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ONE-DIMENSIONAL REPRESENTATIONS OF THE CYCLE SUBALGEBRA OF A SEMI-SIMPLE LIE ALGEBRA

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0. **Introduction.** Let L denote a semi-simple, finite dimensional Lie algebra over an algebraically closed field K of characteristic zero. If $\mathscr{H}$ denotes a Cartan subalgebra of L and $\mathscr{C}$ denotes the centralizer of $\mathscr{H}$ in the universal enveloping algebra U of L, then it has been shown that each algebra homomorphism $\gamma: \mathscr{C} \to K$ (called a "mass-function" on $\mathscr{C}$) uniquely determines a linear irreducible representation of L. The technique involved in this construction is analogous to the Harish–Chandra construction [2] of dominated irreducible representations of L starting from a linear functional $\lambda: \mathscr{H} \to K$. The difference between the two results lies in the fact that all linear functionals on $\mathscr{H}$ are readily obtained, whereas since $\mathscr{C}$ is in general a noncommutative algebra the construction of mass-functions is decidedly nontrivial. For the simple Lie algebras A_1 and A_2, Bouwer [1] has computed all mass functions. In this paper we investigate a means of constructing more general mass-functions for arbitrary semi-simple Lie algebras.

1. **Complete subsystems of the system of roots of L.** Let Δ denote the system of roots(1) of the semi-simple Lie algebra L relative to the Cartan subalgebra $\mathscr{H}$. A subset $\Phi=\{\alpha_1, \ldots, \alpha_n\}$ of Δ is said to be *fundamental* iff Φ is free and for each $\beta \in \Delta$, $\beta=\sum_{i=1}^n m_i\alpha_i$ where the coefficients m_i are integers which are either all ≥ 0 or all ≤ 0. As is well known the root system Δ of a semi-simple Lie algebra admits at least one fundamental subset and moreover the number of roots in any such fundamental subset of Δ is an invariant called the *rank* of L. Any fundamental subset Φ of Δ induces a partial order on Δ. In fact, if $\alpha, \beta \in \Delta$ we say that $\alpha > \beta$ relative to Φ iff $\alpha-\beta=\sum_{i=1}^n m_i\alpha_i$ where the m_i are all nonnegative integers, at least one being greater than zero.

Definition 1. A subset Γ of Δ is said to be *closed* in Δ iff

(i) $0 \in \Gamma$;
(ii) $\alpha \in \Gamma \Rightarrow -\alpha \in \Gamma$; and
(iii) $\alpha, \beta \in \Gamma$, $\alpha+\beta \in \Delta \Rightarrow \alpha+\beta \in \Gamma$.

Definition 2. A subset Γ of Δ is said to be *complete* in Δ iff Γ is closed in Δ and in addition there exists a fundamental subset Φ of Δ such that if $\alpha+\beta \in \Gamma$ with $\alpha, \beta \in \Delta$ and $\alpha, \beta > 0$ relative to Φ then $\alpha, \beta \in \Gamma$.

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(1) For basic facts concerning the system of roots of a semi-simple Lie algebra see [4].

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REMARK. The concepts of closed and complete subsystems in a system of roots Δ are adaptations of "sous-systèmes fermés" and "sous-systèmes saturés" utilized by J. de Siebenthal in [5].

We now list a few relevant properties of closed and complete subsystems of Δ.

LEMMA 1. *Every closed subsystem Γ of Δ is contained in a complete subsystem of minimal rank.*

Proof. This follows since Δ is complete in itself.

