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On Fredholm Operators Between Non-archimedean Fréchet Spaces

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Abstract. It is proved that the index of a Fredholm operator between non-Archimedean Fréchet spaces is preserved under compact perturbations. A similar result is shown for Fredholm operators between non-Archimedean polar regular LF-spaces.

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Introduction

In this paper all linear spaces are over a non-archimedean nontrivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot|: \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (l.c.s) and normed spaces we refer to [6, 9] and [8].

The problem of perturbations of continuous linear operators between Banach spaces has been studied in [5, 11, 12] and [1]. In [1], J. Araujo, C. Perez-Garcia and S. Vega proved that the index of a Fredholm operator between Banach spaces is preserved under compact perturbations. In this paper we extend this result to Fredholm operators between Fréchet spaces. We show the following (Theorem 4). Let X and Y be Fréchet spaces. If T is a Fredholm operator from X to Y and K is a compact operator from X to Y, then $T + K$ is a Fredholm operator, and the index of $T + K$ is equal to the one of T. We prove a similar result for Fredholm operators from a polar regular LF-space to an LF-space (Theorem 8).

Preliminaries

Let X and Y be linear spaces. The set of all linear operators from X to Y we denote by $\mathcal{L}(X, Y)$. We say that $T \in \mathcal{L}(X, Y)$ has an index if $\dim \ker T + \dim(Y/TX) < \infty$. In this case the index of T is defined as $\chi(T) = \dim \ker T - \dim(Y/TX)$. If $T \in \mathcal{L}(X, Y)$ has an index and $F \in \mathcal{L}(X, Y)$ is a finite-dimensional operator (that is $\dim FX < \infty$), then $T + F$ has an index and $\chi(T + F) = \chi(T)$ ([1], Theorem 3.5).

Let X, Y and Z be linear spaces. If two of the three operators $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(Y, Z)$ and $ST \in \mathcal{L}(X, Z)$ have indexes, then the third one also has an index and $\chi(ST) = \chi(T) + \chi(S)$ ([7], Proposition 7.1.6).

The identity operator on a linear space X is indicated by I_X.

By a *seminorm* on a linear space E we mean a function $p: E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}$, $x \in E$ and $p(x + y) \leqslant \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is *norm* if $\ker p := \{x \in E : p(x) = 0\} = \{0\}$.

The set of all continuous seminorms on a l.c.s E is denoted by $\mathcal{P}(E)$. A l.c.s E is of *countable type* if for every $p \in \mathcal{P}(E)$, the normed space $(E/\ker p, \bar{p})$, where $\bar{p}(x + \ker p) = p(x)$ for $x \in E$, contains a linearly dense countable subset.

The set of all continuous linear functionals on a l.c.s X is denoted by X'. If X is of countable type, then for any $x \in (X \backslash \{0\})$ there is $f \in X'$ with $f(x) \neq 0$ ([9]).

A subset B of a l.c.s E is *compactoid* if for each neighbourhood U of 0 in E there exists a finite subset $A = \{a_1, \ldots, a_n\}$ of E such that $B \subset U + \operatorname{co} A$, where $\operatorname{co} A = \{\sum_{i=1}^{n} \alpha_i a_i : \alpha_1, \ldots, \alpha_n \in \mathbb{K}, |\alpha_1|, \ldots, |\alpha_n| \leqslant 1\}$ is the *absolutely convex hull* of A. Any compactoid subset in a l.c.s E is bounded and its linear span is of countable type (see [10], Corollary 1.5).

Let X and Y be locally convex Hausdorff spaces. The set of all continuous linear operators from X to Y we indicate by $L(X, Y)$. A mapping $T \in L(X, Y)$ is an *isomorphism* if T is injective and surjective and $T^{-1} \in L(Y, X)$. A map $T \in L(X, Y)$ is a *Fredholm operator* if it has an index and TX is a closed subspace of Y. The family of all Fredholm operators from X to Y is denoted by $\Phi(X, Y)$. For any $T \in \Phi(X, Y)$, TX is a complemented subspace of Y. Put $\Phi_0(X, Y) = \{T \in L(X, Y) : T \text{ has an index}\}$. By the open mapping theorem ([6], Corollary 2.74), $\Phi_0(X, Y) = \Phi(X, Y)$, whenever X and Y are Fréchet spaces (i.e. complete metrizable locally convex spaces).

