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FIXED POINT OF SUM FOR CONCAVE AND CONVEX OPERATORS WITH APPLICATIONS

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In this paper we study fixed points of sums of α-concave and $(-\alpha)$-convex operators in Υ-complete partially ordered linear spaces. As an application we obtain existence and uniqueness theorems for solutions of a certain type of nonlinear integral equation.

I INTRODUCTION

The concept of α-concave and $(-\alpha)$-convex operators was first introduced by Potter [**5**]. Then Guo Dajun, [**1**] studied fixed points and intrinsic elements of the two kinds of operators. Ortega [**4**] and Leggett [**3**] studied the fixed points of the sum and product of operators. In this paper we extended the real partially ordered Banach spaces in [**5**], [**1**] to Υ-complete partially ordered linear spaces, and we study the fixed points of the sum of α-concave and $(-\alpha)$-convex operators. We use the above result to obtain an existence and uniqueness theorem for the solution of a kind of nonlinear integral equation. Obviously the results in this paper are more general than those in [**3**] and [**5**].

II MAIN RESULTS

DEFINITION 1: Let P be a positive cone in a Υ-complete partially ordered linear space E (see [**2**]). Φ is the set of interior points of P. An operator $f : \Phi \to \Phi$ $(0 < \alpha < 1)$ is called *α-concave* (or *$(-\alpha)$-convex*) if it satisfies the following condition:

$$f(tx) \geqslant t^{\alpha} fx \qquad (\text{or } f(tx) \leqslant t^{\alpha} fx) \qquad \forall x \in \Phi, \ \ 0 < t < 1$$

It is easy to see that f is α-concave (or $(-\alpha)$-convex) if and only if

$$f(sx) \leqslant s^{\alpha} fx \qquad (\text{or } f(sx) \geqslant s^{-\alpha} fx), \quad \forall x \in \Phi, \ \ s > 1$$

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THEOREM 1. *Let P be a positive cone in a Υ-complete partially ordered linear space E. $g, h : \Phi \to \Phi$ are the increasing α-concave and decreasing $(-\alpha)$-convex operators respectively, $0 < \alpha < 1$. Then the operator*

$$Ax = gx + hx + C \quad (x \in \Phi,\ C \in P) \tag{1}$$

has a unique fixed point $x^\star$ in Φ, and for any $x_o \in \Phi$, we have

$$x^\star = \vee\{x_n\} = \wedge\{x_n\}, \tag{2}$$

where $x_n = Ax_{n-1}$, and we have the estimate

$$0 \leqslant x^\star - x_n \leqslant (1 - S_0^{-2\alpha^n})S_0 x_0 \tag{3}$$

where

$$S_0 = \max\{S_1, S_2\}$$

$$S_1 = \sup\{S > 1 : S^{\alpha-1}x_0 \leqslant gx_0 + hx_0\} \tag{4}$$

$$S_2 = \inf\{S > 1 : gx_0 + hx_0 \leqslant S^{1-\alpha}x_0\}$$

PROOF: First let $c = 0$. Then it is clear that $S_0 > 1$. For any $x_0 \in \Phi$, from [4] we have

$$\frac{1}{2}S_0^{\alpha-1}x_0 \leqslant gx_0 + hx_0 \leqslant \frac{1}{2}S_0^{1-\alpha}x_0 \tag{5}$$

Let $U_0 = S_0^{-1}x_0$, $V_0 = S_0x_0$ then $V_0 >> U_0$. Put

$$U_n = gU_{n-1} + hV_{n-1}, \quad V_n = gV_{n-1} + hU_{n-1}. \tag{6}$$

We may prove by induction that

$$[U_n, V_n] \subseteq [U_{n-1}, V_{n-1}], \quad (n = 1, 2, \dots). \tag{7}$$

Since E is Υ-incomplete, there exists $u^\star$, $v^\star \in E$ such that

$$u^\star = \vee\{u_n\}, \quad v^\star = \wedge\{v_n\},$$

and $u_n \leqslant u^\star \leqslant v^\star \leqslant v_n$. Thus

$$u_{n-1} = gu_n + hv_n \leqslant gu^\star + hv^\star \leqslant gv_n + hu_n = v_{n+1}.$$

