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ON THE STRUCTURE OF POLYNOMIALLY NORMAL OPERATORS

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We present some results concerning the structure of polynomially normal operators. It is shown, among other things, that if T^n is normal for some $n > 1$, then T is quasi-similar to a direct sum of a normal operator and a compact operator and if $p(T)$ is normal with T essentially normal, then T can be written as the sum of a normal operator and a compact operator. Utilizing the direct integral theory of operators we finally show that if $p(T)$ is normal and T^*T commutes with $T + T^*$, then T must be normal.

0. Introduction

Let H be a separable, infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators on H. An operator $T \in B(H)$ is called normal if $T T^* = T^*T$ where T^* denotes the adjoint of T. It is clear that if T is normal then any polynomial of T is also normal. However, the converse is not true. To see this take any non-normal T such that $T^2 = 0$. Roots of normal operators have been extensively studied and many beautiful results have been obtained (see [5] and its references).

This paper has two purposes, the first is to give some structure

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theorems for power normal operators, for example, we will show that if T^n is normal for some $n > 1$, then T is quasi-similar to a direct sum of a normal operator and a compact operator. Using a result of F. Gilfeather and some of the B-D-F results we will show that if $P(T)$ is normal for some non-zero polynomial P and T is essentially normal, then T can be written as a sum of a normal operator and a compact operator. The second purpose is to give some sufficient conditions to insure the normality of T whenever $P(T)$ is normal. As an example we will utilize the direct integral theory of operators to show that if $P(T)$ is normal and T^*T commutes with $T + T^*$, then T is normal.

1. Structure Theorems

In this section, we shall present several structure theorems and representations for T whenever some polynomial of T is normal. These results depend heavily upon the beautiful representation theorem of H. Radjavi and P. Rosenthal [8].

Our first result can be stated as follows:

THEOREM 1. *Let $T \in B(H)$ be such that T^2 and $P(T)$ are normal.*

(a) If $P(Z) = a_0 + a_1Z + a_2Z^2 + \ldots + a_nZ^n$ where $n > 2$ and at least two odd powers appear, then $T = V \oplus S$ with V normal and S algebraic.

(b) If $p(Z) = a_0 + a_1Z + a_2Z^2 + \ldots + a_nZ^n$ where $n > 2$ and one and only one odd power appears, then $T = V \oplus S$ with V normal and S nilpotent of index 2.

Proof. Since T^2 is normal, by [8] T can be written as

$$T = \begin{bmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & 0 & -B \end{bmatrix}$$

where A, B are normal, $C \geq 0$, C is one to one and $BC = CB$. Furthermore, B can be chosen so that $\sigma(B)$ lies in the closed upper half-plane.

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Let $p(Z) = a_0 + a_1 Z + a_2 Z^2 + \dots + a_n Z^n$; then

$$p(T) = \begin{bmatrix} p(A) & 0 & 0 \\ 0 & p(B) & X \\ 0 & 0 & p(-B) \end{bmatrix}$$

for some X. If k is the only odd integer such that $a_k \neq 0$, then it is not hard to see that $X = a_k B^{k-1} C$. $P(T)$ being normal implies

$$p(B^*)p(B) = p(B)p(B^*) + |a_k|^2 C^2 B^{k-1} B^{*k-1} = 0.$$

Since C is one to one, $B^{k-1} = 0$. Thus $B = 0$ (the only nilpotent normal operator is the zero operator).

Let $V = A$ and $S = \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}$. Hence $T = V \oplus S$ with V normal and $S^2 = 0$, which proves (b). The proof of (a) can be completed similarly.

COROLLARY 1. *If $T \in B(H)$, T^2 and $a_0 + a_1 T + a_2 T^2 + a_1 T^3 + a_4 T^4$ are normal, then T is similar to a normal operator.*

Proof. Using the same notation as in the proof of Theorem 1, we have

$$p(T) = \begin{bmatrix} p(A) & 0 & 0 \\ 0 & p(B) & X \\ 0 & 0 & p(-B) \end{bmatrix},$$

where $p(Z) = a_0 + a_1 Z + a_2 Z^2 + a_1 Z^3 + a_4 Z^4$, and $X = a_1 C + a_1 BC^2$.