LEMMA 2. *A closed subsystem Γ of Δ is complete in Δ iff Γ admits a fundamental subset Φ_1 contained in a fundamental subset Φ of Δ.*

Proof. Assume first that $\Phi_1=\{\alpha_1, \ldots, \alpha_r\}$ is a fundamental subset of Γ contained in a fundamental subset $\Phi=\{\alpha_1, \ldots, \alpha_r, \ldots, \alpha_n\}$ of Δ. Then if $\alpha, \beta \in \Delta$ with $\alpha, \beta>0$ relative to Φ we have $\alpha=\sum_{i=1}^n m_i\alpha_i$ and $\beta=\sum_{i=1}^n k_i\alpha_i$ where $m_i, k_i \geq 0$. Since Φ_1 is a fundamental subset of Γ if $\alpha+\beta \in \Gamma$ we have $m_i=k_i=0$ for $i=r+1, \ldots, n$ and hence $\alpha, \beta \in \Gamma$.

Conversely, if Γ is complete in Δ relative to the fundamental subset Φ of Δ then $\Phi \cap \Gamma$ is a fundamental subset of Γ.

DEFINITION 3. A pair of complete subsystems Γ_1 and Γ_2 of Δ are said to be *disconnected* iff $\Gamma_1 \cup \Gamma_2$ is a complete subsystem of Δ and

$$(\Gamma_1+\Gamma_2) \cap \Delta = \{0\}.$$

REMARK. In terms of the Dynkin diagram of Δ relative to a fundamental subset Φ the disconnectedness of Γ_1 and Γ_2 can be translated into the property that there exists no direct line joining a simple root of Γ_1 and a simple root of Γ_2.

2. **Subalgebras of $\mathscr{C}$ associated with complete subsystems of Δ.** Let Γ be a complete subsystem of Δ and let Φ be a fundamental subset of Δ such that $\Phi \cap \Gamma$ is a fundamental subset of Γ. If Δ^+ denotes the Φ-positive roots of Δ then, as is well known, the underlying linear space of L admits a basis $B(\Delta, \Phi) =\{Y_\beta, X_\beta, H_\alpha \mid \beta \in \Delta^+, \alpha \in \Phi\}$ called the Cartan basis with the usual Lie product.

In terms of the basis $B(\Delta, \Phi)$ of L the Birkhoff–Witt theorem provides a basis of U consisting of all monomials of the form

$$\prod_{\beta\in\Delta^+} Y_\beta^{m(\beta)} \prod_{\beta\in\Delta^+} X_\beta^{n(\beta)} \prod_{\alpha\in\Phi} H_\alpha^{k(\alpha)} \tag{1}$$

where the exponents $m(\beta)$, $n(\beta)$ and $k(\alpha)$ are nonnegative integers and each product preserves a predetermined order on its index set. We observe that $\mathscr{C}$, the centralizer of the Cartan subalgebra $\mathscr{H}$ in U, is generated as a linear subspace of U by the set of all basis elements of U of the form

$$\prod_{\beta\in\Delta^+} Y_\beta^{m(\beta)} \prod_{\beta\in\Delta^+} X_\beta^{n(\beta)} \prod_{\alpha\in\Phi} H_\alpha^{k(\alpha)} \tag{2}$$

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where

$$\sum_{\beta\in\Delta^+} (n(\beta)-m(\beta))\beta = 0.$$

DEFINITION 4. With Γ and Φ as above we define $\mathscr{C}(\Gamma)$ to be the linear subspace of $\mathscr{C}$ generated by all basis elements of $\mathscr{C}$ of the form

$$\prod_{\beta\in\Gamma^+} Y_\beta^{m(\beta)} \prod_{\beta\in\Gamma^+} X_\beta^{n(\beta)} \prod_{\alpha\in\Phi} H_\alpha^{k(\alpha)}. \tag{3}$$

LEMMA 3. *$\mathscr{C}(\Gamma)$ is a subalgebra of $\mathscr{C}$.*

Proof. It suffices to observe that since Γ is closed in Δ the commutant of any two elements from the set $B(\Gamma, \Phi)=\{Y_\beta, X_\beta, H_\alpha \mid \beta\in\Gamma^+; \alpha\in\Phi\}$ is either z ro or can be expressed as a linear combination of elements from $B(\Gamma, \Phi)$.