An operator $T \in L(X, Y)$ is *compact* if there exists a neighbourhood U of 0 in X such that $T(U)$ is compactoid in Y. The set of all compact operators from X to Y we denote by $C(X, Y)$.

An LF-space (E, τ), i.e. a l.c.s which is the inductive limit of an inductive sequence $((E_n, \tau_n))$ of Fréchet spaces, is *regular* (respectively, *strict*) if for every bounded subset B of E, there is $k \in \mathbb{N}$ such that $B \subset E_k$ and B is τ_k-bounded (respectively, if $\tau_{n+1} | E_n = \tau_n$ for any $n \in \mathbb{N}$).

Every strict LF-space is regular ([3], Theorem 1.4.13); in particular the direct sum $\bigoplus_{n=1}^{\infty} E_n$ of any sequence (E_n) of Fréchet spaces is a regular LF-space.

Any continuous linear map from an LF-space X onto an LF-space Y is open (see [4], Theorem 3.1).

Results

To prove our main result (Theorem 4) we shall need two lemmas.

Let D be a finite-dimensional subspace of a l.c.s X. If X' separates points of X (in particular, if the field $\mathbb{K}$ is spherically complete), then D is complemented in X; so for any $K \in L(X, X)$ there is a finite-dimensional operator $F \in L(X, X)$ such that

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$F|D = K|D$ (i.e. $F(x) = K(x)$ for any $x \in D$). For arbitrary l.c.s X we have the following lemma:

LEMMA 1. *Let X and Y be locally convex Hausdorff spaces. Let $K \in C(X, Y)$ and let D be a finite-dimensional subspace of X. Then there exists a finite-dimensional operator $F \in L(X, Y)$ such that $F|D = K|D$.*

Proof. Put $D_0 = D \cap \bigcap\{\ker f : f \in X'\}$. Let D_1 be an algebraic complement of D_0 in D. Clearly, for any $x \in (D_1 \setminus \{0\})$ there is $f \in X'$ with $f(x) = 1$.

We shall show that there exists a continuous linear projection from X onto D_1. Let $r = \dim D_1$. It is enough to consider the case when $r > 1$. Assume that $1 \leqslant k < r$ and $(x_1, f_1), \ldots, (x_k, f_k) \in D_1 \times X'$ with $f_i(x_j) = \delta_{i,j}$ for $1 \leqslant i, j \leqslant k$. Then there are $x_{k+1} \in (D_1 \cap \bigcap_{i=1}^k \ker f_i)$ and $f \in X'$ such that $f(x_{k+1}) = 1$. Let $f_{k+1} = f - \sum_{i=1}^k f(x_i) f_i$. Then $f_{k+1}(x_{k+1}) = 1$ and $f_{k+1}(x_i) = 0 = f_i(x_{k+1})$ for $1 \leqslant i \leqslant k$. It follows that there exist $(x_1, f_1), \ldots, (x_r, f_r) \in D_1 \times X'$ with $f_i(x_j) = \delta_{i,j}$ for all $1 \leqslant i, j \leqslant r$. Clearly, the operator $P: X \to X, x \to \sum_{i=1}^r f_i(x) x_i$ is a continuous linear projection from X onto D_1.

Put $F = KP$ and $S = (K - F)$. Suppose that $x \in (D_0 \setminus \ker S)$. Since $S(X)$ is of countable type, there exists $g \in (S(X))'$ with $g(S(x)) \neq 0$. But $g \circ S \in X'$ and $x \in D_0$, a contradiction. It follows that $D_0 \subset \ker S$. Clearly, $D_1 \subset \ker S$. Thus $D \subset \ker S$, hence $F(x) = K(x)$ for $x \in D$. □

To get Theorem 4 in a special case, when $Y = X$, $T = I_X$ and $K \in C(X, X)$, it is enough to show that there exists a finite-dimensional operator $F \in L(X, X)$ such that the operator $(I_X + K - F): X \to X$ is an isomorphism. In the proof of Theorem 3 we will need a more general fact.

