$. Since $P^g = P$, we have $P^k = P^h$ and this subgroup is contained in both $H$ and $K$. By Sylow's theorem in the group $D = H\cap K$, we have $P^h = P^d$ for some element $d \in D$, and thus $hd^{-1} \in \mathbf{N}_H(P)$. Also $P^k = P^d$, so $dk^{-1} \in \mathbf{N}_K(P)$. We see now that

$$g = (hd^{-1})(dk^{-1}) \in \mathbf{N}_H(P)\mathbf{N}_K(P),$$

and the proof is complete. ∎

Now we are ready to prove Theorem A.

PROOF OF THEOREM A: Let $P \in \mathrm{Syl}_p(G)$, and write $N = \mathbf{N}_G(P)$. Suppose that $H$ and $K$ are normal subgroups of $G$, each maximal with the property that its intersection with $N$ is equal to its intersection with $P$. We must show that $H = K$. By Lemma 2.1. we have

$$N \cap HK = \mathbf{N}_{HK}(P) = \mathbf{N}_H(P)\mathbf{N}_K(P).$$

Then $N \cap HK$ is a product of two $p$-subgroups, and so it is a $p$-subgroup of $N$. Since $P$ is the unique Sylow $p$-subgroup of $N$, it follows that $N \cap HK = P \cap HK$. Now by the maximality of $H$ and $K$, we conclude that $H = HK = K$, and the proof is complete. $\square$

To prove Theorem B, we choose to work with the $p'$-special characters of the $p$-solvable group $G$. (Their properties can be found in [**1**]. In particular, these members of $\mathrm{Irr}(G)$ form a set of lifts for the irreducible Brauer characters of $G$ having $p'$-degree.)

**THEOREM 2.2.** *Let $G$ be a $p$-solvable group and let $K$ be the intersection of the kernels of the $p'$-special characters of $G$. Then $K$ is the largest normal subgroup of $G$ such that $K \cap P = K \cap \mathbf{N}_G(P)$, where $P \in \mathrm{Syl}_p(G)$.*

PROOF: Write $N = \mathbf{N}_G(P)$. First, we prove by induction on $|G|$ that $K \cap P = K \cap N$. We may assume that $K > 1$, and we choose a minimal normal subgroup $M$ of $G$ with $M \subseteq K$. Now, $K/M$ is the intersection of the kernels of the $p'$-special characters of $G/M$ and $PM/M$ is a Sylow $p$-subgroup of $G/M$ with normaliser $NM/M$. By the inductive hypothesis, we deduce that

$$(K/M) \cap (NM/M) = (K/M) \cap (PM/M),$$

or equivalently, $K \cap NM = K \cap PM$. If $M$ is a $p$-group, then $PM = P$ and $NM = N$, and we are done in this case. We may therefore assume that $M$ is a $p'$-group. Since $M \subseteq K$, Dedekind's lemma yields that

$$(K \cap P)M = K \cap PM = K \cap NM = (K \cap N)M.$$

and therefore, if we can show that $(K \cap P) \cap M = (K \cap N) \cap M$, it will follow that $|K \cap P| = |K \cap N|$, and thus $K \cap P = K \cap N$, as required. In particular, since $M \subseteq K$, it suffices to show that $N \cap M = 1$. As $M$ is a normal $p'$-subgroup of $G$, it follows that $N \cap M = \mathbf{C}_N(P)$, and if this is nontrivial, then by the Glauberman character correspondence, (see [**3**, Chapter 13]), there exists a nonprincipal $P$-invariant character $\theta \in \mathrm{Irr}(M)$. Then there exists a $p'$-special character $\chi \in \mathrm{Irr}(G)$ lying over $\theta$ by [**1**, Corollary (4.8)]. However, $M \subseteq K \subseteq \ker(\chi)$ and this is a contradiction.

Finally, we need to show that if $K < L \lhd G$, then $L \cap P < L \cap N$, and for this purpose, we can assume that $L/K$ is a chief factor of $G$. Assuming that $L \cap N = L \cap P$, we work to derive a contradiction. Since $K < L$, there exists a $p'$-special character $\chi \in \mathrm{Irr}(G)$ such that $L$ is not contained in $\ker(\chi)$. But $\chi$ has $p'$-degree, and this implies that $\chi_L$ has a nonprincipal $P$-invariant irreducible constituent $\theta$, and $\theta$ is necessarily $p'$-special since it lies under $\chi$. Also, $K \subseteq \ker(\theta)$, and thus $L/K$ cannot be a $p$-group because it has a nonprincipal $p'$-special character. We deduce that $L/K$ is a $p'$-group, and thus $L \cap N = L \cap P \subseteq K$ and we have $L \cap NK = (L \cap N)K = K$. Observe, however, that $NK/K$ is the full normaliser of $PK/K$ in $G/K$, and so it follows that $\mathbf{C}_{L/K}(P)$ is trivial. By the Glauberman correspondence, however, $\mathbf{C}_{L/K}(P)$ must be nontrivial since $L/K$ has a nonprincipal $P$-invariant irreducible character. This is a contradiction and the theorem is proved. ∎

Finally, we complete the proof of Theorem B.

Proof of Theorem B: By [**2**, Lemma (5.4) and Corollary (10.3)], we know that restriction to $p$-regular elements defines a bijection from the set of $p'$-special characters of $G$ onto the irreducible Brauer characters of $G$ having $p'$-degree. It follows that the intersection $K$ of the kernels of all $p'$- special characters of $G$ is contained in the intersection $L$ of the kernels of all irreducible Brauer characters having $p'$-degree. By Theorem 2.2, therefore, it suffices to show that $L = K$.

Every $p$-regular element of $L$ must lie in $K$, and thus $L/K$ is a $p$-group. By Theorem 2.2, we know that $K \cap N = K \cap P$, where $P \in \mathrm{Syl}_p(G)$ and $N = \mathrm{N}_G(P)$. As $N \cap K$ is a $p$-group and $L/K$ is a $p$-group, it is easy to see that $N \cap L$ is also a $p$-group, and thus $N \cap L = P \cap L$. By the maximality of $K$ in Theorem 2.2, we conclude that $L = K$, as desired. ∎

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