Hence

$$u_n \leqslant u^\star \leqslant gu^\star + hv^\star \leqslant v^\star \leqslant v_n. \tag{8}$$

By induction it is easy to prove that

$$u_n \geqslant S_0^{-2\alpha^n} v_n \quad (n = 0, 1, 2, \dots). \tag{9}$$

From (8) and (9) we have

$$0 \leqslant v^\star - u^\star \leqslant v_n - u_n \leqslant (1 - S_0^{-2\alpha^n}) v_0.$$

By the Archimedean property we deduce that $v^\star = u^\star$. So by (8) it follows that $u^\star$ is a fixed point of A.

Now we prove uniqueness. Suppose $\overline{x}, \overline{\overline{x}} \in \Phi$ are two distinct fixed points of A. Then there exists $\mu > 1$, such that

$$\mu^{-\alpha} \overline{x} \leqslant \overline{\overline{x}} \leqslant \mu^{-\alpha} \overline{x}.$$

We may prove by induction that

$$\mu^{-\alpha''} \overline{x} \leqslant \overline{\overline{x}} \leqslant \mu^{\alpha''} \overline{x}.$$

In the foregoing inequality we take the limit as $n \to \infty$ and obtain $\overline{x} \leqslant \overline{\overline{x}} \leqslant \overline{x}$. So $\overline{x} = \overline{\overline{x}}$.

Next we prove that $x^\star$, defined by (2), is a fixed point of A. Hence it is a unique fixed point. First, we may prove by induction that

$$u_n \leqslant x_n \leqslant v_n \quad (n = 0, 1, 2, \dots). \tag{10}$$

Let $x_\star = \wedge\{x_n\}$, $x^\star = \vee\{x_n\}$. From (10) and (7), we obtain

$$u_n \leqslant v^\star \leqslant x_\star \leqslant x^\star \leqslant v^\star \leqslant v_n. \tag{11}$$

Since $u^\star = v^\star$, so $x_\star = x^\star = u^\star$. Hence $x^\star$ is a fixed point of A. By (11) and (9), we know that (3) is true. Finally, let $c \neq 0$. Since $Gx = gx + \frac{1}{2}C$, $Hx = hx + \frac{1}{2}C$ are increasing α-concave and decreasing $(-\alpha)$-convex operators respectively, so the theorem is still valid in the case $c \neq 0$. ∎

III Applications

From Theorem 1 we obtain immediately:

Thoerem 2. *Under the conditions of Theorem 1, the equation*

$$Bx = g(\frac{1}{x}) + h(\frac{1}{x}) + C = x$$

has a unique solution.

Theorem 3. *Under the conditions of Theorem 1, we use x_λ to denote the unique solution of the equation $Ax = gx + hx + C = \lambda x$. Then x_λ is decreasing for λ (that is, $0 < \lambda_1 < \lambda_2 \Rightarrow x_{\lambda_1} > x_{\lambda_2}$), o-continuous (that is, for $\lambda_0 > 0$, $0 - lim_{\lambda \to \lambda_0} x_\lambda = x_{\lambda_0}$) and*

$$(12) \qquad 0 - \lim_{\lambda \to +\infty} x_\lambda = 0, \quad 0 - \lim_{\lambda \to 0^*} x_\lambda = +\infty \qquad \text{(the infinite element in } \Phi.$$

Proof: For $\lambda > 0$, by Theorem 1 we know that $Ax = \lambda x$ has a unique solution x_0 in Φ. Let $0 < \lambda_1 < \lambda_2$, if $x_{\lambda_2} \not\leqslant x_{\lambda_2}$, put

$$(13) \qquad \begin{aligned} M &= \inf\{u : x_{\lambda_2} \leqslant u x_{\lambda_1}\} \\ m &= \sup\{\Theta : \Theta x_{\lambda_1} \leqslant x_{\lambda_2}\} \end{aligned}$$

