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Subdirectly irreducible Rees matrix semigroups

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Minimal congruences on a Rees matrix semigroup S having at least one proper congruence are described. Necessary and sufficient conditions for S to be subdirectly irreducible are given in two cases according to whether the structure group of S is trivial.

1. Introduction

Congruences on a Rees matrix semigroup (or a completely 0-simple semigroup) have been described in various ways. The aim of this paper is to show that the recent characterization by Lallement [2] in terms of admissible triples can be used to solve a problem which the other descriptions did not seem to permit. Namely, we will give necessary and sufficient conditions for a Rees matrix semigroup to be subdirectly irreducible; that is, to have the least nontrivial congruence.

Section 2 contains several properties of admissible triples and a restatement of Lallement's Theorem. Our results on subdirect irreducibility are contained in Section 3. Obviously every congruence-free semigroup is subdirectly irreducible, so congruence-free Rees matrix semigroups are described first. Next we list the three possible forms of minimal congruences on a Rees matrix semigroup S which is not congruence-free. Then we determine when S is subdirectly irreducible in terms of the sandwich matrix, when the structure group G is trivial, and in terms of reductivity and the subdirect irreducibility of G when G is non-trivial.

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All undefined terms and notation can be found in [4].

2. Admissibility

Let $S = M^0(I, G, M; P)$ be a regular Rees matrix semigroup. We will define an admissible triple on S and relate this concept to the reductivity of S and the entries of P.

DEFINITION. Let r be an equivalence relation on I, N be a normal subgroup of G, and π be an equivalence relation on M. Then (r, N, π) is called an *admissible triple* on S if the following conditions are satisfied:

(A1) if irj then for all $\mu \in M$, $p_{\mu i} \neq 0$ implies $p_{\mu j} \neq 0$;

(A2) if irj, $p_{\mu i} \neq 0$, and $p_{\nu i} \neq 0$, then

$$p_{\mu i}p_{\nu i}^{-1}p_{\nu j}p_{\mu j}^{-1} \in N ;$$

(A3) if $\mu\pi\nu$ then for all $i \in I$, $p_{\mu i} \neq 0$ implies $p_{\nu i} \neq 0$;

(A4) if $\mu\pi\nu$, $p_{\mu i} \neq 0$, and $p_{\nu j} \neq 0$, then

$$p_{\mu i}p_{\nu i}^{-1}p_{\nu j}p_{\mu j}^{-1} \in N .$$

For any set A we will denote the identity relation by ε_A and the universal relation by ω_A. Where no ambiguity exists we will omit the subscripts. Also if $a, b \in A$, $a \neq b$, we will let $R(a, b)$ denote the equivalence relation whose only nontrivial class is the set $\{a, b\}$.

Our first two results will be fundamental to later considerations. Their proofs follow immediately from the admissibility conditions.

LEMMA 1. *Let (r, N, π) be an admissible triple on S. If $r' \subseteq r$, $N \subseteq N'$, and $\pi' \subseteq \pi$ then (r', N', π') is also admissible.*

LEMMA 2. *For each normal subgroup N of G the triple $(\varepsilon, N, \varepsilon)$ is admissible.*

Recall that the ith and jth columns of the sandwich matrix P are *right proportional* if there exists some element $c \in G$ such that $p_{\mu i} = p_{\mu j}c$ for all $\mu \in M$.

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LEMMA 3. *If* (r, e, ε) *is admissible for some* $r \neq \varepsilon_I$ *then two distinct columns of* P *are right proportional.*

Proof. Since $r \neq \varepsilon_I$ there exist $i \neq j \in I$ such that irj. We will show that the ith and jth columns of P are right proportional.

Since P is regular, $p_{\nu i} \neq 0$ for some $\nu \in M$. Then $p_{\nu j} \neq 0$ by (A1) so put $c = p_{\nu i}^{-1}p_{\nu j}$. Let $\mu \in M$. If $p_{\mu i} \neq 0$ then (A2) implies that $p_{\mu i}p_{\nu i}^{-1}p_{\nu j}p_{\mu j}^{-1} = e$ whence $p_{\mu i}^{-1}p_{\nu j} = p_{\nu i}^{-1}p_{\nu j} = c$. On the other hand if $p_{\mu i} = 0$ then $p_{\nu i} = 0$ by (A1). Hence $p_{\mu j} = p_{\mu i}c$ for all $\mu \in M$; so the ith and jth columns have the desired property.

