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On Non-Integral Dehn Surgeries Creating Non-Orientable Surfaces

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Abstract. For a non-trivial knot in the 3-sphere, only integral Dehn surgery can create a closed 3-manifold containing a projective plane. If we restrict ourselves to hyperbolic knots, the corresponding claim for a Klein bottle is still true. In contrast to these, we show that non-integral surgery on a hyperbolic knot can create a closed non-orientable surface of any genus greater than two.

1 Introduction

For a knot K in the 3-sphere, let $E(K)$ denote its exterior $S^3 - \operatorname{Int} N(K)$, where $N(K)$ is a regular neighborhood of K. A *slope* on $\partial E(K)$ is the unoriented isotopy class of an essential simple closed curve on $\partial E(K)$. Then the slopes can be parameterized by the set $\mathbb{Q} \cup \{1/0\}$ in the usual way [8]. In particular, a slope corresponding to an integer is called an *integral slope*, otherwise it is a *non-integral slope*. For a slope r, $K(r)$ denotes the closed orientable 3-manifold obtained from S^3 by r-surgery. That is, $K(r) = E(K) \cup V$, where V is a solid torus glued to $E(K)$ along their boundaries so that r bounds a meridian disk in V.

In this short note, we consider the situation where Dehn surgery on a knot creates a 3-manifold containing an embedded closed non-orientable surface. Recall that any closed non-orientable surface is a connected sum of projective planes, and its genus is defined to be the number of summands. Thus a projective plane has genus one, a Klein bottle has genus two, etc. We remark that it is well known that $K(p/q)$ contains a closed non-orientable surface if and only if p is even (*cf.* [1]).

Let K be a non-trivial knot. If $K(r)$ contains a projective plane, then $K(r)$ is either real projective 3-space P^3 or a reducible 3-manifold with P^3-summand. Recently, the former was shown to be impossible by [6]. Hence r must be integral by [3]. For some non-hyperbolic knot, non-integral surgery can create a Klein bottle. But, if K is hyperbolic and $K(r)$ contains a Klein bottle, then r is integral by [4]. Along the line, Matignon and Sayari [7, Conjecture A] conjecture that only integral surgery can produce a closed non-orientable surface of genus three. However, we can give a counterexample to this conjecture. In fact, for any integer $n \geq 3$, we will give infinitely many knots (most of them are hyperbolic), each of which admits non-integral surgery creating a closed non-orientable surface of genus n. Note that if a 3-manifold contains a closed non-orientable surface of genus n then it also contains a closed non-orientable surface of genus $n + 2h$ for any $h > 0$, by attaching tubes locally.

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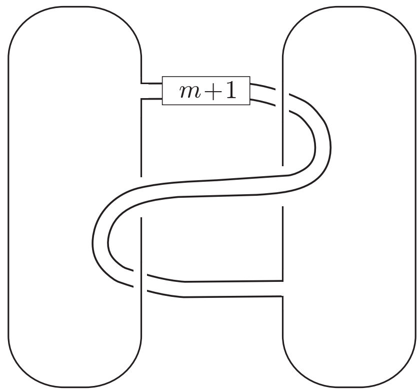

Figure 1

Theorem 1 *Let K be the pretzel knot $p(-3,3,m)$ for $m \in \mathbb{Z}$. For any integers $n \geq 3$ and $h \geq 0$, the 3-manifold $K(\frac{2n-4}{2n-3})$ obtained from S^3 by performing $(2n-4)/(2n-3)$-surgery on K contains a closed non-orientable surface of genus $n+2h$.*

By [5], $K = p(-3,3,m)$ is hyperbolic if $m \neq 0$. Of course, if $m = 0$, then K is the square knot, and if $m = \pm 1$, then K is Stevedore's knot or its mirror image, which is 2-bridge. In fact, we will see that the core of the attached solid torus intersects a closed non-orientable surface of genus n only once as in Figure 4. Thus the cases $m = 0, \pm 1$ show that Lemmas 6.1 and 6.2 (hence Lemma 1.3 and Corollary 1.5) of [7] are not correct. (In Section 3 of [7], S might be boundary-compressible when $s = 1$.)