LEMMA 2. *Let X and Y be Fréchet spaces. Assume that $K \in C(X, Y)$ and $S \in L(Y, X)$. Then there exists a finite-dimensional operator $F \in L(X, Y)$ such that the operator $(I_X + S(K - F)): X \to X$ is an isomorphism.*

Proof. Let U be a neighbourhood of zero in X such that $K(U)$ is compactoid in Y. For some $p \in \mathcal{P}(Y)$ we have $S(W_p) \subset U$, where $W_p = \{y \in Y : p(y) \leqslant 1\}$. Take $\alpha \in \mathbb{K}$ with $0 < |\alpha| < 1$. Since $K(U)$ is compactoid, $E = K(X)$ is of countable type and there exists a finite-dimensional subspace D_0 of E such that $K(U) \subset \alpha^2 W_p + D_0$. Let D be an algebraic complement of $\ker p \cap D_0$ in D_0. Then

$$K(U) \subset \alpha^2 W_p + \ker p \cap D_0 + D \subset \alpha^2 W_p + \alpha^2 W_p + D \subset \alpha^2 W_p + D.$$

It follows that

$$\forall x \in U \, \exists x' \in D : (Kx - x') \in \alpha^2 W_p \cap E.$$

Now, we prove that there exists a continuous linear projection P from E onto D such that $p(Px) \leqslant |\alpha|^{-1} p(x)$ for any $x \in E$. Put $E_p = (E / \ker p)$ and $\bar{p}(x + \ker p) = p(x)$ for $x \in E$. Denote by π the quotient map from E onto E_p. Let $\tilde{E}_p = (\tilde{E}_p, \tilde{p})$ be the completion of the normed space $(E_p, \bar{p})$ of countable type.

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Clearly, $\pi(D)$ is a closed subspace of the Banach space $\tilde{E}_p$ of countable type, so there exists a continuous linear projection Q from $\tilde{E}_p$ onto $\pi(D)$ such that $\tilde{p}(Qz) \leqslant |\alpha|^{-1}\tilde{p}(z)$ for any $z \in \tilde{E}_p$ (see [8], Theorem 3.16 and its proof). It is easy to see that $G = \pi^{-1}(\ker Q \cap E_p)$ is a closed subspace of E and $G + D = E$. Since $\ker Q \cap \pi(D) = \{0\}$ and $D \cap \ker p = \{0\}$, then $G \cap D = \{0\}$. The linear projection $P: G + D \to D, g + d \to d$ is continuous because G is closed and $\dim D < \infty$. Let $x \in E$. Since $Q(\pi(x - Px)) = 0$, then $p(Px) = \bar{p}(\pi(Px)) = \bar{p}(Q(\pi(Px))) = \bar{p}(Q(\pi(x))) \leqslant |\alpha|^{-1}\bar{p}(\pi(x)) = |\alpha|^{-1}p(x)$. Thus $p(Px) \leqslant |\alpha|^{-1}p(x)$ for any $x \in E$.

It follows that $P(W_p \cap E) \subset \alpha^{-1}W_p$, so $(I_E - P)(W_p \cap E) \subset \alpha^{-1}W_p$. Put $F = P \circ K$ and $L = S(K - F)$. For any $x \in U$ we have $L(x) = S(I_E - P)K(x) = S(I_E - P)(Kx - x')$; hence $L(U) \subset S(\alpha W_p) \subset \alpha U$. Thus $L^n(U) \subset \alpha^{n-1}L(U)$ for any $n \in \mathbb{N}$. Since $L(U)$ is compactoid and $|\alpha| < 1$, then $\lim_n L^n(x) = 0$ for any $x \in U$ (and $L^n(x) = \alpha^n x_n, n \in \mathbb{N}$, for some bounded sequence $(x_n) \subset X$). It follows that the series $\sum_{n=0}^{\infty}(-1_{\mathbb{K}})^n L^n(x)$ is convergent in X for any $x \in X$. Let

$$M: X \to X, x \to \sum_{n=0}^{\infty}(-1_{\mathbb{K}})^n L^n(x).$$

By the continuity of L we get $M(I_X + L) = I_X = (I_X + L)M$. Hence, by the open mapping theorem, the operator $(I_X + L): X \to X$ is an isomorphism. □