LEMMA 4. *If the* ith *and* jth *columns of* P *are right proportional then* $(R(i, j), N, \varepsilon)$ *is an admissible triple for each normal subgroup* N *of* G.

Proof. By hypothesis there exists some $c \in G$ such that $p_{\mu i} = p_{\mu j}c$ for all $\mu \in M$, so (A1) obviously holds. If $p_{\mu i} \neq 0$ and $p_{\nu i} \neq 0$ then $p_{\mu i}p_{\nu i}^{-1}p_{\nu j}p_{\mu j}^{-1} = (p_{\mu j}c)(p_{\nu j}c)^{-1}p_{\nu j}p_{\mu j}^{-1} = e$, hence (A2) holds. The remaining admissibility conditions are easy to verify.

It is well-known (for example, [4, Theorem V.3.14]) that S is left reductive if and only if no two distinct columns of P are right proportional. Hence

COROLLARY 5. *The following conditions are equivalent on* S;

(i) S *is not left reductive;*

(ii) $(R(i, j), e, \varepsilon)$ *is admissible for some* $i \neq j$;

(iii) the ith *and* jth *columns of* P *are right proportional for some* $i \neq j$.

Denote the lattice of congruences on a semigroup S by $C(S)$. A congruence $\sigma \in C(S)$ is called *proper* if it is different from the universal relation. Put

$$C'(S) = \{\sigma \in C(S) : \sigma \neq \varepsilon \text{ and } \sigma \neq \omega\} .$$

We will conclude this section by stating the very basic result of

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Lallement [2] linking congruences on S to admissible triples. The notation introduced will be used throughout the remainder of this paper.

THEOREM 6 (Lallement). *Let* $S = M^0(I, G, M; P)$. *If* (r, N, π) *is an admissible triple on* S *then the relation* $\theta = \theta(r, N, \pi)$ *defined on* S *by*

$$(i, a, \mu)\theta(j, b, \nu) \text{ iff } a \neq 0,\ b \neq 0,\ irj,\ \mu\pi\nu, \text{ and}$$
$$p_{\alpha i} a p_{\mu k} \equiv p_{\alpha j} b p_{\nu k} \pmod{N} \text{ for some } \alpha \in M,\ k \in I \text{ such that}$$
$$p_{\alpha i} \neq 0,\ p_{\mu k} \neq 0,\ 0\theta 0,$$

is a proper congruence on S . *Conversely every proper congruence on* S *can be written in the form* $\theta(r, N, \pi)$ *for some admissible triple* (r, N, π) .

It can easily be verified that $\theta(r, N, \pi) \subseteq \theta(s, K, \rho)$ if and only if $r \subseteq s$, $N \subseteq K$, and $\pi \subseteq \rho$. Moreover, using Lemma 2 we see that $\theta_N = \theta(\varepsilon, N, \varepsilon) \in C(S)$ for every normal subgroup N of G .

3. Subdirect irreducibility

In this section we make use of Lallement's Theorem to find all subdirectly irreducible Rees matrix semigroups. Recall that a semigroup is *congruence-free* if $C'(S) = \emptyset$. (The term h-simple was used in [5].) We first dispose of those Rees matrix semigroups which are congruence-free since they are always subdirectly irreducible. For those which are not congruence-free we will consider two cases according to whether the structure group is trivial. First we will use the above results to give an alternative proof of a result due to Munn ([3, Theorem 2.1]; see also [6]).

THEOREM 7. *A Rees matrix semigroup* $S = M^0(I, G, M; P)$ *is congruence-free if and only if*

(1) G *is a simple group and* $S \simeq G$ *or*

(2) G *is the trivial group and no two distinct rows or columns of* P *are identical.*

Proof. Let S be congruence-free. It follows from Lemma 2 that $\theta = \theta(\varepsilon, G, \varepsilon) \in C(S)$, so $\theta = \omega$ or $\theta = \varepsilon$. The former case implies that $|I| = |M| = 1$. Since $\theta(\varepsilon, N, \varepsilon) \in C(S)$ for every normal subgroup N of G it follows that G is simple. From the latter case we see immediately that G is the trivial group, and that P is of the desired form follows from Lemma 3.

That such semigroups are congruence-free is obvious.

We will now proceed to describe those subdirectly irreducible Rees matrix semigroups S which are not congruence-free. First we will characterize their minimal congruences.