Since $p(T)$ is normal, it follows that $X = 0$. C being one to one implies that $1 + B^2 = 0$. But $\sigma(B)$ being contained in the closed upper half-plane implies that $B + i$ is invertible, and so $B = i$. Hence

$$T = \begin{bmatrix} A & 0 & 0 \\ 0 & i & C \\ 0 & 0 & -i \end{bmatrix}$$

which is similar to the normal operator

$$\begin{bmatrix} A & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}.$$

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The similarity is implemented by the invertible operator

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & C/2i \\ 0 & 0 & 1 \end{bmatrix}.$$

COROLLARY 2. *Let T be as in Theorem 1(b) and T be essentially normal (that is, $T^*T - TT^* \in K(H)$ the ideal of compact operators), then T is the direct sum of a normal operator and a compact operator.*

Proof. Since $T = V \oplus S$, with V normal and $S^2 = 0$, $T^*T - TT^* \in K(H)$ implies that $S^*S - SS^* \in K(H)$. Therefore $\pi(S)$ is a normal nilpotent element of the Calkin algebra $B(H)/K(H)$, with π being the canonical map of $B(H)$ onto $B(H)/K(H)$. Hence $\pi(S) = 0$, and so S is compact.

An operator $T \in B(H)$ is said to be quasidiagonal, or quasitriangular if there exists a sequence $\{P_n\}_{n=0}^{\infty}$ of finite-rank projections converging strongly to 1 such that $\|P_nT - TP_n\| \to 0$ or $\|P_nTP_n - TP_n\| \to 0$, respectively (see [6] and [7]). It is known that a normal operator is both a quasidiagonal and a quasitriangular operator [6]. Now we give the following generalization of this result.

COROLLARY 3. *Let T be as in Theorem 1(b), then T is quasidiagonal.*

Proof. We have $T = V \oplus S$ with V normal and $S^2 = 0$. It is known that every normal operator is quasidiagonal, and by [10] every nilpotent of index 2 is quasidiagonal. Since the direct sum of two quasidiagonal operators is quasidiagonal [6], the result follows.

From the quasitriangularity case we require the following remarkable result due to F. Gilfeather.

THEOREM 2. *Let $T \in B(H)$ be such that $p(T)$ is normal for some polynomial p. Then there exist reducing subspaces $\{H_n\}_{n=0}^{\infty}$ for T such that $H = \bigoplus_{n=0}^{\infty} H_n$, $T_0 = T|H_0$ is algebraic, and $T_n = T|H_n$ is similar to a normal operator.*

Proof. See [5].

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COROLLARY 4. *If* $T \in B(H)$ *is polynomially normal, then* T *is quasitriangular.*

Proof. In the theory of quasitriangular operators it is known that (see [7]) an operator with countable spectrum is quasitriangular, an operator similar to a quasitriangular one is quasitriangular and any countable direct sum of quasitriangular operators is quasitriangular. In view of these properties, the result now follows by Theorem 2.

Remark. Applying the corollary to T^* as well, we conclude that T is biquasitriangular, that is, T and T^* are quasitriangular.

COROLLARY 5. *If* T *is a polynomially normal operator, then* T *is a norm-limit of algebraic operators.*

Proof. The result follows from a remarkable characterization given by Voiculescu [12] which asserts that the set of biquasitriangular operators coincides with the norm-closure of the set of algebraic operators.

COROLLARY 6. *If* T *is polynomially normal and essentially normal, then* T *can be written as the sum of a normal operator and a compact one, hence* T *is quasidiagonal.*

Proof. The corollary follows from a result of [1] which states that if T is essentially normal such that both T and T^* are quasitriangular, then T is normal plus compact.

Corollary 6 admits the following generalization.