The proof of Lemma 2 shows that for any sequentially complete l.c.s X and any $K \in C(X, X)$ there exists a finite-dimensional operator $F \in L(X, X)$ such that the operator $(I_X + K - F): X \to X$ is an algebraic isomorphism. Hence we get

COROLLARY 3. *Let X be a sequentially complete l.c.s X. Then for any $K \in C(X, X)$ we have $I_X + K \in \Phi_0(X, X)$ and $\chi(I_X + K) = 0$; in particular, the operator $I_X + K$ is injective if and only if it is surjective.*

Now, we shall prove our main result.

THEOREM 4. *Let X and Y be Fréchet spaces. If $T \in \Phi(X, Y)$ and $K \in C(X, Y)$, then $T + K \in \Phi(X, Y)$ and $\chi(T + K) = \chi(T)$.*

Proof. Denote by $\hat{X}$ the quotient space $X/\ker T$ and by Q the quotient map from X onto $\hat{X}$. Let $\hat{T}: \hat{X} \to Y$ with $\hat{T}(Qx) = Tx, x \in X$. Clearly, $Q \in \Phi(X, \hat{X})$ and $\hat{T} \in \Phi(\hat{X}, Y)$. Since $\hat{T}\hat{X}$ is a closed subspace of Y with $\dim(Y/\hat{T}\hat{X}) < \infty$, then $\hat{T}\hat{X}$ is complemented in Y and by the open mapping theorem there exists $S \in L(Y, \hat{X})$ with $S\hat{T} = I_{\hat{X}}$. By Lemma 1 there is a finite-dimensional operator $F \in L(X, Y)$ such that $\ker T \subset \ker(K - F)$. Let $G: \hat{X} \to Y$ with $G(Qx) = (K - F)(x), x \in X$; clearly, $G \in C(\hat{X}, Y)$. By Lemma 2 there exists a finite-dimensional operator $H \in L(\hat{X}, Y)$ such that the operator $(I_{\hat{X}} + S(G - H)): \hat{X} \to \hat{X}$ is an isomorphism. Since $S\hat{T} = I_{\hat{X}}$, $I_{\hat{X}}+ S(G - H) = S(\hat{T} + G - H)$ and $\hat{T} \in \Phi(\hat{X}, Y)$, then $S \in \Phi(Y, \hat{X})$,

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$(\hat{T}+G-H) \in \Phi(\hat{X}, Y)$, $\chi(S)=-\chi(\hat{T})$ and $\chi(\hat{T}+G-H)=\chi(\hat{T})$. Hence $(\hat{T}+G) \in \Phi(\hat{X}, Y)$ and $\chi(\hat{T}+G)=\chi(\hat{T}+G-H)=\chi(\hat{T})$. It follows that $(T+K)-F=(\hat{T}+G)Q \in \Phi(X, Y)$ and $\chi(T+K-F)=\chi(\hat{T}+G)+\chi(Q)=\chi(\hat{T})+\chi(Q)=\chi(\hat{T}Q)=\chi(T)$. Thus $T+K \in \Phi(X, Y)$ and $\chi(T+K)=\chi(T+K-F)=\chi(T)$. □

If X is a regular LF-space, then for any sequence $(\alpha_n) \subset \mathbb{K}$ with $\lim \alpha_n = 0$ and every bounded sequence $(x_n) \subset X$, the series $\sum_{n=1}^{\infty} \alpha_n x_n$ is convergent in X (see [3], Propositions 2.3.2 and 2.3.3).

Using the proof of Lemma 2 we obtain

LEMMA 5. *Let X be a regular LF-space and let Y be a l.c.s. Assume that $K \in C(X, Y)$ and $S \in L(Y, X)$. Then there exists a finite-dimensional operator $F \in L(X, Y)$ such that the operator $(I_X + S(K-F)): X \to X$ is an isomorphism.*

(Note, that Lemma 5 is a generalization of Lemma 2, since we do not assume that LF-spaces are proper, so Fréchet spaces are regular LF-spaces.)