In general, it is interesting to find the minimum genus of which a closed non-orientable surface can be embedded in a given 3-manifold. For example, Bredon and Wood [1] determined this for lens spaces. See also [2]. It might be true that the minimum genus of closed non-orientable surfaces in our $K(\frac{2n-4}{2n-3})$ is n. Indeed, this can be confirmed when $n = 3$ and 4, but we could not prove it generally.

2 Proof of Theorem 1

Let K be the pretzel knot $p(-3,3,m)$. Then it has a ribbon knot presentation as shown in Figure 1, where the box with $m+1$ denotes $m+1$ right-handed half-twists. Let $n \geq 3$ be an integer, and let $M = K(\frac{2n-4}{2n-3})$ be the resulting manifold obtained from S^3 by $(2n-4)/(2n-3)$-surgery on K. To prove Theorem 1, it is sufficient to show that M contains a closed non-orientable surface of genus n.

Take an unknotted circle C as in Figure 2, and perform (-1)-twisting on C. Then the result can be deformed as in Figure 3, and $(2n-3)$-twisting yields the surgery description of M there.

Finally, Figure 4 shows a non-orientable surface S of genus n whose boundary circle has slope $2n-4$. Here, S can be seen as the union of $n-2$ Möbius bands and

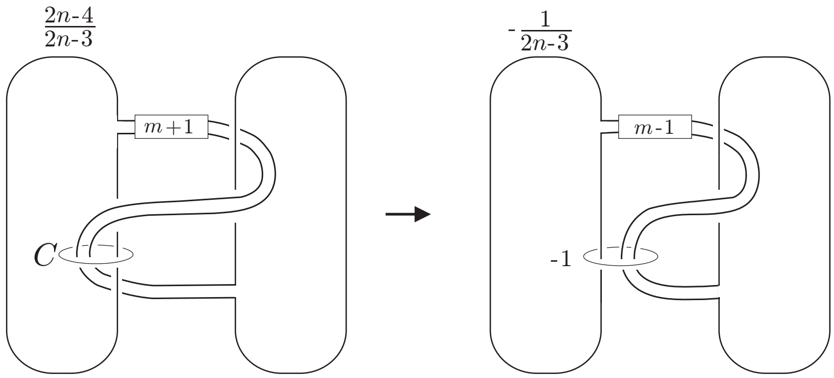

Figure 2

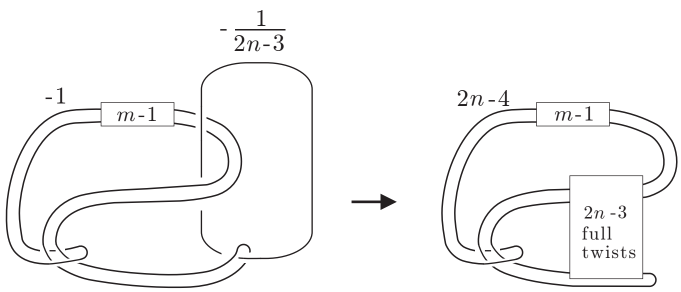

Figure 3

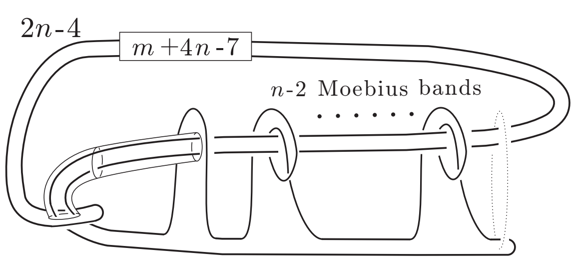

Figure 4

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a twice-punctured disk with a tube attached. Then S can be capped off by a disk of the attached solid torus. Hence M contains a closed non-orientable surface of genus n. Also, the dotted circle indicates the core of the attached solid torus of M, which intersects S in one point. This completes the proof of Theorem 1. ■

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