LEMMA 4. *The complementary linear subspace $\overline{\mathscr{C}(\Gamma)}$ of $\mathscr{C}(\Gamma)$ in $\mathscr{C}$ determined by the basis* (2) *of $\mathscr{C}$ is an ideal in $\mathscr{C}$.*

Proof. Again it suffices to note that for any element $z\in B(\Gamma, \Phi)$ and any $w\in\{Y_{\beta'}, X_{\beta'} \mid \beta'\in\Delta\Gamma\}$ the commutant $[z, w]$ is either zero or is a linear combination of elements from $\{Y_{\beta'}, X_{\beta'} \mid \beta'\in\Delta-\Gamma\}$.

THEOREM 5. *If Γ is a complete subsystem of Δ then every algebra homomorphism $\gamma: \mathscr{C}(\Gamma)\to K$ can be trivially extended to a mass function $\bar{\gamma}: \mathscr{C}\to K$.*

Proof. Since $\overline{\mathscr{C}(\Gamma)}$ is an ideal of $\mathscr{C}$ it is clear that an algebra homomorphism $\gamma: \mathscr{C}(\Gamma)\to K$ can be extended to a mass function $\bar{\gamma}: \mathscr{C}\to K$ simply by setting $\bar{\gamma}$ equal to zero on elements of $\overline{\mathscr{C}(\Gamma)}$.

This theorem permits the construction of mass functions on $\mathscr{C}$ by extending algebra homomorphism on suitable subalgebras $\mathscr{C}(\Gamma)$ of $\mathscr{C}$. The next theorem provides sufficient conditions for combining algebra homomorphisms on different subalgebras of $\mathscr{C}$ to obtain a mass function on $\mathscr{C}$.

THEOREM 6. *If Γ_1 and Γ_2 are two disconnected complete subsystems of Δ and $\gamma_i: \mathscr{C}(\Gamma_i)\to K$ are algebra homomorphisms for $i=1, 2$ with $\gamma_1=\gamma_2$ on $\mathscr{C}(\{0\})$ then γ_1 and γ_2 admit a common extension to a mass function on $\mathscr{C}$.*

Proof. Since by assumption $\Gamma_1\cup\Gamma_2$ is a complete subsystem of Δ, it suffices to find a common extension of γ_1 and γ_2 to $\mathscr{C}(\Gamma_1\cup\Gamma_2)$. To this end we note that since $[X_\beta, X_{\beta'}]=[X_\beta, Y_{\beta'}]=[Y_\beta, Y_{\beta'}]=0$ for all $\beta\in\Gamma_1^+$ and all $\beta'\in\Gamma_2^+$ we can express any basis element $c\in\mathscr{C}(\Gamma_1\cup\Gamma_2)$ of the form (3) as a commuting product of a basis element $c_1\in\mathscr{C}(\Gamma_1)$ and a basis element $c_2\in\mathscr{C}(\Gamma_2)$ both of the form (3). Since this representation is unique up to factors from $\mathscr{C}(\{0\})$ we can define a map $\gamma: (\Gamma_1\cup\Gamma_2)\to K$ by setting for any basis element c of $\mathscr{C}(\Gamma_1\cup\Gamma_2)\gamma(c)=\gamma_1(c_1)\gamma_2(c_2)$ where $c_i\in\mathscr{C}(\Gamma_i)$ as above and extending linearly to all of $\mathscr{C}(\Gamma_1\cup\Gamma_2)$. It is clear that γ is the required extension of γ_1 and γ_2.

3. **Examples.** (A) It is clear that $\Gamma=\{0\}$ is a complete subsystem of Δ and

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moreover $\mathscr{C}(\{0\})$ is a commutative subalgebra of $\mathscr{C}$. Since $\Phi(\{0\})$ is generated as an algebra by $\{H_\alpha | \alpha \in \Phi\}$ for some fundamental subset Φ of Δ, the algebra homomorphisms $\gamma: \mathscr{C}(\{0\}) \to K$ are in one-one correspondence with the linear functionals on $\mathscr{H}$. In fact it is easily seen that the irreducible representations of L determined by the trivial extensions of such γ's to a mass function on $\mathscr{C}$ is simply the irreducible representation of L having "highest weight function" λ = the restriction of γ to $\mathscr{H}$.