To prove our last theorem we will also need the following proposition:

PROPOSITION 6. (a) (*See* [2]). *If X and Y are LF-spaces, then $\Phi_0(X, Y) = \Phi(X, Y)$.*

(b) *Let D be a finite-dimensional subspace of an LF-space X. Then X/D is an LF-space, too. If X is polar and regular, then X/D is regular.*

(c) *Let M be a closed subspace of an LF-space X with* $\dim(X/M) < \infty$. *Then M is an LF-space.*

Proof. (a) Let $T \in L(X, Y)$ with $\dim(Y/TX) < \infty$ and let D be an algebraic complement of TX in Y. Clearly, $X \times D$ is an LF-space and the linear continuous map $S: X \times D \to Y, (x, d) \to x + d$ is surjective. By the open map theorem for LF-spaces, S is an isomorphism; so $TX = S(X \times \{0\})$ is a closed subspace of Y.

(b) Let $(X, \tau) = \lim \operatorname{ind}(X_n, \tau_n)$. Without loss of generality we can assume that $D \subset X_1$. Let $n \in \mathbb{N}$. Let π_n: $X_n \to X_n/D$ and π: $X \to X/D$ be quotient maps. Put φ_n: $X_n \to X, x \to x$, and ψ_n: $X_n/D \to X/D, x+D \to x+D$. Clearly, $\pi \circ \varphi_n = \psi_n \circ \pi_n$; hence for any $B \subset X/D$ we have $\varphi_n^{-1}(\pi^{-1}(B)) = \pi_n^{-1}(\psi_n^{-1}(B))$. Let γ be a locally convex linear topology on X/D such that $(X/D, \gamma) = \lim \operatorname{ind}(X_n/D, \tau_n/D)$. We shall show that $\gamma = \tau/D$.

Let $n \in \mathbb{N}$. Let $U \in \tau/D$. Since $\pi_n^{-1}(\psi_n^{-1}(U)) = \varphi_n^{-1}(\pi^{-1}(U))$, then $\psi_n^{-1}(U) = \pi_n(\varphi_n^{-1}(\pi^{-1}(U))) \in \tau_n/D$. Thus for any $n \in \mathbb{N}$ the map ψ_n: $(X_n/D, \ \tau_n/D) \to (X/D, \tau/D)$ is continuous. Hence $\tau/D \subset \gamma$.

Clearly, $\pi^{-1}(\gamma) = \{\pi^{-1}(U) : U \in \gamma\}$ is a locally convex linear topology on X. Let $n \in \mathbb{N}$. Let $V \in \pi^{-1}(\gamma)$. Then for some $U \in \gamma$ we have $V = \pi^{-1}(U)$; hence $\varphi_n^{-1}(V) = \varphi_n^{-1}(\pi^{-1}(U)) = \pi_n^{-1}(\psi_n^{-1}(U)) \in \tau_n$. Thus for any $n \in \mathbb{N}$ the map $\varphi_n: (X_n, \tau_n) \to (X, \pi^{-1}(\gamma))$ is continuous. Hence, $\pi^{-1}(\gamma) \subset \tau$, so $\gamma \subset \tau/D$.

Thus $\tau/D = \gamma$. Clearly, τ/D is a Hausdorff topology. It follows that $(X/D, \tau/D)$ is an LF-space.

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If X is polar, then D is complemented in X; thus for any bounded subset B in X/D there exists a bounded subset A in X such that $\pi(A) = B$. Therefore X/D is regular, if X is polar and regular.

(c) It follows from (b), since X is isomorphic to the product $M \times D$, where D is an algebraic complement of M in X. □

Using Lemma 5 and Proposition 6(a), we get the following corollary:

COROLLARY 7. *Let X be a regular LF-space. Then for any $K \in C(X, X)$ we have $I_X + K \in \Phi(X, X)$ and $\chi(I_X + K) = 0$; in particular, the operator $I_X + K$ is injective if and only if it is surjective.*

By Lemma 5, Proposition 6 and the proof of Theorem 3 we obtain the following theorem:

THEOREM 8. *Let X be a polar regular LF-space and let Y be an LF-space. If $T \in \Phi(X, Y)$ and $K \in C(X, Y)$, then $T + K \in \Phi(X, Y)$ and $\chi(T + K) = \chi(T)$.*

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