THEOREM 3. *If* T *is polynomially normal and essentially hyponormal, then* T *can be written as the sum of a normal operator and a compact one.*

Proof. Suppose that $p(T)$ is normal, for some polynomial p. Since $\pi(p(T)) = p(\pi(T))$, it follows that $p(\pi(T))$ is normal in $B(H)/K(H)$. Since $B(H)/K(H)$ is a C*-algebra, there exist a Hilbert space H_0 and an isometric *-isomorphism ν of $B(H)/K(H)$ into $B(H_0)$, (see [4]). Since $p(\nu \circ \pi(T)) = \nu(p(\pi(T)))$, and $p(\pi(T))$ is normal, then $\nu \circ \pi(T)$ is polynomially normal and hyponormal operator in $B(H_0)$. Thus $\nu \circ \pi(T)$ is normal [11], and so $\pi(T)$ is normal. The result now follows by Corollary 6.

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Two operators S and T in $B(H)$ are said to be *quasi-similar* if there exist two operators X and Y in $B(H)$, which are one to one and of dense range, such that $SX = XT$ and $YS = TY$. The importance of quasi-similarity for the invariant subspace problem lies in the fact that for two quasi-similar operators S and T, if one of them has a proper hyperinvariant subspace, so does the other one. The following well-known lemma [7] enables us to prove that a power normal operator is quasi-similar to a direct sum of a normal operator and a compact operator.

LEMMA. *Suppose* $\{H_n\}_{n=0}^{\infty}$ *is a sequence of Hilbert spaces and for each* n, $S_n A_n S_n^{-1} = B_n$, *where* $A_n, B_n \in B(H_n)$, *and* S_n *is an invertible operator in* $B(H_n)$. *Then the operators* $A = \bigoplus_{n=0}^{\infty} A_n$ *and* $B = \bigoplus_{n=0}^{\infty} B_n$ *acting on the Hilbert space* $H = \bigoplus_{n=0}^{\infty} H_n$ *are quasi-similar.*

THEOREM 4. *Let* $T \in B(H)$ *be such that* T^n *is normal for some* $n > 1$. *Then* T *is quasi-similar to a direct sum of a normal operator and a compact one.*

Proof. By Theorem 2, $T = \bigoplus_{n=0}^{\infty} T_n$, where T_0 is nilpotent and T_n is similar to a normal operator N_n. It is known that every nilpotent operator is quasi-similar to a compact operator (see [7]). Thus T_0 is quasi-similar to some compact operator K in $B(H_0)$. By the Lemma $\bigoplus_{n=1}^{\infty} T_n$ is quasi-similar to the normal operator $N = \bigoplus_{n=1}^{\infty} N_n$ on the Hilbert space $\bigoplus_{n=1}^{\infty} H_n$. Therefore T is quasi-similar to $N \oplus K$ as required.

2. Conditions implying the normality of a polynomially normal operator

It has been shown by Stampfli [11] that a power normal operator which is hyponormal must be normal. The direct integral representation [9] enables us to give a different proof for this fact and to establish an analogous result for the class of operators considered by Campbell in [2]; where a special case when $p(Z) = Z^2$ was proved.

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THEOREM 5. *If T is polynomially normal and T^*T commutes with $T + T^*$, then T is normal.*

Proof. Assume that $p(T)$ is normal for some polynomial p. Let A be the abelian von Neumann algebra generated by $p(T)$. Then the underlying Hilbert space H can be written as a direct integral $H = \int^{\oplus} H(x)d\mu(x)$ such that each operator in A is diagonal and each operator in A', the commutant of A, is decomposable relative to this representation. Since $T \in A'$, we have $T = \int^{\oplus} T(x)d\mu(x)$ for almost every x. Since $p(T)$ can be expressed as $P(T) = \int^{\oplus} f(x)d\mu(x)$ for some $f \in L^{\infty}(\mu)$, it follows that $p(T(X)) = f(x)$ for almost all x and so $T(x)$ is algebraic. From [2] and [3] one can conclude that if T^*T commutes with $T + T^*$ and T has a countable spectrum, then T is normal. Since for almost all x, $T(x)$ has a finite spectrum and $T^*(x)T(x)$ commutes with $T(x) + T^*(x)$, we conclude that almost every $T(x)$ is normal. Hence T is normal and the proof is complete.

THEOREM 6. *If T is a polynomially normal operator which is hyponormal, then T is normal.*

Proof. Since it is known [11] that a hyponormal operator whose spectrum is countable must be normal, the proof can be completed as that of Theorem 5.

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