(B) If $\beta_0 \in \Phi$ a fixed fundamental subset of Δ, then $\Gamma = \{0, \pm\beta_0\}$ is a complete subsystem of Δ. In this case $\mathscr{C}(\{0, \pm\beta_0\})$ is a commutative subalgebra of $\mathscr{C}$ generated by $\{H_\alpha | \alpha \in \Phi\} \cup \{Y_{\beta_0}X_{\beta_0}\}$. All algebra homomorphisms $\gamma: \mathscr{C}(\{0, \pm\beta_0\}) \to K$ are obtained by setting $\gamma(1)=1$; $\gamma(H_\alpha)$ = arbitrary scalar for each $\alpha \in \Phi$; and $\gamma(Y_{\beta_0}X_{\beta_0})$ = arbitrary scalar and extending linearly and multiplicatively to all of $\mathscr{C}(\{0, \pm\beta_0\})$. It is interesting to note here that for appropriate values of $\gamma(Y_{\beta_0}X_{\beta_0})$ the irreducible representation determined by the trivial extension of γ does not admit a highest weight function (cf. [1] or [3]).

(C) Let $\beta_1, \beta_2 \in \Phi$ a fixed fundamental subset of Δ, such that

$$\Gamma = \{0, \pm\beta_1, \pm\beta_2, \pm(\beta_1+\beta_2)\}$$

forms a complete subset of Δ. (In terms of the Dynkin diagram of L relative to Φ, this requires only that the simple roots β_1 and β_2 are directly connected by a single line.) In this case $\mathscr{C}(\Gamma)$ is a noncommutative subalgebra of $\mathscr{C}$ generated by

$$\{H_\alpha |\ \alpha \in \Phi\} \cup \{Y_{\beta_1}X_{\beta_1}, Y_{\beta_2}X_{\beta_2}, Y_{\beta_1+\beta_2}X_{\beta_1+\beta_2}, Y_{\beta_1+\beta_2}X_{\beta_1}X_{\beta_2}, Y_{\beta_2}Y_{\beta_1}X_{\beta_1+\beta_2}\}.$$

Using some calculations of Bouwer [1] related to the mass functions of A_2 we obtain all mass functions of $\mathscr{C}(\Gamma)$ by setting $\gamma(1)=1$; $\gamma(H_\alpha)$ = arbitrary scalar for each $\alpha \in \Phi$; and

$$\begin{aligned}
\gamma(Y_{\beta_1}X_{\beta_1}) &= s(s-1-\gamma(H_{\beta_1}))\\
\gamma(Y_{\beta_2}X_{\beta_2}) &= (s-1)(s+\gamma(H_{\beta_2}))\\
s\gamma(Y_{\beta_1+\beta_2}X_{\beta_1+\beta_2}) &= \gamma(Y_{\beta_1+\beta_2}X_{\beta_1}X_{\beta_2})\\
&= \gamma(Y_{\beta_2}Y_{\beta_1}X_{\beta_1+\beta_2})\\
&= s(s-1-\gamma(H_{\beta_1}))(s+\gamma(H_{\beta_2}))
\end{aligned}$$

and extending linearly and multiplicatively to all of $\mathscr{C}(\Gamma)$ where s is an arbitrary scalar.

(D) Using Theorem 6 we observe that we can combine any two disconnected complete subsystems Γ_1 and Γ_2 of Δ and again give an explicit means of obtaining all algebra homomorphisms $\gamma: \mathscr{C}(\Gamma_1 \cup \Gamma_2) \to K$ provided we know all algebra homomorphisms on $\mathscr{C}(\Gamma_1)$ and $\mathscr{C}(\Gamma_2)$ respectively.

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