It is easy to see that $M > 1$ and $m > 1$ and

$$m x_{\lambda_1} \leqslant x_{\lambda_2} \leqslant M x_{\lambda_1}, \quad m \leqslant M.$$

If $m^{-1} \geqslant M^{-1}$, then $m \leqslant M^{-1}$. This contradicts $m > 1$. Hence, $m^{-1} < M$, i.e., $M^{-1} < m$. Thus we have

$$M^{-1} x_{\lambda_1} \leqslant x_{\lambda_2} \leqslant M x_{\lambda_1},$$

$$x_{\lambda_2} \leqslant \frac{1}{\lambda_2}[g(M x_{\lambda_1}) + h(M^{-1} x_{\lambda_1}) + C] \leqslant \frac{\lambda_1}{\lambda_2} M^\alpha x_{\lambda_1}.$$

By (13) we have $M = \frac{\lambda_1}{\lambda_2} M^\alpha$. Thus $\lambda_2 < \lambda_1$. This contradicts the hypothesis of the theorem. Hence $x_{\lambda_2} \leqslant x_{\lambda_1}$. Since the fixed point of $\frac{1}{\lambda}A$ is unique, so $x_{\lambda_2} < x_{\lambda_1}$.

Now we prove o-continuity. We observe that $0 < \lambda_1 < \lambda_2 \Rightarrow x_{\lambda_2} < x_{\lambda_1}$. Put

$$(14) \qquad \begin{aligned} m &= \sup\{\Theta : \Theta x_{\lambda_1} \leqslant x_{\lambda_2}\} \\ M &= \inf\{\mu : x_{\lambda_2} \leqslant \mu x_{\lambda_1}\} \end{aligned}$$

It is easy to see that $0 < m < 1$,

$$m x_{\lambda_1} \leqslant x_{\lambda_2} \leqslant M x_{\lambda_1}, \qquad m \leqslant M.$$

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If $m^{-1} < M$, then $M > 1$, $M^{-1} < m$. Hence

$$M^{1} x_{\lambda_1} \leqslant x_{\lambda_2} \leqslant M x_{\lambda_1},$$

$$x_{\lambda_2} \leqslant \frac{1}{\lambda_2}[g(M x_{\lambda_1}) + h(M^{-1} x_{\lambda_1}) + C] \leqslant \frac{\lambda_1}{\lambda_2} M^{\alpha} x_{\lambda_1}.$$

By (14) we have

$$M \leqslant \frac{\lambda_1}{\lambda_2} M^{\alpha}, \qquad \frac{\lambda_1}{\lambda_2} \geqslant M^{1-\alpha} > 1$$

so $\lambda_1 > \lambda_2$, which contradicts our hypothesis. Hence $M \leqslant m^{-1}$. Then we have

$$m x_{\lambda_1} \leqslant x_{\lambda_2} \leqslant m^{-1} x_{\lambda_1}$$

$$x_{\lambda_2} \geqslant \frac{1}{\lambda_2}[g(m x_{\lambda_1}) + h(m^{-1} x_{\lambda_1}) + C] \geqslant \frac{\lambda_1}{\lambda_2} m^{\alpha} x_{\lambda_1}. \tag{15}$$

By (14) we have

$$\frac{\lambda_1}{\lambda_2} m^{\alpha} \leqslant m, \qquad (\frac{\lambda_1}{\lambda_2})^{\frac{1}{1-\alpha}} \leqslant m.$$

Hence

$$x_{\lambda_2} \geqslant m x_{\lambda_1} \geqslant (\frac{\lambda_1}{\lambda_2})^{\frac{1}{1-\alpha}} x_{\lambda_1}$$

$$0 < x_{\lambda_1} - x_{\lambda_2} \leqslant x_{\lambda_1} - (\frac{\lambda_1}{\lambda_2})^{\frac{1}{1-\alpha}} x_{\lambda_1} = [1 - (\frac{\lambda_1}{\lambda_2})^{\frac{1}{1-\alpha}} x_{\lambda_1}.$$