LEMMA 8. *A proper congruence* σ *on* S *is minimal if and only if* σ *has one of the following three forms:*

(1) $\sigma = \theta(R(i, j), e, \varepsilon)$ *for some* $i, j \in I$, $i \neq j$;

(2) $\sigma = \theta(\varepsilon, e, R(\mu, \nu))$ *for some* $\mu, \nu \in M$, $\mu \neq \nu$;

(3) $\sigma = \theta_N$ *for some minimal normal subgroup* N *of* G .

Proof. Let $\sigma = \theta(r, N, \pi)$ be minimal on S . Since $\theta_N \in C(S)$ and $\theta_N \subseteq \sigma$ we have either $\sigma = \theta_N$ or $N = e$. Thus all minimal congruences on S are of the form $\theta(\varepsilon, N, \varepsilon)$ or $\theta(r, e, \pi)$.

First we show that θ_N is minimal if and only if N is minimal. Suppose θ_N be minimal. If K is a normal subgroup of G and $K \subseteq N$ then $\theta_K \subseteq \theta_N$ so minimality implies $\theta_K = \varepsilon_S$ or $\theta_K = \theta_N$. Thus $K = e$ or $K = N$, respectively, so N is minimal. Conversely if N is minimal and $\sigma = \theta(r, K, \pi) \subseteq \theta_N$ then $r = \pi = \varepsilon$ and $K \subseteq N$. The last inclusion implies that $K = e$ or $K = N$, so $\sigma = \varepsilon_S$ or $\sigma = \theta_N$ respectively. Hence θ_N is minimal.

It remains to determine when $\sigma = \theta(r, e, \pi)$ is minimal. Let σ be minimal and suppose that $r \neq \varepsilon$. Then $\theta(r, e, \varepsilon) \in C'(S)$ by Lemma 1, and $\theta(r, e, \varepsilon) \subseteq \sigma$, so $\pi = \varepsilon$. Further, irj for some $i \neq j \in I$, so $\theta(R(i, j), e, \varepsilon) \subseteq \theta(r, e, \varepsilon) = \sigma$. But then the minimality of σ implies $R(i, j) = r$. Thus $\sigma = \theta(r(i, j), e, \varepsilon)$; similarly $\pi \neq \varepsilon$ implies $\sigma = \theta(\varepsilon, e, R(\mu, \nu))$ for some $\mu \neq \nu \in M$. Since $\sigma \neq \varepsilon$ we must have either $r \neq \varepsilon$ or $\pi \neq \varepsilon$, so σ is of the desired form. That such congruences are minimal is obvious.

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The standard way of introducing subdirectly irreducible semigroups is via the direct product (see, for example, [5]). However it suits our purposes here to adopt the definition that a semigroup is *subdirectly irreducible* if the intersection of any set of nonidentical congruences is nonidentical.

THEOREM 9. *The following conditions are equivalent on a Rees matrix semigroup* $T = M^0(I, e, M; P)$ *over the trivial group:*

(i) T *is subdirectly irreducible;*

(ii) exactly two distinct rows or two distinct columns of P *are identical;*

(iii) T *has precisely one congruence different from* ε *and* ω .

Proof. *(i) implies (ii).* If T is subdirectly irreducible with least congruence σ , then according to Lemma 8 we can say without loss of generality that $\sigma = \theta(R(i, j), e, \varepsilon)$ for some $i \neq j$. It follows from Corollary 5 that the ith and jth columns of P are identical. Suppose two other columns of P , say the kth and lth columns, are also identical. Using Corollary 5 again, put $\rho = \theta(R(k, l), e, \varepsilon) \in C(T)$. If $\{i, j\} \neq \{k, l\}$ then $\sigma \cap \rho = \varepsilon_T$. But T is subdirectly irreducible and $\sigma \neq \varepsilon_T$, so $\rho = \varepsilon_T$. This means that $k = l$. A similar approach shows that no other column of P is equal to either the ith or jth column, hence these are the only distinct identical columns of P .

Now suppose that two rows, say the μth and νth rows, are equal. Then Corollary 5 implies that $\tau = \theta(\varepsilon, e, R(\mu, \nu)) \in C(T)$. But $\sigma \cap \tau = \varepsilon_T$, so $\tau = \varepsilon_T$ or $\tau = \sigma$. The first equality implies that $\mu = \nu$ while the latter is impossible since $R(i, j) \neq \varepsilon$. Thus no two distinct rows of P are identical.

(ii) implies (iii). Suppose that the only distinct identical columns are the ith and jth , and that no two distinct rows are identical. According to Corollary 5, $\sigma = \theta(R(i, j), e, \varepsilon) \in C'(T)$. Let $\tau = \theta(r, e, \pi) \in C'(T)$. If $\pi \neq \varepsilon$ then it follows easily from Lemma 1 that $(\varepsilon, e, R(\mu, \nu))$ is an admissible triple for some $\mu \neq \nu$. However this implies that the μth and νth rows of P are identical,

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contradicting the hypothesis. Thus $\tau = \theta(r, e, \varepsilon)$, so $r \neq \varepsilon$, which means krl for some $k \neq l$. But then $(R(k, l), e, \varepsilon)$ is an admissible triple, so the kth and lth columns are identical by Corollary 5. By hypothesis we conclude that $\{k, l\} = \{i, j\}$, so $\tau = \sigma$. Thus σ is the only congruence on T different from ε and ω.

That *(iii)* implies *(i)* is obvious.

For the remainder of this paper let $S = M^0(I, G, M; P)$ where G is a nontrivial group and e denotes the identity of G. Recall that $\theta_N = \theta(\varepsilon, N, \varepsilon)$ for each normal subgroup N of G.

PROPOSITION 10. *If S is subdirectly irreducible then it is reductive.*

Proof. We know from Lemma 2 that $\theta_G \in C'(S)$. Since S is subdirectly irreducible $\sigma \cap \theta_G \neq \varepsilon_S$ for all $\sigma \in C'(S)$. Thus no triple of the form $(R(i, j), e, \varepsilon)$ or $(\varepsilon, e, R(\mu, \nu))$ can be admissible since each induced congruence intersects θ_G nontrivially. That S is reductive now follows from Corollary 5 and its dual.

PROPOSITION 11. *If S is subdirectly irreducible then G is a subdirectly irreducible group.*

Proof. Let $\sigma = \theta(r, N, \pi)$ be the least congruence on S. It suffices to show that $\sigma = \theta_N$. For in such a case if $K \neq e$ is a normal subgroup of G then $\theta_K \in C'(S)$ by Lemma 2. But the minimality of σ implies that $\theta_N \subseteq \theta_K$, whence $N \subseteq K$. Thus N is the least normal subgroup of G, so G is subdirectly irreducible.

Now we will show that $\sigma = \theta_N$. First, suppose that $N = e$; that is, $\sigma = \theta(r, e, \pi)$. If $r \neq \varepsilon$ then $R(i, j) \subseteq r$ for some $i \neq j$, so $(R(i, j), e, \varepsilon)$ is an admissible triple by Lemma 1. It follows from Corollary 5 that S is not left reductive and from Proposition 10 that S is not subdirectly irreducible, contradicting the hypothesis. The assumption $\pi \neq \varepsilon$ will lead analogously to the same contradiction. Since $\sigma \neq \varepsilon_S$ it follows that $N \neq e$, so that $\sigma = \theta_N$.

THEOREM 12. *A Rees matrix semigroup S over a nontrivial group G*

is subdirectly irreducible if and only if S *is reductive and* G *is subdirectly irreducible.*

Proof. In view of Propositions 10 and 11 it suffices to prove the necessity. So let S be reductive and G be subdirectly irreducible with least normal subgroup K. We will show that θ_K is the least non-identical congruence on S.

Suppose that (r, e, π) is an admissible triple. If $r \neq \varepsilon_I$ then $R(i, j) \subseteq r$ for some $i \neq j$, so $(R(i, j), e, \varepsilon_M)$ is admissible by Lemma 1. But Corollary 5 indicates that S is not right reductive, which contradicts the hypothesis. Hence $r = \varepsilon_I$; similarly $\pi = \varepsilon_M$. Therefore no nonidentical congruence on S has the trivial subgroup for its middle entry.

Now let $\sigma \in C'(S)$, $\sigma = \theta(r, N, \pi)$. We have seen above that $N \neq e$, so the minimality of K implies that $K \subseteq N$. It is clear that $\theta_K \subseteq \sigma$. Finally, Lemma 2 insures that $\theta_K \neq \varepsilon_S$, so θ_K is a non-identical congruence which is contained in every such congruence.

We might point out that the proofs of the last two results indicate that the least congruence on S is θ_K where K is the least normal subgroup of G. Moreover if $\theta(r, N, \pi) \in C'(S)$ then $N \neq e$.

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