In this inequality, let $\lambda_1 = \lambda_0$, $\lambda_2 = \lambda$. Then x_λ is o-continuous with respect to λ. As in (15) we have

$$(\frac{\lambda_1}{\lambda_2}) M^{-\alpha} x_{\lambda_1} \leqslant x_{\lambda_2}.$$

Hence

$$\frac{\lambda_1}{\lambda_2} M^{-\alpha} x_{\lambda_1} \leqslant x_{\lambda_2} \leqslant \frac{\lambda_1}{\lambda_2} M^{\alpha} x_{\lambda_1}.$$

In this inequality, let $\lambda_2 = \lambda$, and we see that (12) holds. ∎

THEOREM 4. *Let E be a Υ-complete Riesz space of Banach type and Φ be a non-empty positive cone of E. With operator A defined as in Theorem 1, we have that A is a contraction on Φ. That is, there exists r, R $(0 < r < R)$, such that*

$$\forall x \in \Phi, \quad 0 \leqslant \| x \| < r \Rightarrow Ax \nleqslant x,$$

$$\forall x \in \Phi, \quad \| x \| > R \Rightarrow Ax \ngeqslant x.$$

PROOF: By Theorem 1 we deduce that A has a fixed point $x^\star$. $\forall x \in \Phi$, put

$$(16) \qquad \begin{aligned} t_0 &= \sup\{t : tx^\star \leqslant x\} \\ s_0 &= \inf\{s : x \leqslant sx^\star\}. \end{aligned}$$

Obviously,

$$(17) \qquad t_0 x^\star \leqslant x \leqslant s_0 x^\star, \qquad t_0 \leqslant s_0.$$

First we prove that

$$(18) \qquad x \in \Phi, \qquad x \geqslant Ax \Rightarrow x \geqslant x^\star.$$

By (17), $s_0^{-1} \nleqslant t_0$. Hence $s_0 < t_0^{-1}$. By (17), we have

$$t_0 x^\star \leqslant x \leqslant t_0^{-1} x^\star.$$

If $t_0 < 1$, then $x \geqslant Ax \geqslant t_0^\alpha x^\star$. By (16), we have $t_0^\alpha \leqslant t_0$, which is a contradiction. So $t_0 \geqslant 1$, and (18) holds.

Similarly, we have

$$(19) \qquad x \in \Phi,\ x \leqslant Ax \Rightarrow x \leqslant x^\star.$$

Since the interval in a Υ-complete Riesz space of Banach type is bounded, from (18), (19) we see that the Theorem holds. ∎

THEOREM 5. *Consider the integral equation*

$$(20) \qquad \lambda x(t) = \int_{R^n} \{k_1(t,s) \sum_{i=1}^{\infty} a_i(s)[x(s)]^{\alpha_i} + k_2(t,s) \sum_{i=1}^{\infty} b_i(s)[x(s)]^{-\beta_i}\} ds$$

where $\lambda > 0$, R^n is an n-dimensional Euclidean space. If

(i) $\alpha_i, \beta_i > 0$ and $\sup_i \alpha_i = \sup_i \beta_i = \alpha > 1$;

(ii) $k_i(\ell, s)$ $(i = 1, 2)$ are nonnegative measureable functions on R^{2n}, and there exist constants m, M $(0 < m < M)$ such that

$$m \leqslant \int_{R^n} k_i(\ell, s) ds \leqslant M, \quad i = 1, 2, \quad \forall t \in R^n;$$

(iii) $a_i(s)$, $b_i(s)$ are nonnegative measurable functions on R^n and there exist constants Υ_i, Θ_i $(i = 1, 2)$, $0 < \Theta_i < \Upsilon_i$ such that

$$\Theta_i \leqslant \sum_{i=1}^{\infty} a_i(s) \leqslant \Upsilon, \quad \Theta_2 \leqslant \sum_{i=1}^{\infty} b_i(s) \leqslant \Upsilon_2,$$

then equation (20) has a unique continuous solution $x_\lambda(t)$ satisfying the condition

$$0 < \inf_{t \in R^n} x_\lambda(t) \leqslant \sup_{t \in R^n} x_\lambda(t) < +\infty.$$

PROOF: The proof is an easy application of Theorem 1. ∎

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